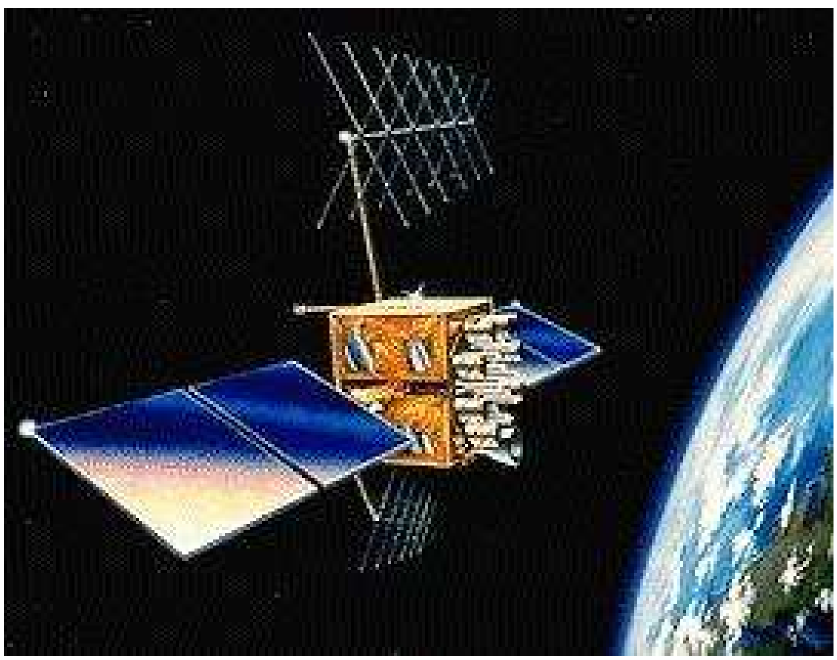
Relativistic effects
in GPS and LEO



Mikkjal Gulklett

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University of Copenhagen, Denmark
Department of Geophysics
The Niels Bohr Institute
for Physics, Astronomy and Geophysics
Supervisors: Carl Christian Tscherning and Poul Olesen

Preface

This rapport appears in partial fulfillment of the requirement for the Cand. Scient. degree at the University of Copenhagen, Niels Bohr Institute.
Thanks to: My supervisors Carl Christian Tscherning and Poul Olesen for stimulating and interesting discussions.

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Summary and introduction

Typically a problem in a first year course in special relativity starts with the phrase: *Consider a spacecraft moving with* 90% *of the speed of light...* In other words; we are dealing with an extreme situation. Often leading people to think of the theory of relativity as something exotic, which only has theoretical academic interest.

However, with the precision now available from modern atomic clocks, high precision timekeeping requires the application of the general and special theory of relativity. In fact the Global Positioning System is an example of an engineering system in which the theory of relativity is essential to its performance.

This thesis is an investigation of the relativistic effects in the GPS. Although emphasis is on satellite navigation, the effects on other satellite configurations, such as Low Earth Orbiting satellites, is considered. Chapter by chapter, the most important results are:

Chapter 1 Is a brief introduction to the theory of relativity, in order to list some key results and to introduce important nomenclature used in the remainder of the thesis. The description is founded on the principle of equivalence, rather than modern geometry and the mathematical formalism is introduced when it is needed. Einstein's field equations is introduced along with some important solutions. The Newtonian limit is considered and finally, the post Newtonian approximation is introduced.

Chapter 2 Contains a description of the gravitational field of the Earth, and essential orbital mechanics needed in order to describe the relativistic effects on satellite - and Earthbound clocks.

Chapter 3 Here, the physical foundation of positioning is investigated. The problem of synchronization in the vicinity of the Earth is considered. It is found that - due to a cancellation of effects - clocks on Earth's geoid all beat at the same (proper) rate. This is used to establish a coordinate time scale, which can be used as a basis of synchronization in the vicinity of the Earth.

Chapter 4 The satellites carry atomic clocks that maintain time to a precision of a few parts in 10^{14}. In the GPS, there are three relativistic effects that are taken into account. First, the combined effect of second order doppler shift (equivalent to time dilation) and gravitational red shift phenomena cause the clock to run fast by 38 μs per day. Second, although the orbits are nominally circular, the inevitable residual eccentricity causes a sinusoidal variation over one revolution between the time readings of the satellite clock and the time registered by a similar clock on the ground. This eccentricity effect has typically a peak - to - peak amplitude of 60 - 90 ns. Finally, because of the universality of the speed of light, there is a correction in the Earth's rotating frame of reference called the Sagnac effect, which is equivalent to the effect due to the change in position of the receiver during the time of propagation of the signal as analyzed in an inertial frame. The value for a receiver at rest on the equator is 133 ns, it may be larger for moving receivers. At the sub-nanosecond level additional corrections apply, including the contribution from Earth's oblateness, tidal effects and the Shapiro time delay, whereas post Newtonian effects are shown to be negligible. From these examples we clearly see that the theory of relativity has become a tool of engineering necessity. Furthermore, since 1 ns is equivalent to 30 cm, an error in time determination will lead to an error in position and *vise versa.*

Chapter 5 Is a summary of timescales used in geodesy.

Chapter 6 In this postscript, I outline what I believe to be some promising venues for future research in the field of relativistic geodesy.

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Chapter 1

Introduction to General Relativity

1.1 The principle of equivalence

The special theory of relativity is founded on the following principle (freely cited from "The Foundation of The General Theory of Relativity" [22]): *"If a coordinate system K, is chosen so that in relation to it the laws of physics hold good in their simplest form, then the same laws also hold good in any other system K' moving in uniform translation with respect to K"*.

This is called the *special principle of relativity.* But it is seen to be fully equivalent with the Newtonian *relativity principle of mechanics* (se eg. [24]). Thus the special theory of relativity does not depart from classical mechanics through the principle of relativity, but through the postulate of the constancy of the velocity of light in every inertial frame, which leads to the Lorentz transformation. The special theory of relativity is actually a continuation and completion of the ideas of Galileo and Newton.

Following his success with the special theory of relativity, Einstein sought a relativity theory of gravitation. Just as the special theory was founded on the universality of the speed of light, Einstein founded his general theory of relativity on the universality of the free fall:

Classically the equation of motion of a particle is governed by Newton's second law[1]

$$\mathbf{F} = m_i \ddot{\mathbf{x}}, \tag{1.1}$$

where $\ddot{\mathbf{x}}$ is the acceleration, and m_i is called the *inertial mass* of the particle.

[1] All of chapter 1 is founded on [17], unless otherwise stated. Furthermore, the velocity of light is chosen to be $c = 1$ throughout chapter 1, whereas SI units are used in the rest of the thesis, unless otherwise stated.

The inertial mass is a quantitative measure of how much a particle resists acceleration. In a constant gravitational field, we have

$$\mathbf{F} = m_g \mathbf{g}, \tag{1.2}$$

where $\mathbf{g}$ is the gravitational field strength and m_g is the gravitational mass of the particle, which is a quantitative measure of how much a particle attracts, - and is attracted by - other material bodies. Now if we assume that $m_i = m_g$ then $\ddot{\mathbf{x}} = \mathbf{g}$, and we see that the effect of gravity can be transformed away, by choosing new coordinates

$$\mathbf{y} = \mathbf{x} - \frac{1}{2}\mathbf{g}t^2, \tag{1.3}$$

which leads to

$$\ddot{\mathbf{y}} = \ddot{\mathbf{x}} - \mathbf{g} = 0. \tag{1.4}$$

This means that in the $\mathbf{y}$-system there's is no effect of gravity. Einstein generalizes this to the well-known *equivalence principle*:"*In an arbitrary gravitational field it is possible at each space-time point to select locally inertial systems (freely falling small elevators) such that the laws of physics in these are the same as in special relativity.*"

Special relativity is based on the form-invariance[2] of the laws of physics under a Lorentz transformation, which is a special group of linear transformations. To include accelerated reference frames non-linear transformations must be admitted. Einstein introduced an even more general principle, - known as the *principle of general covariance*, which states that "*the laws of nature are to be expressed by equations, which must be form-invariant under an arbitrary continuous transformation of coordinates*". The mathematical machinery in which this principle is embodied is the theory of tensor analysis. A tensor is defined by its transformation properties under a coordinate transformation, see eg. appendix C.

1.2 Geometry

From special relativity we know that an event is described by a four vector[3], and since the freely falling elevators *a priori* are small, we consider an infinitesimal four vector $dx^\alpha = (dx^0, d\mathbf{x}) = (dt, d\mathbf{x})$. From special relativity we

[2] This is what Einstein called "covariance".

[3] Hermann Minkowski gave the special theory of relativity a powerful mathematical generalization. He studied the equation of an expanding spherical wavefront in two inertial systems, and found that it resembled the Phytagorean theorem, which is invariant under a rotation in $\mathbb{R}^3$. Then he reasoned that the four dimensional spacetime interval is an invariant under a Lorentz transformation.

also have that the proper time is Lorentz invariant, with respect to a change of coordinates from $x^\alpha \mapsto x'^\alpha$, i. e.

$$d\tau^2 = dt^2 - d\mathbf{x}^2 = dt'^2 - d\mathbf{x}'^2, \tag{1.5}$$

which can be rewritten as

$$d\tau^2 = dt^2(1 - \mathbf{v}^2). \tag{1.6}$$

In the rest system $\mathbf{v} = 0$, so $d\tau^2 = dt^2$. The physical interpretation of proper time is thus that it is the time readings of a clock which is in rest with respect to a freely falling elevator.

In the moving system we get the familiar formula for time dilation

$$dt' = \frac{d\tau}{\sqrt{1 - v'^2}}. \tag{1.7}$$

At this point an observation regarding the time dilation may be useful. Very often it is described in a way, which suggests that we are dealing with a physical effect upon the moving clock. This is misleading, because clocks always measure proper time. It is not a physical effect in the clock the time measures appear to slow down as its velocity increases relative to the observer. It is merely a metric effect due to the transformation from one system to another. The same clock will appear to tick at different rates if seen from different reference frames. Thus the effect is not located in the clock, but is purely a metric effect due to the relative motion of the chosen reference frames.

In four vector notation it can be shown that the proper time can be written as

$$d\tau^2 = -\eta_{\alpha\beta} dy^\alpha dy^\beta. \tag{1.8}$$

Here Einstein's summation convention is used: Whenever an index occurs twice in a product, it is to be summed over. Greek indices assume the range 0, 1, 2, 3, and whenever Latin indices are used, they assume the range 1, 2, 3. Here $\eta_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $\eta_{11} = \eta_{22} = \eta_{33} = 1$, and $\eta_{00} = -1$.

The next step is to generalize the metric given by equation (1.8), which is valid in any freely falling elevator. In arbitrary coordinates x^μ we obtain - by using the fact that the freely falling coordinates y^α are functions of the x^μ coordinates - that

$$d\tau^2 = -\eta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} dx^\mu dx^\nu \equiv -g_{\mu\nu}(x) dx^\mu dx^\nu, \tag{1.9}$$

where the *metric tensor* is defined by

$$g_{\mu\nu}(x) = \eta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}. \tag{1.10}$$

According to the principle of equivalence it follows that special theory of relativity can always be applied to an infinitely small region, in which the metric can be expressed by equation (1.8). The space described by this metric is called Minkowskian or sometimes flat Minkowski space. The physical interpretation of equation (1.9) is that the geometry of space is in general not flat Minkowski space on a global scale. The deviation from flat space is represented by the effects of gravity. Thus the geometry of space can be thought of as a pseudo Riemannian 4-manifold, which locally is Minkowskian.

1.3 The equation of motion

Let us now consider a particle which moves under the influence of a gravitational field. Then according to the equivalence principle, there exist a freely falling system of coordinates y^α, in which the equation of motion is that of uniform linear motion, i. e.

$$\frac{d^2y^\alpha}{d\tau^2} = 0, \tag{1.11}$$

where the proper time is given by equation (1.8). Now if we want to express the equation of motion in arbitrary coordinates $x^\mu(\tau)$, we have to apply the chain rule when performing the differentiation in equation (1.11), because the freely falling coordinates y^α are functions of the coordinates $x^\mu(\tau)$. Hence we get that

$$\begin{aligned} 0 &= \frac{d}{d\tau}\Big(\frac{dy^\alpha}{d\tau}\Big) = \frac{d}{d\tau}\Big(\frac{\partial y^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\Big) \\ &= \frac{\partial y^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \end{aligned} \tag{1.12}$$

Using the rules of differentiation[4], equation (1.12) can be rewritten as

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{1.14}$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\lambda}{\partial y^\alpha} \tag{1.15}$$

[4] By the rules of differentiation we have (from [17])

$$\frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^\mu} = \delta^\lambda_\mu, \tag{1.13}$$

where δ^λ_μ is the Kronecker delta defined by $\delta^\lambda_\mu = 1$ when $\lambda = \mu$ and $\delta^\lambda_\mu = 0$ else.

is the so called Christoffel symbol, sometimes called the affine connection. Equation (1.14) is the relativistic equation of motion, and is called the geodesic equation. Here the derivation of the geodesic equation was based on the equivalence principle. We could also have used Euler-Lagrange's variational principle. It turns out that the later method is much more fruitful in practical applications. We have now seen that the gravitational field which determines the force on freely falling bodies, is determined by the Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$. Mathematically, the Christoffel symbols may be expressed in terms of the derivatives of the metric tensor

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \Big(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \Big), \tag{1.16}$$

where $g^{\lambda\sigma}(x)$ is the inverse of $g_{\lambda\sigma}(x)$, and is given by

$$g^{\lambda\sigma} = \eta^{\alpha\beta} \frac{\partial x^{\lambda}}{\partial y^{\alpha}} \frac{\partial x^{\sigma}}{\partial y^{\beta}}. \tag{1.17}$$

The physical interpretation of equation (1.16) is that $g_{\mu\nu}$ also plays the role of a gravitational potential. That is the derivatives of $g_{\mu\nu}$ determines the "field" $\Gamma^{\lambda}_{\mu\nu}$.

1.4 The Newton limit

In this section we consider the so called Newton limit. First we make some physical assumptions:

(1) All velocities are small compared to the velocity of light, i. e.

$$\Big| \frac{d\mathbf{x}}{d\tau} \Big| \ll 1 \equiv c. \tag{1.18}$$

(2) The metric tensor $g_{\mu\nu}(x)$ is independent of time, i. e. static.

(3) The gravitational field is weak. This means that the deviations from flat Minkowski space are small i. e.

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \qquad |h_{\mu\nu}| \ll 1. \tag{1.19}$$

If we apply condition (1) to the equation of motion (1.14), we obtain to lowest order:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{00} \Big(\frac{dt}{d\tau} \Big)^2 \approx 0. \tag{1.20}$$

In order to proceed we have to determine Γ^{μ}_{00}, which is given by equation (1.16):

$$\Gamma^{\mu}_{00} = \frac{1}{2} g^{\mu\sigma} \Big(\frac{\partial g_{0\sigma}}{\partial x^0} + \frac{\partial g_{0\sigma}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\sigma} \Big). \tag{1.21}$$

By condition (2), $g_{\mu\nu}(x)$ is independent of time ($x^0 = t$), and therefore equation (1.21) reduces to

$$\Gamma^{\mu}_{00} = -\frac{1}{2} g^{\mu\sigma} \frac{\partial g_{00}}{\partial x^\sigma}. \tag{1.22}$$

Inserting condition (3) we get

$$\begin{aligned} \Gamma^{\mu}_{00} &= -\frac{1}{2} \eta^{\mu\sigma} \frac{\partial}{\partial x^\sigma} (\eta_{00} + h_{00}) + \frac{1}{2} h^{\mu\sigma} \frac{\partial}{\partial x^\sigma} (\eta_{00} + h_{00}) \\ &\approx -\frac{1}{2} \eta^{\mu\sigma} \frac{\partial h_{00}}{\partial x^\sigma}. \end{aligned} \tag{1.23}$$

For $\mu = 0$ one finds that $\Gamma^{0}_{00} \approx 0$, because h_{00} is independent of time. Thus we get

$$\frac{d^2 t}{d\tau^2} \approx 0 \qquad \Longrightarrow \qquad \frac{dt}{d\tau} = \text{constant}. \tag{1.24}$$

For $\mu = 1, 2, 3$ we get, using equation (1.24), that

$$\frac{d^2 \mathbf{x}}{dt^2} = \frac{1}{2} \nabla h_{00}(\mathbf{x}). \tag{1.25}$$

This can be compared with Newton's equation

$$\frac{d^2 \mathbf{x}}{dt^2} = -\nabla \phi(\mathbf{x}), \tag{1.26}$$

where $\phi(\mathbf{x})$ is the gravitational potential. From classical mechanics we have for a point mass M, that

$$\phi = -\frac{GM}{r}, \tag{1.27}$$

where G is the gravitational constant. Comparing equation (1.25) and (1.26) gives: $h_{00} = \text{constant} - 2\phi$. But since we are dealing with a isolated point mass, we require that space should be flat Minkowskian for $r \longrightarrow \infty$ and $\phi \longrightarrow 0$. Thus the constant must vanish i. e. $h_{00} = -2\phi$. Inserting this result in equation (1.19) thus gives:

$$g_{00}(\mathbf{x}) = -(1 + 2\phi(\mathbf{x})). \tag{1.28}$$

Thus we see that space around a point mass - in the Newton limit - is slightly curved.

1.5 The gravitational frequency shift

In the preceding section we saw that space around a point mass is curved. The curvature of space around a point mass can indeed be observed. Let us therefore consider a clock at rest in a weak gravitational field. Then the proper time will be given by

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu = -g_{00}(\mathbf{x}) dt^2. \tag{1.29}$$

Thus the time measured in this system in the point x is

$$dt = \frac{d\tau}{\sqrt{-g_{00}(x)}}. \tag{1.30}$$

The coordinate time interval between two events is the same throughout space, whereas the proper time only has local meaning, and is different for clocks in different gravitational potentials. The coordinate time interval depends on the chosen coordinates, but the proper time interval at a given point is an invariant. Thus we can not observe the effect of gravity directly by equation (1.30), because all clocks in the same point will suffer the same shift. But we can compare two different points in space. If the clock is a physical system with frequency $f = 2\pi/dt$, then we get (using the weak field limit (1.28))

$$\frac{f_2}{f_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} \approx 1 + (\phi(x_2) - \phi(x_1)). \tag{1.31}$$

Thus the fractional frequency shift for a spherically symmetric body with mass M, is

$$\frac{\Delta f}{f_1} = GM\Big(\frac{1}{r_1} - \frac{1}{r_2}\Big). \tag{1.32}$$

In 1960 Pound and Rebka used the Mössbauer effect to measure the frequency shift of a photon, emitted from a $h = 22.6$ m high tower and then observed at the ground. From equation (1.32) the frequency shift should be (using $r_1 \approx r_2 \approx r$)

$$\Delta\phi = \frac{GM}{r_1 r_2}(r_2 - r_1) \approx \frac{gh}{c^2} = 2.46 \cdot 10^{-15}, \tag{1.33}$$

where $GM/r^2 = g/c^2$ in ordinary SI units. This is in good agreement with the experimental value, which is $(2.57 \pm 0.26) \cdot 10^{-15}$ [17].

1.6 Einstein's equations

Einstein's field equations is a generalization of Poisson's equation

$$\nabla^2 \phi = 4\pi G\rho, \tag{1.34}$$

where ϕ is the Newtonian potential, and ρ is a mass density. In the Newton limit we have

$$\nabla^2 g_{00} = -8\pi G\rho = -8\pi G T_{00}, \tag{1.35}$$

where T_{00} represent the mass density, and is a component of the so called *energy - momentum tensor*. Equation (1.35) is naturally generalized to

$$E_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{1.36}$$

in arbitrary coordinates. The tensor $E_{\mu\nu}$ is referred to as the Einstein tensor. We now demand that

(1) The energy - momentum theorem should hold, i. e. the covariant derivative of $T_{\mu\nu}$ should vanish.

(2) $E_{\mu\nu}$ should only depend on the metric and its first and second derivative.

It turns out that the so called *Bianci identities* provides such a tensor. The result is the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}, \tag{1.37}$$

where $R_{\mu\nu}$ is the so called *Ricci tensor*, and R is the Ricci scalar, which is obtained from the Ricci tensor by contraction

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{1.38}$$

Einstein's field equations were first derived by Einstein in 1916 [22]. The detailed derivation is beyond the scope of this text, and can be found in most books on general relativity, c. f. [16], [17] and [21]. Equation (1.37) is a tensor equation, and looks deceivingly simple, but a closer look reveals a system of complicated differential equations. It has thus only been possible to solve Einsteins field equations in some special cases, where the most important one is the static gravitational field around a central mass. In the empty space around such a mass, equation (1.37) has the form

$$R_{\mu\nu} = 0, \tag{1.39}$$

which represents 10 differential equations, of which 6 are independent. The solution to equation (1.39) is the famous Schwarzschild - solution, which is given by

$$d\tau^2 = \Big(1 - \frac{2GM}{r}\Big)dt^2 - \frac{dr^2}{1 - \frac{2GM}{r}} - r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{1.40}$$

in Schwarzschild coordinates (t, r, θ, φ), where the spatial part of the metric is expressed in spherical polar coordinates.

In 1963 Roy P. Kerr found another useful solution - which includes rotation - to Einstein's field equations. The derivation of the Kerr solution is far from simple and beyond the scope of this text (for details the reader can consult [33]). In so called *Boyer - Lindquist coordinates*, the Kerr metric assumes the form (from [33]):

$$\begin{aligned} d\tau^2 = \Big(1 - \frac{2GM\rho}{\rho^2 + a^2\cos^2\theta}\Big)dt^2 - \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 - 2GM\rho}d\rho^2 \\ -(\rho^2 + a^2\cos^2\theta)d\theta^2 - \Big((\rho^2 + a^2)\sin^2\theta + \frac{2GM\rho a^2\sin^4\theta}{\rho^2 + a^2\cos^2\theta}\Big)d\varphi^2 \\ -\frac{4GM\rho a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}dtd\varphi, \end{aligned} \tag{1.41}$$

where

$$\rho^2 = \frac{r^2 - a^2}{2} + \Big(\frac{(r^2 - a^2)^2}{4} + a^2z^2\Big)^{1/2}, \quad r^2 = x^2 + y^2 + z^2, \quad \cos\theta = \frac{z}{\rho}. \tag{1.42}$$

The Kerr metric has all the characteristics of the field of a rotating axially symmetric body: If $a = 0$ it reduces to the normal Schwarzschild solution. Furthermore, it is axially symmetric and time independent, and if the sign of t and φ are changed the metric is left unchanged (spinning in the opposite direction, and time running backward). More importantly the metric is unchanged if the signs of a and φ are reversed. This suggests that a specifies a spin direction. Finally it contains a cross term $d\varphi dt$, which also is present in the metric of rotating flat space (this is known from special relativity).

The expression given in equation (1.41) is rather complicated to work with, but in the limit of large distances (compared to the gravitational field in question), we can expand the metric to the order $\sim a/\rho$, which gives [33]:

$$\begin{aligned} d\tau^2 = \Big(1 - \frac{2GM}{r}\Big)dt^2 - \frac{dr^2}{1 - \frac{2GM}{r}} \\ -r^2(d\theta^2 + sin^2\theta d\varphi^2) - \frac{4GMa}{r}\sin^2\theta d\varphi dt, \end{aligned} \tag{1.43}$$

where we have used that to first order $\rho \approx r$. This is actually the Schwarzschild solution plus a cross term. Physically a is related to the angular momentum J of the source (for details see [33]):

$$Ma = -\frac{J}{c^3} \tag{1.44}$$

in ordinary SI units.

1.7 The post Newtonian approximation

Considering the complexity of Einstein's field equations, it is astonishing how many exact solutions there has been found. However, apart from the Schwarzschild solution, the usefulness of them is questionable, since the connection with field sources is rather unclear. For instance consider the Kerr solution. It describes the outer solution of a rotating point mass (eg. a rotating black hole), but it is questionable whether it can be thought of as describing the field of a non-collapsed rotating body. Thus, one faces the necessity of finding approximate solutions to Einstein's field equations. One way of doing this in the Solar system, is the so called *post Newtonian approximation*[5]. I will not go into the details of how to obtain the post Newtonian metric, because the derivation is rather lengthy and beyond the scope of this thesis.

1.7.1 Foundations

In the Solar system, the typical kinetic energy equals the potential energy. Thus if the typical mass scale, length scale, and velocity is given by M, r and v, then the order of magnitude of v^2 will roughly be of the same order as GM/r, i. e.

$$v^2 \sim \frac{GM}{r}. \tag{1.45}$$

This theory makes use of the fact that the gravitational potential is weak in the Solar system. One therefore expands the metric coefficients in terms of the smallness parameter v^2. The expansions are carried to the 4'th order (v^4) for g_{00}, 3'rd order for g_{i0} and 2'nd order for g_{ij}. Odd powers of v are only present in g_{i0}. The reason that the g_{i0} component only contains odd powers of

[5]This section about the post Newtonian approximation and the following one, about the equation of motion to post Newtonian order is based on the book "Gravitation and Cosmology" by S. Weinberg, reported in [21], which as far as I know is the only book that contains a detailed derivation of the post Newtonian metric.

v is that it must change sign under time reversal $t \to -t$. The contravariant form of the metric tensor is found by $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$.

1.7.2 The post Newtonian metric

The post Newtonian metric is in the so called *harmonic gauge*[6], given by

$$g_{00} = 1 - 2\phi - 2\phi^2 - 2\psi \quad (1.46)$$

$$g_{i0} = \zeta_i \quad (1.47)$$

$$g_{ij} = \delta_{ij}(1 - 2\phi), \quad (1.48)$$

where ϕ is the Newtonian potential, and where ψ and ζ_i are certain post Newtonian potentials, which are unanticipated by the Newtonian theory. These new potentials depend on the internal structure and motion of the matter distribution (for details see [21]). The harmonic coordinate condition leads to the gauge condition

$$4\frac{\partial \phi}{\partial t} + \nabla \cdot \zeta = 0. \quad (1.49)$$

The post Newtonian scalar potential ψ only leads to physical effects in the expression for the g_{00} component of the metric, and it can be accounted for by replacing ϕ with the physically significant field $\phi + \psi$ in (1.46).

Far from a finite spherical distribution of energy and momentum, it can be shown that the field $\phi + \psi$ is given as

$$\phi + \psi \to -\frac{GM}{r} - \frac{G\mathbf{r} \cdot \mathbf{D}}{r^3} + \mathcal{O}\Big(\frac{1}{r^3}\Big), \quad (1.50)$$

where $r = |\mathbf{r}|$, and where $\mathbf{r}$ is the field point. The quantity $\mathbf{D}$ does not represent an effect of any physical importance, but just a displacement of

[6]The symmetric Einstein tensor $E_{\mu\nu}$ has 10 independent components, thus Einstein's field equations are in fact 10 algebraically independent equations. The unknown metric tensor $g_{\mu\nu}$ has also 10 independent components, but the 10 $E_{\mu\nu}$ are related by the so called *Bianchi identities*, which are four differential identities. Thus there are only 6 independent equations in 10 unknowns, leaving us with four degrees of (gauge) freedom. The failure of Einstein's field equations in determining the metric tensor uniquely, is analogues to the failure of Maxwell's equation's in determining the vector potential uniquely. Thus, as in electrodynamics, one can lift the degrees of freedom by choosing some suitable gauge condition. In relativity this can be done by introducing a particular coordinate system, - usually one that will simplify the computational problem. When deriving the post Newtonian metric it leads to a tremendous simplification, if four coordinate conditions, known as the harmonic gauge conditions, given by $g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$ is introduced. This gauge condition implies that the coordinates themselves are harmonic functions (see [21] for details), hence the name *harmonic coordinates*.

the field. This follows from the fact that (1.50) can be rewritten as [21]

$$\phi + \psi \to -\frac{GM}{|\mathbf{r} - \mathbf{D}/M|} + \mathcal{O}\Big(\frac{1}{r^3}\Big). \tag{1.51}$$

Thus the $\mathbf{D}$ term can be avoided, by choosing origo at the center of energy.

Further, it can be shown that the post Newtonian vector potential ζ is related to the angular momentum $\mathbf{J}$, as

$$\zeta \to \frac{2GM(\mathbf{r} \times \mathbf{J})}{r^3} + \mathcal{O}\Big(\frac{1}{r^3}\Big). \tag{1.52}$$

The post Newtonian vector potential ζ gives rise to the famous Lense-Thirring effect.

1.7.3 Equation of motion to post Newtonian order

Let us now look at a particle (eg. a satellite) in free fall with respect to the gravitational field of the Earth. By definition, the proper time τ measured in the freely falling satellite, is related to the harmonic coordinate time t by

$$d\tau^2 = -g_{00}dt^2 - g_{i0}dx^i dt - g_{ij}dx^i dx^j, \tag{1.53}$$

which leads to

$$\Big(\frac{d\tau}{dt}\Big)^2 = -g_{00} - g_{i0}v^i - g_{ij}v^i v^j. \tag{1.54}$$

Inserting (1.46) - (1.48) into equation (1.54), one gets to 4'th order

$$\Big(\frac{d\tau}{dt}\Big)^2 = 1 + (2\phi - \mathbf{v}^2) + 2(\phi^2 + \psi - \zeta \cdot \mathbf{v} + \phi\mathbf{v}^2), \tag{1.55}$$

where the first set of parentheses encloses terms to order 2, and the second set of parentheses encloses terms to order 4. Choosing the positive square root and applying the binomial expansion[7] we obtain

$$\frac{d\tau}{dt} = 1 - \frac{1}{2}\mathbf{v}^2 + \phi - \frac{1}{8}(2\phi - \mathbf{v}^2)^2 + \phi^2 + \frac{3}{2}\phi\mathbf{v}^2 + \psi - \zeta \cdot \mathbf{v}. \tag{1.56}$$

This can be rewritten as

$$\frac{d\tau}{dt} = 1 - L, \tag{1.57}$$

where

$$L = \frac{1}{2}\mathbf{v}^2 - \phi + \frac{1}{8}(2\phi - \mathbf{v}^2)^2 - \phi^2 - \frac{3}{2}\phi\mathbf{v}^2 - \psi + \zeta \cdot \mathbf{v}. \tag{1.58}$$

[7] $\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$.

The integral $\int (d\tau/dt)dt$ is stationary, so L can be thought of as the Lagrange function $L = T - V$, where T is the kinetic energy, and V is the potential. The equation of motion can thus be obtained in a standard way from the Euler - Lagrange's equations [37]

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial x^i}, \tag{1.59}$$

where it is understood that acting on ϕ or ζ, d/dt is to be taken as $\partial/\partial t + \mathbf{v}\cdot\nabla$. Thus, by inserting (1.58) into Euler-Lagrange's equation (1.59), we obtain the equation of motion to post Newtonian order

$$\begin{aligned}\frac{d\mathbf{v}}{dt} = -\nabla(\phi + 2\phi^2 + \psi) - \frac{\partial \zeta}{\partial t} + \mathbf{v} \times (\nabla \times \zeta) \\ +3\mathbf{v}\frac{\partial \phi}{\partial t} + 4\mathbf{v}(\mathbf{v}\cdot\nabla)\phi - \mathbf{v}^2\nabla\phi.\end{aligned} \tag{1.60}$$

Chapter 2

Classical Mechanics

2.1 The Newtonian gravitational field of the Earth

In this section, I will show that the Earth's gravitational potential can be expanded into spherical harmonics. Before doing so, a slight excursion into potential theory is needed, in order to find the mutual gravitational potential of a mass point and a non spherical distribution of matter. The method is a standard procedure in geophysics, and can be found in most textbooks on geodesy, eg. [29]. Thus I will only give an outline of how this is done, and leave out most of the detailed calculations.

In the following it is assumed that the Earth is composed of infinitely many mass points M_i. Consider now a mass point m at the distance r_i from the i'th point in the mass distribution. Then the mutual gravitational potential between the two bodies is

$$U = -\sum_{i=1}^{\infty} \frac{GmM_i}{r_i}. \tag{2.1}$$

Here U is naturally chosen to be negative, because one has to apply a work in order to escape the gravity field of the Earth. This is contrary to the common convention in geodesy where the potential is taken to be positive.

Let's assume, that m is a unit mass, and let M denote the total mass of the Earth. Now if the small particles are conglomerated to form a continuous body of density $\rho(x, y, z)$, the summation over the i particles can be replaced by an integral over the volume of the body. Hence the potential at m's location is

$$U = -G \int_V \frac{\rho(x, y, z)}{r} dV. \tag{2.2}$$

Now it can easily be shown that, for any mass element ρdV, the potential is a C^2-function (for details see [29]) and that it satisfies Laplace's equation

$$\nabla^2 U = 0. \tag{2.3}$$

Hence U is harmonic [30]. It immediately follows that the potential can be expanded into a series of spherical harmonics, (for details see [29]):

$$U = -\frac{GM}{r}\left\{1+\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n P_{nm}(\cos\theta)(C_{nm}\cos m\varphi + S_{nm}\sin m\varphi)\right\}, \tag{2.4}$$

where θ is the colatitude, and φ the longitude. P_{nm} is the associated Legendre functions of degree n and order m. The coefficients C_{nm} and S_{nm} are called the Stokes coefficients and are functions of the size, shape and density distribution of the Earth, and have been estimated using methods of satellite geodesy. The summation over n starts at 2, because the 0'th contribution gives 1, and by choosing the origin of the coordinate system to coincide with the center of mass, it follows that $C_{11} = S_{11} = 0$.

Equation (2.4) can be separated into *Zonal* and *Tesseral* terms [15]:

$$\begin{aligned} U = & - \frac{GM}{r}\left\{1 - \sum_{n=2}^{\infty} J_n\left(\frac{R}{r}\right)^n P_n(\cos\theta)\right. \\ & + \left.\sum_{n=2}^{\infty}\sum_{m=1}^{n} J_{nm}\left(\frac{R}{r}\right)^n P_{nm}(\cos\theta)\cos m(\varphi - \varphi_{nm})\right\}, \end{aligned} \tag{2.5}$$

where $P_n(\cos\theta)$ is the Legendre polynomials of degree n. The coefficients J_n are associated with the oblateness of the Earth, and the J_{nm} coefficients are associated with the elliptical shape of the Earth's equator. The relationship between the coefficients J_{nm} and the phase angle φ_{nm} is defined by

$$C_{nm} = J_{nm}\cos m\varphi_{nm}, \qquad S_{nm} = J_{nm}\sin m\varphi_{nm}. \tag{2.6}$$

Hence

$$J_{nm} = \sqrt{C_{nm}^2 + S_{nm}^2} \qquad \text{and} \qquad \varphi_{nm} = \frac{1}{m}\arctan\left(\frac{S_{nm}}{C_{nm}}\right). \tag{2.7}$$

For GPS satellites the tesseral terms tend to average out [15], and can therefore - as a first approximation - be neglected.

2.2 The WGS-84 system

The World Geodetic System - 1984 is the reference system of the GPS satellites, and was constructed by the US Department of Defence (DoD). It is defined by adopting Cartesian coordinates to the 10 GPS monitoring stations, derived by Doppler measurements on these sights. The WGS-84 system is realized by the ephemeris of the GPS satellites. In order to compute the GPS orbits some additional constants have to be adopted.

An ellipsoidal coordinate system is attached to the WGS-84 system by locating an ellipsoid at the origin of the WGS-84 system, and letting the rotation axis coincide with the z-axis of the WGS-84. Some of the most important parameters of the WGS-84 system is listed in table 2.1, including the defining parameters and some derived geometric constants (for more details see DMA Technical Report Part I [31]).

Parameter	Symbol	Value
Semi-major axis	R	6378137 m
Reciprocal flattening	1/f	298.257223563
Angular velocity	ω	$7.292115 \cdot 10^{-5} s^{-1}$
Geocentric gravitational constant	GM	$398600.5\ km^3 s^{-2}$
Second zonal harmonic	J_2	0.00108262998905
3'th zonal harmonic	J_3	-0.00000253215307
4'th zonal harmonic	J_4	-0.00000161098761
5'th zonal harmonic	J_5	-0.00000023578564
6'th zonal harmonic	J_6	0.00000054316985
Second tesseral harmonic	$J_{2,2}$	0.0000018155

Table 2.1: Some of the main parameters of WGS-84.

The WGS-84 is a military reference system. Therefore certain aspects of it are classified (Part III of DMA Technical Report is classified). Thus it is only the lowest order harmonic coefficients which have been published. However, it turns out that we will only need the J_2 coefficient (this will be clarified later), so this does not limitate the study of relativistic effects. Furthermore, the WGS-84 is comprised of a consistent set of parameters. Thus it may lead to a less accurate result in the calculations, if one substitutes one parameter with a newer and more accurate one. Therefore in applications like the GPS it is best to use the parameters of the WGS-84, and not try to make any adjustments yourself. Hence I will consistently use WGS-84 parameters throughout the rest of this thesis.

2.3 Orbital mechanics

In this section I will discuss some essential features - which are necessary in satellite geodesy - of classical orbital mechanics. There are a lot of standard text books on this subject, eg. [1], [15] and [29].

A characteristic feature of a satellite orbit is the altitude. The chosen altitude of a satellite is based on both physical and geometrical considerations, including effects on the signal, avoidance of the Van Allen radiation belts, satellite visibility, coverage area, etc. The conventional altitude regimes are listed in table 2.2.

Category	Acronym	Altitude regime	Definition	Example
Near Earth Orbit.	NEO	150km -500km	Affected by the atmosphere, finite life time.	Space shuttle
Low Earth Orbit.	LEO	500km -1500km	Above atmosphere, but below first Van Allen belt.	CHAMP
Medium Earth Orbit.	MEO	5000km −15000km	Between first and second Van Allen belt.	ICO
High Earth Orbit.	HEO	20000km −50000km	Above second Van Allen belt.	GPS and Glonass
Geo-stationary Orbit.	GEO	35786km	Geosynchronous circular orbit in the equatorial plane.	Communications satellites

Table 2.2: Altitude regimes. Table from [15].

The second characteristic of a satellite orbit is the inclination i. The inclination angle i is measured with respect to equatorial plane at the ascending node where the satellite crosses the equator from the Southern Hemisphere to the Northern Hemisphere (see figure 2.1). The orbit is called direct if $0 < i < \pi/2$, because it revolves in the same sense as the rotation of the Earth. If the inclination exceeds $\pi/2$ the orbit is called retrograde and if $i = 0$ or $i = \pi/2$, then the orbit is called equatorial or polar, respectively.
On figure 2.1, the x-axis is aligned with the *vernal equinox* Υ, which describes the direction of the Sun as seen from the Earth at the beginning of spring time. The angle Ω is called the right ascension of the ascending node, which describes the angle between the vernal equinox and the point on the orbit at

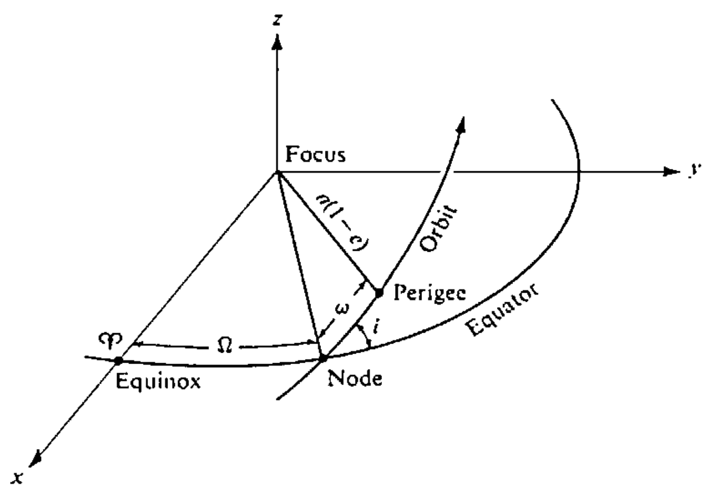

Figure 2.1: Orbital orientation. Here i is the inclination, Ω is the right ascension of the ascending node, ω is the argument of Perigee, e the eccentricity and a the semi-major axis of the orbit. Figure from [29].

which the satellite crosses the equator from south to north, and ω is called the argument of perigee, which describes the angle between the direction of the ascending node and the perigee.

Let's now consider a satellite in an elliptical Kepler orbit. It can, by fairly simple means be shown, that the orbital angular momentum is a constant of the motion [24], since it is always perpendicular to both the velocity and the radius vector. This implies that the orbit takes place in a plane. This plane is called the orbital plane. Another conserved quantity is the mechanical energy and from this it follows that the velocity can be expressed as

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right). \tag{2.8}$$

This is the so called *vis viva* equation. r denotes the radial distance to the satellite and a denotes the semi major axis of the orbital ellipse.

So far nothing has been said about the time dependence of the orbit. In the following it is useful to define an auxiliary circle, which circumscribes the orbital ellipse (see figure 2.2). For this purpose an auxiliary variable E, which is called the *eccentric anomaly* is defined, via the equations:

$$x = r\cos\nu = a(\cos E - e), \tag{2.9}$$

$$y = r\sin\nu = b\sin E, \tag{2.10}$$

where a is the semi-major axis of the orbital ellipse, b the semi-minor axis and the angle ν is the so called *true anomaly*.

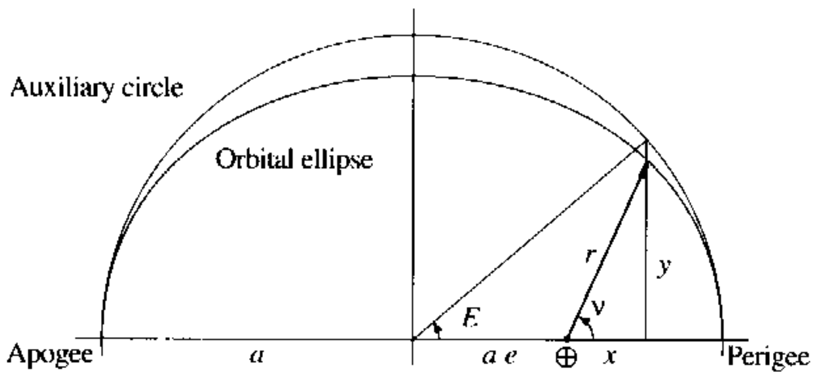

Figure 2.2: The definition of the eccentric anomaly E. Figure from [25].

Furthermore, the radial distance r can be expressed by E as

$$r = \sqrt{x^2 + y^2} = a(1 - e\cos E). \tag{2.11}$$

From the geometry we also have

$$\cos\nu = \frac{\cos E - e}{1 - e\cos E}, \tag{2.12}$$

and

$$\sin\nu = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}. \tag{2.13}$$

This pair of equations determines the true anomaly ν in terms of the eccentric anomaly E. The relationship between ν and E, can also be expressed through Gauss's equation

$$\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}, \tag{2.14}$$

but since the tangent function has a period of 180°, one has to be careful to avoid quadrant ambiguities, when using equation (2.14).

A third anomaly used in satellite geodesy is the *mean anomaly* M, defined by the equation

$$M = n(t - t_p) = M_0 + n(t - t_e), \tag{2.15}$$

where

$$n = \frac{2\pi}{T} = \sqrt{\frac{GM}{a^3}} \tag{2.16}$$

is the mean motion of the satellite, t_p is the time of perigee, and M_0 is the mean anomaly at some arbitrary epoch t_e. The relationship between M and

E can be found upon integration with respect to time

$$M = n\int_{t_p}^{t} dt = \int_{0}^{\nu} (1 - e\cos E)dE, \tag{2.17}$$

which leads to Kepler's equation

$$M = E - e\sin E. \tag{2.18}$$

Kepler's equation is transcendental, so it can not be solved analytically. In practice a numerical algorithm must be used. This could for instance be based on Newton's method of finding roots. For small eccentricities, an approximate solution can be found [1]:

$$E = M + e\sin M + \frac{1}{2}e^2\sin 2M + \frac{1}{8}e^3(3\sin 3M - \sin M) + ... \tag{2.19}$$

The six numbers $(a, e, i, \Omega, \omega, M_0)$ introduced here are called Keplerian orbital elements, and they completely specify the satellite orbit. The size and shape of the orbit are specified by the semi-major axis a and the eccentricity e. The orientation of the orbital plane is specified by the inclination i, and right ascension of the ascending node Ω. The orientation of the orbit in its own orbital plane is specified by the argument of perigee ω. Finally, the time dependence of the orbit is specified by the mean anomaly M_0 at a given epoch t_e. These six numbers are equivalent to specifying the initial position and velocity in an Cartesian inertial frame.

2.4 Orbital perturbations

In this section I we will consider some of the disturbing forces on a satellite. In figure 2.3 the order of magnitude of various perturbing forces is depicted, as a function of radial distance from the Earth's center.
The leading contribution is seen to arise from the mass of the Earth and this means that a Keplerian orbit provide a reasonable first approximation. The the most important perturbations is seen to come from the quadrupole moment of the Earth (J_2), atmospheric drag, solar radiation pressure, and gravitational attractions of the Sun and Moon.

For a satellite, the type of the leading perturbations, depends on the altitude regime in which the satellite operates. Atmospheric drag is the largest of the non-conservative forces acting on a Near - Earth satellite. The modelling of aerodynamic forces is extremely difficult. This is due to three reasons. Firstly, the properties of the upper atmosphere. Especially the density

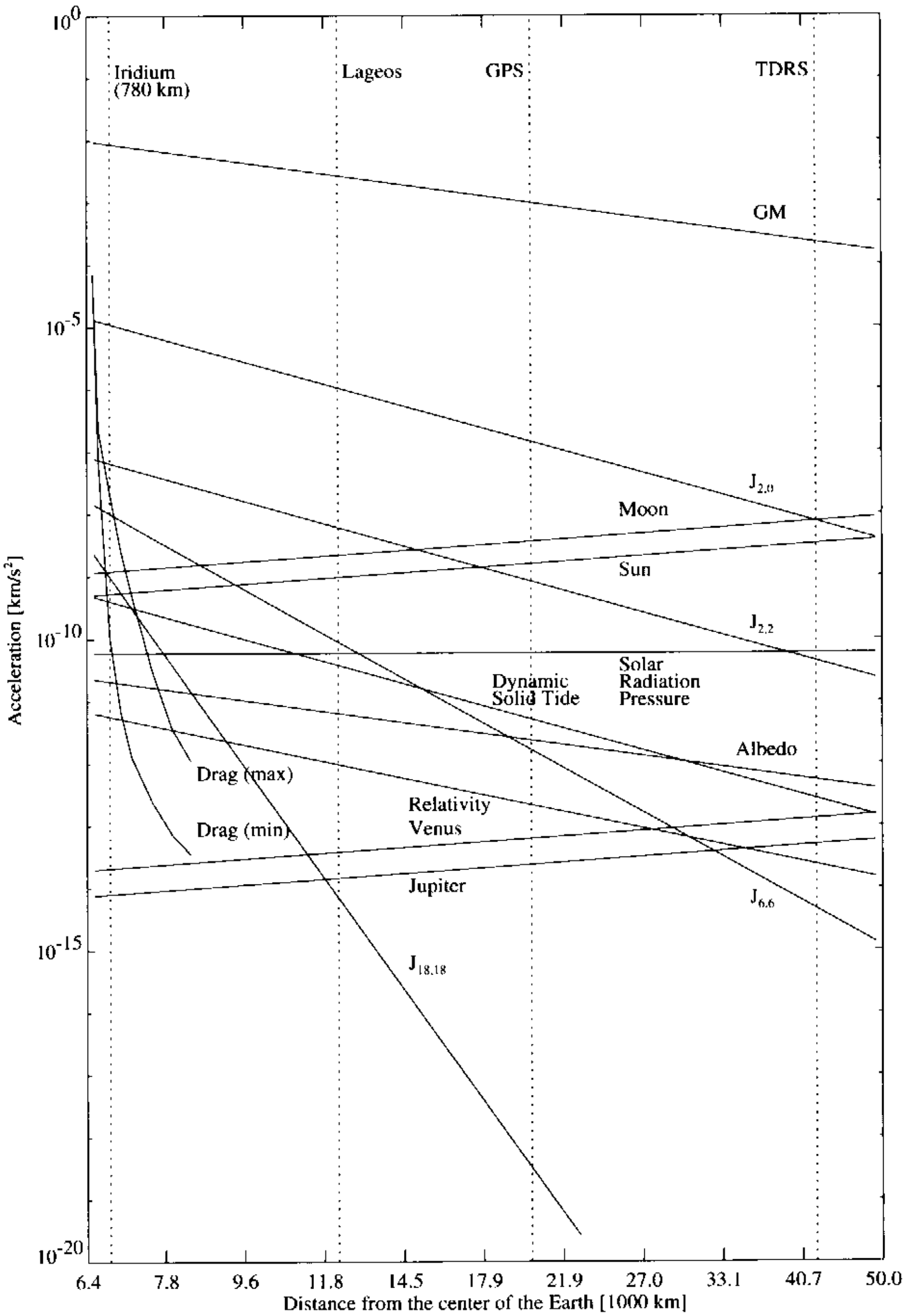

Figure 2.3: Order of magnitude of various perturbations on a satellite orbit, as explained in the text. Figure from [25].

and its dependence of the solar radiation pressure, is not accurately known. Secondly, the modelling of these forces requires detailed knowledge of the interaction of neutral gasses - as well as charged particles - with the different space vehicles surfaces. Thirdly, the attitude with respect to the atmospheres particle flux, depends on the shape of the satellite. This means that in practice, it is easier to equip the satellite with a accelerometer, which measures the non-conservative forces on the satellite. This has for instance been done for CHAMP.

Solar radiation pressure tends to change the orbit shape (c. f. [25] for details), and can be a considerable perturbation, especially for geostationary communication satellites with large solar panels.

2.4.1 Oblateness

If we only include the first order oblateness correction, which is related to the quadrupole moment of the Earth, then the potential at distance r from Earth's center becomes

$$V = -\frac{GM}{r}\Big(1 - J_2\Big(\frac{R}{r}\Big)^2 P_2(\cos\theta)\Big). \tag{2.20}$$

The gravitational force is $\mathbf{F} = -m\nabla V$, and the equation of motion is $\mathbf{a} = -\nabla V$, which can be written as

$$\mathbf{a} = \nabla\Big(\frac{GM}{r}\Big) + \nabla P, \tag{2.21}$$

where P is the perturbing potential, given by

$$P = -\frac{GM}{r}\frac{J_2}{2}\Big(\frac{R}{r}\Big)^2(1 - 3\cos^2\theta). \tag{2.22}$$

By spherical trigonometry we have the identity

$$\cos\theta = \sin(\omega + \nu)\sin i, \tag{2.23}$$

where θ is the co-latitude ω is the altitude of perigee, ν the true anomaly, and i the inclination. We can therefore write equation (2.22) as

$$P = -\frac{GM}{r}\frac{J_2}{2}\Big(\frac{R}{r}\Big)^2(1 - 3\sin^2(\omega + \nu)\sin^2 i). \tag{2.24}$$

Furthermore, using the trigonometrical relation

$$2\sin^2(\omega + \nu) = 1 - \cos(2\omega + 2\nu), \tag{2.25}$$

the disturbing potential P can be written as

$$P = -\frac{GMJ_2}{2r}\Big(\frac{R}{r}\Big)^2\Big(1 - \frac{3}{2}\sin^2 i\Big) - \frac{3GMJ_2}{4r}\Big(\frac{R}{r}\Big)^2\sin^2 i\cos(2\omega + 2\nu). \tag{2.26}$$

The differential equations, which describe the variation with time of the orbital elements are called Lagrange's Planetary Equations, and they are given by (c. f. [29]):

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial P}{\partial M} \tag{2.27}$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2e}\frac{\partial P}{\partial M} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial P}{\partial \omega} \tag{2.28}$$

$$\frac{di}{dt} = \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial P}{\partial \omega} - \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial P}{\partial \Omega} \tag{2.29}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial P}{\partial i} \tag{2.30}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial P}{\partial i} + \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial P}{\partial e} \tag{2.31}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2e}\frac{\partial P}{\partial e} - \frac{2}{na}\frac{\partial P}{\partial a} \tag{2.32}$$

In general the orbit will experience secular and short periodic perturbations due to J_2. Secular perturbations grow linearly in time and short periodic perturbations are periodic in multiples of the true anomaly. The first order secular terms may be obtained by averaging the short periodic terms out of P, by integrating over the mean anomaly from 0 to 2π:

$$\overline{P} = \frac{1}{2\pi}\int_0^{2\pi} P dM. \tag{2.33}$$

Furthermore, it can be shown that (see appendix B)

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 dM = (1-e^2)^{-3/2}, \tag{2.34}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 \sin(2\nu) dM = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 \cos(2\nu) dM = 0. \tag{2.35}$$

This leads to

$$\overline{P} = \frac{GMJ_2R^2}{2a^3(1-e^2)^{3/2}}\Big(1 - \frac{3\sin^2 i}{2}\Big), \tag{2.36}$$

which only depend on a, e and i. Thus - by inspecting Lagrange's equations - we see that oblateness produces three principal effects: (1) a nodal regression,

(2) rotation of the semi-major axis, and (3) a change in the period of revolution. Inserting equation (2.36) into Lagrange's Planetary Equations yields that there are no changes in the size, shape, or inclination of the orbit

$$\frac{da}{dt} = \frac{de}{dt} = \frac{di}{dt} = 0, \tag{2.37}$$

but there is a drift of the right ascension of the ascending node Ω. The rate of nodal regression is

$$\frac{d\Omega}{dt} = -\frac{3}{2}\frac{nJ_2R^2}{a^2(1-e^2)^2}\cos i. \tag{2.38}$$

For polar orbits ($i = \pi/2$) there is no drift, whereas the drift is westward for direct orbits ($i < \pi/2$), and eastward for retrograde orbits ($i > \pi/2$). Secondly, there is a rotation of the semi-major axis

$$\frac{d\omega}{dt} = \frac{3}{4}\left(\frac{R}{a}\right)^2\frac{J_2}{(1-e^2)^2}(5\cos^2 i - 1). \tag{2.39}$$

We see that the variation of ω vanishes for $\cos i \approx \pm\sqrt{5}/5$, or equivalently $i \approx 63.4°$ and $116.6°$. This situation is called critical inclination, and is used for highly elliptical communication satellites. Finally,

$$\frac{dM}{dt} = n + \frac{3}{2}\frac{nJ_2}{(1-e^2)^{3/2}}\left(\frac{R}{a}\right)^2(1 - 3\cos^2 i). \tag{2.40}$$

The above equation can be reformulated in terms of the orbital period, by defining the nodal period as [15]

$$T_N = T + \Delta T_N, \tag{2.41}$$

where T denotes the ideal Keplerian period, and

$$\Delta T_N = -\frac{3}{8}\left(\frac{R}{a}\right)^2\frac{J_2}{(1-e^2)^2}(7\cos^2 i - 1)T \tag{2.42}$$

is the oblateness correction to the nodal period.

An estimation of these effects can be done using CHAMP as an example, thus ignoring nonconservative forces for the moment, and only calculate the effect of J_2, using the WGS-84 parameters for the Earth, and for CHAMP $a = 6828000$ m, $e = 0.004$ and $i = 1.523672437$ radians, gives

$$\begin{aligned}
\frac{d\Omega}{dt} &\approx -0.37 \quad deg/day \\
\frac{d\omega}{dt} &\approx -3.88 \quad deg/day \\
\Delta T_N &\approx 3.30 \quad sek/revolution
\end{aligned}$$

We thus see that for a LEO satellite like CHAMP the oblateness of the Earth produces a significant perturbation.

For a typical GPS satellite the perturbations are considerably smaller. Using the parameters of a sample GPS satellite with $a = 26560$ km, $e = 0.013$ and $i = 0.96$ radians, gives

$$\begin{aligned} \frac{d\Omega}{dt} &\approx -0.0387 \quad deg/day \\ \frac{d\omega}{dt} &\approx 0.0215 \quad deg/day \\ \Delta T_N &\approx -2.20 \quad sek/revolution \end{aligned}$$

So oblateness is seen to play a minor, but not insignificant role for the GPS satellites.

2.4.2 Triaxiality

As discussed before, the tesseral terms tend to average out over the long term, and they can for most applications be neglected in the first approximation. However, for geostationary satellites the non-sphericity of Earth's equator gives rise to a non-negligible perturbation.

The equatorial cross section of the Earth is elliptical rather than circular, with a 65 m deviation from the circular shape. When modelling the Earth as an ellipsoid, one therefore refers to its *triaxiality*. The tesseral gravitational harmonic coefficient ($J_{2,2}$) of the Earth is related to the ellipticity of the Earth's equator. There are four equilibrium points separated by approximately 90° along the equator; two stable points and two unstable points. The two stable points are called gravitational valleys. Satellites placed at other longitudes than the stable points will drift with a certain period of oscillation. Thus they require *east-west* station keeping maneuvers to maintain their orbital position.

This effect can be estimated by considering the longitudinally dependent terms in the Earth's potential. Considering the potential of the Earth in the equatorial plane to degree 2, one gets

$$V = -\frac{GM}{r}\Big\{1 + \Big(\frac{R}{r}\Big)^2\Big(\frac{1}{2}J_2 + 3J_{2,2}\cos 2(\lambda - \lambda_{2,2})\Big)\Big\}. \tag{2.43}$$

The oblateness term J_2 leads to a correction to the geostationary orbit radius [15]:

$$\Delta r = \frac{1}{2}J_2\Big(\frac{R}{r}\Big)^2 r. \tag{2.44}$$

This corresponds to a correction to the radius of approximately one half kilometer. The triaxial perturbation ($J_{2,2}$) gives a longitudinal acceleration

$$a_\lambda = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\lambda}\Big|_{\theta=\pi/2} = -\frac{GM}{r^3}\left(\frac{R}{r}\right)^2(6J_{2,2}\sin 2(\lambda-\lambda_{2,2})). \tag{2.45}$$

Using the WGS-84 parameters i.e., $GM = 398600.5\ km$, $R = 6378137\ m$, $J_{2,2} = 0.0000018155$, $r = 42164\ km$, and (by equation (2.7)) $\lambda_{2,2} = -14.9°$, one finds

$$a_\lambda = -1.76\sin 2(\lambda + 14.9°)m/s/year. \tag{2.46}$$

Triaxiality, thus leads to a maximal annual perturbation of approximately 1.76 m/s, which must be compensated by using thrusters on the satellite.

Geostationary satellites play an important role in both civilian and military applications, for telecommunication and as a part of an alarm system, detecting rocket launches.

2.4.3 Luni-Solar perturbations

The gravitational forces exerted by the Sun and the Moon, on the Earth and a geostationary satellite, produces a considerable perturbation of the inclination. It can, after lengthy calculations, be shown that the net effect is that the orbital inclination of a geostationary satellite increases approximately by one degree per year [15]. In order to keep the orbit in the equatorial plane, the inclination must be corrected by *north-south* station keeping. These maneuvers require approximately 95% of the station keeping fuel budget of the entire lifetime of the satellite.

Chapter 3

Application to GPS and LEO

The Global Positioning System includes 24 satellites in orbit around the Earth with an orbital period of 12 sidereal hours. The 24 satellites are distributed unevenly in six orbital planes with an inclination of 55 ° with respect to the equatorial plane, so that from practically any point on the Earth four or more satellites are visible. GPS is funded by and controlled by the U. S. Department of Defense (DOD). Although the GPS has thousands of civilian users of GPS world-wide, the system was designed for and is operated by the U. S. military. The GPS uses stable highly accurate atomic clocks in the satellites and on the Earth in order to provide position and time determination.

The satellites are affected by relativity in three different ways: In the equation of motion, in the signal propagation, and in the beat rate of the satellite clocks. I will primarily treat the clock effects. In the following it will be shown that these clocks have relativistic frequency shifts, which are so large that - without accounting for them - the system would not work. Furthermore, I will consider the relativistic clock effects on a LEO and the highly eccentric Russian *Molnya* satellite orbit.

3.1 The segments of GPS

The GPS consists of three segments. The *Space Segment* of the system consists of the GPS satellites. These space vehicles (SV's) send radio signals from space. Tied to the clocks are timing signals which can be thought of as sequences of events in spacetime, characterized by positions and times of transmission.

The *Control Segment* consists of a system of tracking stations located around the world. The Master Control facility is located at Schriever Air

Force Base in Colorado. These monitor stations measure signals from the SV's. These signals are incorporated into orbital models for each satellites. The models compute precise orbital data (ephemeris) and SV clock corrections for each satellite. The Master Control station uploads ephemeris and clock data to the SV's. The SV's then send subsets of the orbital ephemeris data to GPS receivers over radio signals (low frequency modulations of the GPS signals).

The *User Segment* consists of the GPS receivers and the user community. GPS receivers convert SV signals into position and time estimates.

3.2 Signal structure

The signals are received on the Earth at two carrier frequencies in the L band. These frequencies are derived from the fundamental 10.23 Mhz frequency [1]; L1 at 154 $\times$ 10.23 Mhz and L2 at 120 $\times$ 10.23 Mhz. The L1 signal contains two codes; one at 1.023 Mhz (the so called C/A-code), and one encrypted at 10.23 Mhz (the so called P-code). The L2 signal only contains the P-code. Only users with access to both frequencies and both codes, can make precise real time positioning. Civilian users only have access to the C/A code and to the phases. Thus there are two levels of accuracy in positioning. More details on the signal structure can be found in [1].

3.3 The physical foundation of positioning

Suppose that we have four SV's each carrying an atomic clock, and let the clocks be synchronized. Let us assume that - at times t_i - light signals are sent from the four satellites, and that the i'th satellite - at time t_i - is located at position $\mathbf{r}_i$ ($i = 1, 2, 3, 4$). Since the velocity of light c is independent of the relative motions of the satellites and the receiver, one faces the problem of four equations in the four unknown spacetime coordinates $(t, \mathbf{r}) = (t, x, y, z)$:

$$c^2(t - t_i)^2 = |\mathbf{r} - \mathbf{r}_i|^2. \tag{3.1}$$

Here it is assumed that the clocks are *ideal*. By this we mean that there is no additional clock bias on the left side of equation (3.1). Therefore Einstein's postulate on the constancy of the speed of light is seen to be the conceptual basis of the GPS.

Timing errors of one nanosecond will lead to positioning errors of $s = c \cdot t \approx 30$ cm. It is also of critical importance to carefully specify the reference system in which the clocks are synchronized so that equation (3.1) is valid [12].

The timing signals in question can be thought of as places in the wave train of the transmitted GPS signal, where there is a phase reversal of the circularly polarized electromagnetic signal. The electromagnetic field tensor transforms in a linear fashion. Hence if **E** (electric field) and **B** (magnetic field) both are zero at some place in spacetime, all observers moving relatively to each other, will agree that at that point in spacetime, **E** and **B** will be zero. Such zeros correspond to the phase reversals of the circularly polarized electromagnetic waves coming from the satellites[1]. Thus when a place where the phase reversal is, (which propagates with speed c) hits an antenna, all observers moving relatively to the antenna, will in principle agree on the occurrence of that event. Any coordinate system can be chosen to analyze the event, but some coordinate systems are more convenient than others.

3.4 Synchronization by clock transport

Clock synchronization is important for practical applications of relativity, because two spatially separated observers must be able to relate the readings of their clocks, in order to compare time dependent observed quantities.

Consider two clocks at A and B. These two clocks can be synchronized by transporting an intermediate clock, which is set according to the clock at A, from A to B, and then adjust the clock at B according to it.

For the portable clock, which is transported along some path from A to B, the metric will be given by (in SI units) [15]

$$d\tau^2 = -g_{00}dt^2 - \frac{2}{c}g_{0j}dtdx^j - \frac{1}{c^2}g_{ij}dx^i dx^j, \tag{3.2}$$

where the metric has been separated into spatial and temporal parts. Hence the elapsed proper time $\Delta\tau$ for a given coordinate time, will be given by the following path integral

$$\Delta\tau = \int_{path} \sqrt{-g_{00} - \frac{2}{c}g_{0j}\frac{dx^j}{dt} - \frac{1}{c^2}g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}dt. \tag{3.3}$$

Since the spacetime interval is a quadratic equation in dt, it follows from equation (3.2) that the elapsed coordinate time Δt for a given proper time,

[1]It is a property of the GPS signals that the signals are circularly polarized, so that when a phase reversal occurs both **E** and **B** will pass through zero.

is given by the following path integral

$$\Delta t = \int_{path} \frac{1}{-g_{00}} \left\{ \frac{g_{0j}}{c} \frac{dx^j}{d\tau} \pm \sqrt{-g_{00} + \frac{1}{c^2}(-g_{ij}g_{00} + g_{0i}g_{oj}) \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}} \right\} d\tau. \quad (3.4)$$

This expression can be rewritten as

$$\Delta t = \int_{path} \left\{ \frac{1}{c} \frac{g_{0j}}{-g_{00}} \frac{dx^j}{d\tau} \pm \frac{1}{\sqrt{-g_{00}}} \sqrt{1 + \frac{1}{c^2} \gamma_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}} \right\} d\tau, \quad (3.5)$$

where $\gamma_{ij} = (g_{ij} - g_{0i}g_{0j}/g_{00})$ is the spatial three dimensional metric tensor. Using the binomial expansion[2], we find that the elapsed coordinate time for clock transport between two points A and B, can be approximated as

$$\Delta t = \int_A^B \left\{ 1 - \frac{1}{2}(-g_{00} - 1) + \frac{1}{2} \frac{1}{c^2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right\} d\tau + \frac{1}{c} \int_A^B g_{0j} \frac{dx^j}{d\tau} d\tau, \quad (3.6)$$

where the last integral is the so-called Sagnac term.

If we operate in an Earth - Centered Inertial (ECI) frame, the components of the metric tensor in the Newton limit is given by

$$\begin{aligned} g_{ij} &= \delta_{ij}, \\ g_{0j} &= 0, \\ g_{00} &= -(1 + 2\phi/c^2), \end{aligned} \quad (3.7)$$

where ϕ is the Newtonian gravitational potential. Hence when considering time transfer from point A to B in the ECI frame, the elapsed coordinate time is given by

$$\Delta t = \int_A^B \left\{ 1 - \frac{\phi}{c^2} + \frac{1}{2} \frac{v^2}{c^2} \right\} d\tau, \quad (3.8)$$

where v is the velocity of the portable clock in the ECI frame.

3.5 Einstein synchronization

In this section (which is based on [15] and [23]), I consider the so-called Einstein synchronization procedure. For the sake of simplicity, we first consider an inertial reference frame in flat Minkowski spacetime. Since light travels on a null geodesic $d\tau = 0$, the propagation equation is

$$c^2 dt^2 - dx^2 - dy^2 - dz^2 = 0. \quad (3.9)$$

[2] $1/\sqrt{-g_{00}} = 1/\sqrt{1 + (-g_{00} - 1)} \approx 1 - \frac{1}{2}(-g_{00} - 1)$.

Thus the elapsed coordinate time is given by

$$\Delta t = \frac{1}{c}\int_{path} \sqrt{dx^2 + dy^2 + dz^2} = \frac{1}{c}\int_{path} d\sigma = \frac{L}{c}, \tag{3.10}$$

where $d\sigma$ is the differential distance along the light path. Now suppose that a light signal is sent from a reference clock at A at the time t_1, reflected at a clock at the point B, and returned to A at the time t_3. If the coordinate time of reflection is denoted t_2, then

$$t_2 = \frac{1}{2}\Big((t_1 + \Delta t) + (t_3 - \Delta t)\Big) = \frac{t_1 + t_3}{2}, \tag{3.11}$$

where $\Delta t = L/c$. The coordinate time of reflection is thus identified with the midpoint between t_1 and t_3. The clock at the point B is thus assigned the proper time $\tau_B = t_2$. This procedure is known as Einstein synchronization.

Let us consider the same procedure, but now in an arbitrary spacetime, where the propagation equation is given by

$$g_{00}c^2dt^2 + 2g_{0j}dx^jcdt + g_{ij}dx^idx^j = 0. \tag{3.12}$$

Solving for dt and integrating along the light path from A to B and back again, gives

$$\Delta t_{AB} = \frac{1}{c}\int_A^B \frac{1}{-g_{00}}\left(g_{0j}dx^j + \sqrt{(g_{0i}g_{0j} - g_{ij}g_{00})dx^idx^j}\right), \tag{3.13}$$

and

$$\Delta t_{BA} = \frac{1}{c}\int_A^B \frac{1}{-g_{00}}\left(g_{0j}dx^j - \sqrt{(g_{0i}g_{0j} - g_{ij}g_{00})dx^idx^j}\right). \tag{3.14}$$

Thus the coordinate time of reflection is

$$t_2 = \frac{1}{2}\Big((t_1 + \Delta t_{AB}) + (t_3 - \Delta t_{BA})\Big) = \frac{t_1 + t_3}{2} + \frac{1}{c}\int_A^B \frac{g_{0j}}{-g_{00}}dx^j. \tag{3.15}$$

That is, if $g_{0j} \neq 0$ in our coordinate system, then it is not possible to Einstein-synchronize the clocks uniquely.

3.6 Synchronization and the Sagnac effect

Since most GPS users are fixed to the Earth, or moving slowly over the surface of the Earth, it was decided by the constructors of the GPS to send the satellite orbital data in a Earth Centered Earth Fixed (ECEF) reference frame, in which the Earth rotates with a fixed rotation rate defined by $\omega = 7.292115 \cdot 10^{-7} s^{-1}$ see table 2.1. This model is as mentioned before the World Geodetic System 1984 (WGS-84). Such a frame has only navigational interest. In fact it is much easier to treat physical phenomena in a ECI frame.

As pointed out by N. Ashby [12], local observations using GPS, are unaffected by the effects on the scales of time and length measurements due to other Solar System bodies, which are time dependent.

Let us therefore, ignore the gravitational potential for the moment, and consider the simplest feasible transformation from a system at rest (inertial system) to a uniformly rotating system.

Thus, let our rest system be the flat Minkowski space, in which the metric is given by (in cylindrical coordinates)

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - dz. \tag{3.16}$$

Let (t', r', θ', z') denote the coordinates in the rotating system. If the axis of rotation coincides with the axis z and z', then the transformation from the rest system to the rotating system is given by

$$t = t', \qquad r = r', \qquad \theta = \theta' + \omega t', \qquad z = z', \tag{3.17}$$

where ω is the angular velocity of rotation. Substituting (3.17) into (3.16), gives the metric in the rotating system:

$$c^2 d\tau^2 = \Big(1 - \frac{\omega^2 r'^2}{c^2}\Big) c^2 dt'^2 - 2\omega r'^2 d\theta' dt' - d\sigma'^2, \tag{3.18}$$

where

$$d\sigma'^2 = dr'^2 + r'^2 d\theta'^2 + dz'^2. \tag{3.19}$$

The time transformation used here $(t = t')$ is deceivingly simple. It means that the coordinate time in the rotating system is determined in the underlying inertial system.

It should be noted that the rotating frame only can be used for distances lower than c/ω, since g_{00} is positive for $r' > c/\omega$ (follows from (3.18)). The physical reason is that at large distances the velocity would become greater than the speed of light, which is inadmissable.

Let's now see what happens if we attempt to use synchronization by clock transport, in the rotating system, in order to make a network of synchronized clocks.

Factoring equation (3.18) we obtain the proper time increment on the transported clock:

$$d\tau^2 = dt'^2 \Big(1 - \frac{\omega^2 r'^2}{c^2} - \frac{2\omega r'^2 d\theta'}{c^2 dt'} - \frac{d\sigma'^2}{c^2 dt'^2}\Big). \tag{3.20}$$

We assume that the clock is moved around so slowly that we can preclude time dilation. In that case, the last term in equation (3.20) can be neglected. Also keeping only first order terms in the small quantity $\omega r'/c$, we get after applying the binomial expansion:

$$d\tau = dt' - \frac{\omega r'^2 d\theta'}{c^2}. \tag{3.21}$$

Now $r'^2 d\theta/2$ can be identified with the infinitesimal area dA in the rotating system swept out by a vector from the axis of rotation to the portable clock, projected onto the equatorial plane. This leads to:

$$\int_{path} dt' = \int_{path} d\tau + \frac{2\omega}{c^2} \int_{path} dA. \tag{3.22}$$

The last term in the above equation is the Sagnac term (c. f. 3.6). Anyone which is unaware of the rotation, would therefore be unaware of the Sagnac contribution, and would only use the first term in equation (3.22), when performing synchronization. Let's for example consider a synchronization process, where we are establishing a network of synchronized clocks on Earth's equator, by carrying a portable clock along the equator in the eastward direction, and set the time of a network of fixed clocks according to it. Using the (WGS-84) values for the Earth, we have that

$$\frac{2\omega}{c^2} = 1.6227 \cdot 10^{-21}\mathrm{s/m}^2 \qquad \text{and} \qquad R = 6378137\mathrm{m},$$

where R is the equatorial radius of the Earth, so the projected area is $\pi R^2 = 1.27802 \cdot 10^{14} m^2$. Therefore the Sagnac term in equation (3.22) is

$$\frac{2\omega}{c^2} \int_{path} dA = 207.4\mathrm{ns}. \tag{3.23}$$

Thus performing synchronization in this way will lead to a considerable discontinuity in the clock system, because the first clock will lead the last clock by 207.4 ns.

Let's now consider what will happen, if we wanted to apply Einstein synchronization instead. Light travels on a null geodesics, thus putting $d\tau^2 = 0$ in equation (3.18) and only retain terms to first order in $\omega r'/c$, leads to

$$c^2 dt'^2 - \frac{2\omega r'^2 d\theta'(c dt')}{c} - d\sigma'^2 = 0. \tag{3.24}$$

Solving for dt' and applying the binomial expansion, gives

$$dt' = \frac{d\sigma'}{c} + \frac{\omega r'^2 d\theta'}{c^2}. \tag{3.25}$$

Hence we get

$$\int_{path} dt' = \int_{path} \frac{d\sigma'}{c} + \frac{2\omega}{c^2} \int_{path} dA. \tag{3.26}$$

Thus we see that whether we use light or portable clocks in the synchronization process, path dependent inconsistencies are un-evadable in the rotating system. Whereas synchronization in the underlying inertial system is self-consistent, whether one uses light or portable clocks. For the GPS this means that all the ground based and orbiting clocks of the entire system, have to be synchronized in the ECI frame, in which self-consistency can be achieved. An other way of putting this is, that at nanosecond level the Earth does not provide an inertial frame.

The Sagnac effect was first experimentally verified by J. C. Hafele and R. E. Keating (1972) [27], using an atomic clock onboard a commercial jet plane.

3.7 The Earth's geoid and coordinate time

Next we need to describe the gravitational field near the Earth due to Earth's mass itself. We assume for the moment that Earth's mass distribution is static, and that there exists a locally inertial, non-rotating, freely falling reference system with origin at the center of the Earth. We write an approximate form of the Kerr solution to Einstein's field equations[3] (c. f. equation (1.43))

$$\begin{aligned} c^2 d\tau^2 = \Big(1+\frac{2V}{c^2}\Big)c^2dt^2 - \Big(1+\frac{2V}{c^2}\Big)^{-1}dr^2 \\ -r^2(d\theta^2 + sin^2\theta d\varphi) + \frac{4GJsin^2\theta}{rc^3}(cdt)d\varphi. \end{aligned} \tag{3.27}$$

This is essentially the Schwarzschild solution plus an off-diagonal term. Here we have used the coordinates (ct, r, θ, φ), where r is the geocentric distance, θ is the co-latitude, J is Earth's angular momentum, which we at the equator can approximate as

$$J = \frac{2MR^2\omega}{5}, \tag{3.28}$$

where R is the equator radius of the Earth. The V in equation (3.27) is the Newtonian gravitational potential of the Earth, which approximately is given

[3] I'm cheating a little bit here, because the Kerr metric was derived for a centrally symmetric rotating massive body. So it is questionable whether it can be interpreted as the outer solution for the Earth, because here the rotational velocity is correlated with the oblateness via the equation of state. But in the following we will se that the angular momentum of the Earth plays a negligible roll.

by

$$V = -\frac{GM}{r}\Big(1 - J_2\Big(\frac{R}{r}\Big)^2 P_2(cos\theta)\Big), \tag{3.29}$$

where P_2 is the Legendre polynomial of degree 2, and J_2 is the quadrupole moment coefficient. The WGS-84 value is $J_2 = 1.0826300 \cdot 10^{-3}$. The next multipole contribution to equation (3.29) is about 1000 times smaller (see table 2.1). Thus higher multipole moment contributions have a negligible effect at the present time.

Now we need to transform the metric (3.16) to the WGS-84 system, which is an ECEF reference system. We use the transformation for spherical polar coordinates:

$$t = t', \qquad r = r', \qquad \theta = \theta', \qquad \varphi = \varphi' + \omega t'. \tag{3.30}$$

By performing this transformation, we obtain

$$\begin{aligned} c^2 d\tau^2 = & \Big(1 + \frac{2V}{c^2} - \frac{r'^2\omega^2 sin^2\theta'}{c^2} + \frac{GJ\omega sin^2\theta'}{r'c^4}\Big)c^2dt'^2 \\ & -\Big(1 + \frac{2V}{c^2}\Big)^{-1} dr'^2 - r'^2 d\theta'^2 - r'^2 sin^2\theta' d\varphi'^2 \\ & -\Big(2r'^2\omega sin^2\theta' - \frac{4GJsin^2\theta'}{r'^2c^2}\Big)dt'd\varphi'. \end{aligned} \tag{3.31}$$

The g_{00} component of the metric tensor in the rotating (WGS-84) frame is thus given by

$$g_{00} = -\Big(1 + \frac{2V}{c^2} - \frac{r'^2\omega^2 sin^2\theta'}{c^2} + \frac{4GJ\omega sin^2\theta'}{r'c^4}\Big), \tag{3.32}$$

where the effective gravitational potential in the rotating frame includes the static gravitational potential of the Earth, a centripetal potential term, and a small contribution from the angular momentum. An estimation of these quantities can be done at the equator i. e. for $\theta' = \pi/2$ and $r' = R$:

$$\begin{aligned} 2V/c^2 &= -1.39 \cdot 10^{-9} \\ -R^2\omega^2 sin^2\theta'/c^2 &= -2.40 \cdot 10^{-12} \\ 4GJ\omega sin^2\theta'/Rc^4 &= 2.6 \cdot 10^{-21} \end{aligned}$$

Thus the angular momentum gives a small positive contribution to the effective potential. This contribution, however, is so small that it can be neglected for most geodetic studies. I will therefore define the effective potential to be:

$$\phi_{eff} \equiv V - \frac{\omega^2 r'^2 sin^2\theta'}{2}. \tag{3.33}$$

In the metric expressions used in this section, the coordinate time is determined by ideal clocks infinitely far away from The Earth. The SI system, however, defines the coordinate time unit (the second) by atomic clocks (which for the sake of argument shall be considered as ideal) at rest at mean sea level on the spinning Earth. Thus we shall consider clocks on Earth's geoid. These clocks are moving because of the Earth's spin, and due to the oblateness of the Earth, they are situated at different distances from the center of the earth. In order to proceed we therefore need an expression for the effective gravitational potential on this surface.

Assuming that the Earth is in hydrostatic equilibrium, mean sea level can be considered as an equipotential surface, and such a surface has to satisfy the condition

$$g_{00} = \text{constant}. \tag{3.34}$$

Using the above definition of the effective potential, we get

$$g_{00} = -\left(1 + \frac{2\phi_{eff}}{c^2}\right) = \text{constant}. \tag{3.35}$$

Therefore we set

$$-\frac{2GM}{c^2 r'}\left\{1 - J_2\left(\frac{R}{r'}\right)^2\left(\frac{3cos^2\theta' - 1}{2}\right)\right\} - \frac{\omega^2 r'^2 sin^2\theta'}{c^2} = \frac{2\phi_0}{c^2}, \tag{3.36}$$

where ϕ_0 is the geopotential evaluated at the geoid. Since ϕ_0 is a constant, it can be evaluated at equator, where $\theta' = \pi/2$ and $r' = R$. This gives:

$$\begin{aligned} \frac{\phi_0}{c^2} &= -\frac{GM}{Rc^2} - \frac{J_2 GM}{2Rc^2} - \frac{\omega^2 R^2}{2c^2} \\ &= -6.95348 \cdot 10^{-10} - 3.764 \cdot 10^{-13} - 1.203 \cdot 10^{-12} \\ &= -6.9693 \cdot 10^{-10}. \end{aligned} \tag{3.37}$$

Hence there are three different contributions to this potential. An $1/r$ central term contribution due to Earth's mass itself, a contribution due to Earth's quadrupole moment, and a centripetal term due to Earth's rotation. It can be seen that the main contribution is due to the mass of the Earth. In the expression (3.37) ϕ_0 has been divided by c^2, because relativistic clock corrections are easily expressed thereby. For a clock at rest on Earth's geoid, we get from equation (3.35)

$$d\tau = dt'\left(1 + \frac{\phi_0}{c^2}\right). \tag{3.38}$$

Thus clocks at rest on Earth's geoid run slow by seven parts in 10^{10} compared to clocks at infinity.

This model geoid, is actually what geo-physicists call the reference ellipsoid, from which the real geoid is determined. Looking at equation (3.38), we see that clocks at rest on Earth's geoid all beat at the same rate. This is due to a cancellation of effects on the geoid: A clock north of the equator will be more red shifted than a clock at equator, because it is closer to the center of the Earth (due to the oblateness of the Earth). On the other hand, it will move slower than the the clock at equator, because it is closer to the axis of rotation, so it suffers much less time dilation. We can now exploit this fact to define a new time coordinate

$$t'' = \Big(1 + \frac{\phi_0}{c^2}\Big)t' = \Big(1 + \frac{\phi_0}{c^2}\Big)t. \tag{3.39}$$

We have now established a coordinate time scale with respect to clocks at rest at Earth's geoid. This is exactly how the SI second is defined, namely in terms of the proper time rate of clocks at rest on Earth's geoid, which by equation (3.39) is identical to the coordinate time t''.

The classical definition of the geoid was given as the equipotential surface closest to mean sea level. But in view of equation (3.38), it seems natural to adopt a new definition of the geoid[4]:

Definition: *The relativistic geoid is the surface closest to mean sea level, on which ideal clocks beat at the same rate.*

This definition is not so impractical as it sounds. The fractional frequency stability of modern atomic clocks is of the order of five parts in 10^{14}. Thus for practical applications of this definition, atomic clocks can be considered as ideal. The geoid can be measured everywhere it it is accessible, - on land or at sea. On land it may be necessary to drill a hole in the ground or make use of existing mine shafts or caves.

3.8 Realization of coordinate time

In the last section, we saw that the Earth's angular momentum introduces a off diagonal term in the metric. This off diagonal term is, however, so small as compared to the terms due to Earth's potential (including quadrupole moment), and the centrifugal potential that it is negligible. Thus it would not be a big mistake, if we use the Schwarzschild metric to describe the curvature of space around the Earth. In the ECI frame we thus have

$$c^2 d\tau^2 = \Big(1 + \frac{2(V - \phi_0)}{c^2}\Big)c^2 dt^2 - \frac{dr^2}{1 + \frac{2V}{c^2}} - r^2(d\theta^2 + sin^2\theta d\varphi^2), \tag{3.40}$$

[4]This definition was first proposed by Arne Bjerhammar (1986) [28].

where we have applied the transformation (3.39), and then dropped the primes on t'' and used the symbol t instead. The unit of time is thus the SI second, as established by standard clocks on the rotating geoid, with synchronization established in the underlying ECI - frame. The factor $(V - \phi_0)$ in the first term of equation (3.40), is due to the fact that - as viewed from the ECI - frame the rate of the SI second is determined by moving clocks in a spatially dependent gravitational field.

Due to the time dilation and gravitational frequency shifts as described by the Schwarzschild metric, the proper time on orbiting GPS clocks can not be used to transfer time from one transmission event to another. As we have seen in the last section, path dependent effects must be accounted for. The Earth is - for the moment - assumed to be isolated, so the time variable t is valid in a coordinate system large enough to cover the whole GPS constellation. Therefore it seems naturall to use the time variable t in equation (3.40) as a basis for synchronization in the neighborhood of the Earth.

Let's now consider what happens for a slowly moving clock. Since the Earth's gravitational field is weak, i. e., $2V/c^2$ is small, we can expand

$$\frac{1}{1+\frac{2V}{c^2}} \approx 1 - \frac{2V}{c^2} + \frac{4V^2}{c^4} - ... \tag{3.41}$$

Retaining only terms to the order $1/c^2$, we obtain after insertion into (3.40):

$$c^2 d\tau^2 = \Big(1 + \frac{2(V-\phi_0)}{c^2}\Big)c^2dt^2 - \Big(1 - \frac{2V}{c^2}\Big)dr^2 \\ -r^2(d\theta^2 + sin^2\theta d\varphi^2). \tag{3.42}$$

Factoring c^2dt^2 out, and retaining only terms to the order $1/c^2$ gives

$$c^2 d\tau^2 = \Big(1 + \frac{2(V-\phi_0)}{c^2} - \frac{dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)}{c^2dt^2}\Big)c^2dt^2. \tag{3.43}$$

The third term in equation (3.43) is just the square of the velocity v with respect to the ECI frame divided by c^2, written in spherical polar coordinates (see appendix A). By using the binomial expansion, we can write the proper time increment as

$$d\tau = \Big(1 + \frac{(V-\phi_0)}{c^2} - \frac{v^2}{2c^2}\Big)dt. \tag{3.44}$$

Solving for the coordinate time and integrating along the path of the clock leads to

$$\int_{path} dt = \int_{path}\Big(1 - \Big(\frac{(V-\phi_0)}{c^2} - \frac{v^2}{2c^2}\Big)\Big)d\tau. \tag{3.45}$$

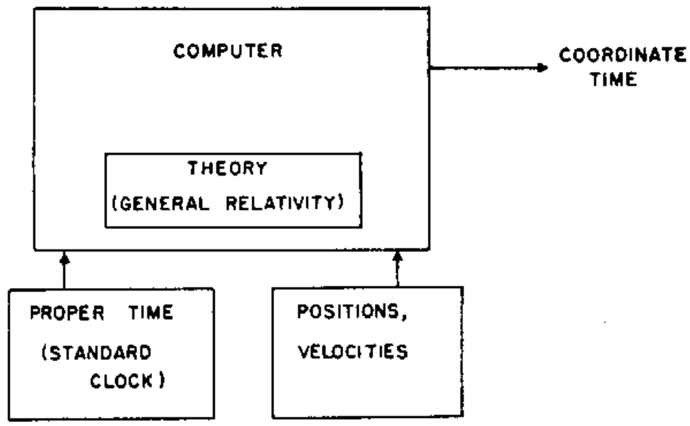

Figure 3.1: Coordinate time as resulting from relativistic corrections applied to the elapsed time measured using standard clocks on the spinning Earth. Figure from [13].

A system of spatially separated clocks, which are synchronized and properly adjusted in order to maintain synchronization, will be referred to as a coordinate time system. In the following I will discuss the principles of how to construct such a system[5] without the path dependent discrepancies discussed in section 3.6. Imagine for the moment that there are no gravitational fields. Then one may introduce an underlying local inertial frame, with origo at the Earth's geocenter. In this non-rotating frame a set of standard clocks is introduced, all of them being synchronized, and running at agreed upon rates, such that synchronization is maintained. Now introduce the rotating Earth, with a set of standard clocks distributed around upon the Earth, possibly moving around. To each of this standard clocks a set of corrections may be applied, based upon the known positions and motions of the clocks, using equation (3.45), such that at each instant the coordinate clock is synchronous with the fictive clock at rest in the local inertial frame, whose location coincides with the Earth based standard clock at that instant. This set of clocks will therefore be keeping coordinate time in the ECEF system, and coordinate time will be equivalent to the time measured by clocks in the underlying inertial frame. When the gravitational field of the Earth is introduced, we can still obtain coordinate time by applying a correction for the red shift, given by the first term in equation (3.45). This procedure is illustrated by figure 3.1.

The practical implementation of such a coordinate time is further compli-

[5]This construction is based on [13].

cated by the gravitational fields of the Sun and the other Solar system bodies, the motion of the geocenter, and whatever motions clocks can have. All these effects must be considered when calculating the relativistic corrections. However, in the vicinity of the Earth (distances under 50.000 km from the geocenter), equation (3.45) provides a good approximation, because at these distances, the potentials of the Moon and Sun are negligible (se section 4.4).

3.9 Some limitations of the determination of coordinate time

The actual geoid of the Earth deviates significantly from the reference ellipsoid. In figure 3.2 the Earth's geoid is seen to deviate up to approximately 105 m from the reference ellipsoid in one place. Figure 3.2 is based upon satellite velocities using the CHAMP satellite. A height difference of 105 m between two clocks will lead to a fractional frequency difference of $g \cdot h/c^2 \approx 1.1 \cdot 10^{-14}$.

Because the actual geoid (not the reference ellipsoid) is the surface of reference when comparing clock rates, the systematic error caused by the deviation of the reference ellipsoid from the actual geoid, can be accounted for

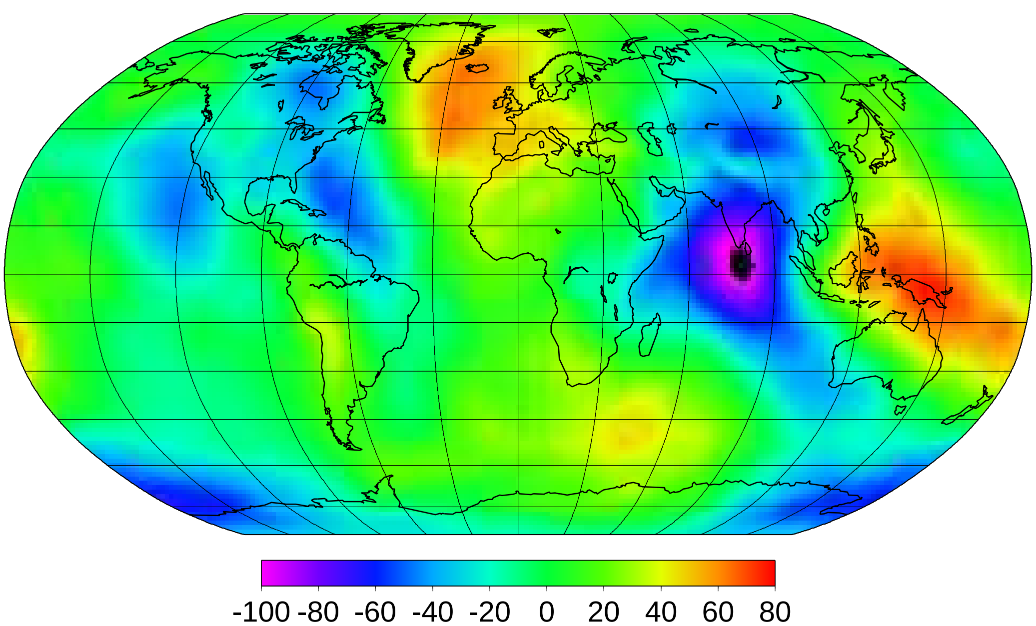

Figure 3.2: The Earth's geoid obtained from the knowledge of the state vector and measurement of nonconservative forces, using the energy conservation method. The numbers in the color scale are in meters. Figure from [6].

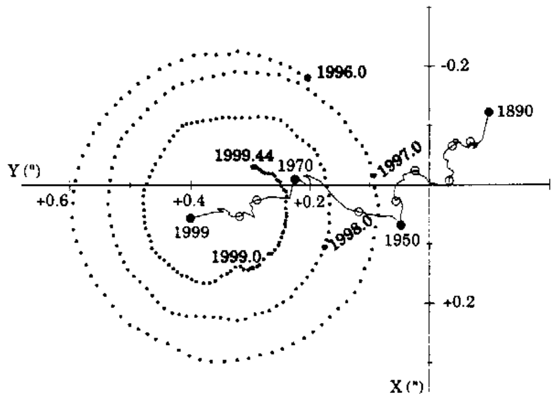

Figure 3.3: Polar motion 1996 - 1999, and mean polar displacement 1890 - 1999. Figure from [32].

in the calculations. A correction of one part in 10^{-14} is, however, negligible because this is below, or in the best case, at the same level of accuracy of modern atomic clocks [7]. However, with the increasing improvement of atomic clock performance it may be possible to include this error in the future. Also the Coriolis force acts on large scale ocean currents, with the net effect that mean sea level can deviate from the geoid, by a few meters. As above this will lead to a negligible error.

More serious limitations arise from the polar wander, because a change in the direction of the Earths axis, will lead to errors in computing the Sagnac effect [7]. As seen before, the Sagnac effect is of the form

$$\Delta t = \frac{2\omega A}{c^2}, \tag{3.46}$$

where A is the area swept out by a vector from the axis of rotation to the portable clock, projected onto the equatorial plane. Consider a time transfer process in which two ground stations are involved with the area to be projected being that of the triangle consisting of the Earth's center and the two stations as vertices.

If the Earth's axis of rotation changes by a small angle $\delta\theta$, then the change in the Sagnac effect is less than $2\omega A\delta\theta/c^2$. Thus if we assume the polar wander is 30 m, then $\delta\theta < 5 \cdot 10^{-6}$, which gives an error $\Delta t \approx 6ps$. Usually the Sagnac effect is of the order of magnitude of $\Delta t \approx 10^2\ ns$ for a typical experiment, so the effect arising from the polar wander, as well as the uncertainties in ω, can be neglected at present time. Here the polar wander is set to 30 m which may be true over a period of 30 years, but the annual variation is maximally a few meters (see figure 3.3).

Chapter 4

Satellite clocks

As mentioned before, physical effects are easiest described in a inertial frame. This applies to the satellite orbits as well, - the relativistic effects on a clock onboard a satellite is easier to deal with in the ECI frame, because then the Sagnac effect becomes irrelevant. From equation (3.45) it is seen that the coordinate time interval depends on the combined effects of second order Doppler shifts and gravitational frequency shifts. The geopotential ϕ_0/c^2 in equation (3.45) includes the coordinate time scale correction needed, when using clocks on the spinning Earth as reference. When calculating ϕ_0/c^2 Earth's quadrupole moment must be included (se equation (3.37)). In equation (3.45) V is the potential at the satellite's position.

4.1 The coordinate time interval

When calculating the coordinate time interval by equation (3.45), it is necessary to know the potential at the satellite's position. We have seen that the Earth's quadrupole moment gives rise to some small orbital perturbations, but since the quadrupole moment dies off rapidly with the distance r from Earth's center, its effects on GPS satellites are small. Let us therefore as a first approximation neglect the quadrupole moment and assume that the satellite orbits are Keplerian (this is however a poor approximation for LEO satellites). Then the velocity v in equation (3.45) can be calculated by the *vis viva* equation. Thus, inserting (2.8) into (3.45) gives

$$\Delta t = \int_{path} \left\{ 1 + \frac{3GM}{2ac^2} + \frac{\phi_0}{c^2} - \frac{2GM}{c^2}\left(\frac{1}{a} - \frac{1}{r}\right) \right\} d\tau. \tag{4.1}$$

The first two terms in equation (4.1) gives a constant rate correction. For a GPS satellite with orbital semi-major axis 26562 km, the constant rate correction is

$$\frac{3GM}{2ac^2} + \frac{\phi_0}{c^2} = 2.5046 \cdot 10^{-10} - 6.9693 \cdot 10^{-10} = -4.465 \cdot 10^{-10}$$
$$= -38.58 \mu s/day.$$

Thus the orbiting clock is beating faster than a clock on the Earth, by approximately $38\mu s$ per day, primarily because it is gravitationally blue-shifted. This effect is enormous compared to the available precision of modern atomic clocks at the nanosecond level. To compensate for this rate difference, the satellite clock is given a fractional rate offset prior to launch. The resulting frequency is

$$(1 - 4.465 \cdot 10^{-10}) \times 10.23 \quad \text{MHz} = 10.2299999954326 \quad \text{MHz}, \tag{4.2}$$

so that it will appear to an observer on the geoid that the clock beats with the chosen frequency 10.23 MHz. Therefore, apart from a periodic correction (the last correction term under the integral in equation (4.1)) the orbiting clock becomes a coordinate clock.

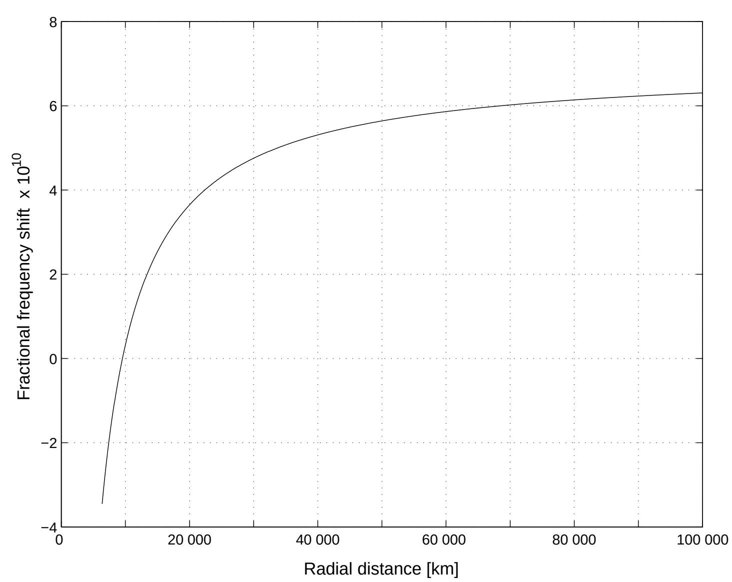

Figure 4.1: Fractional frequency shift for circular orbits versus radial distance a as explained in the text.

If the satellite orbit is circular the third correction term in equation (4.1) vanishes. Thus for circular orbits we will only have a constant fractional frequency shift. On figure 4.1 the fractional frequency shift of a clock in a circular orbit is shown versus orbital radius. For a GPS satellite (orbital mean radius of 26500 km) the gravitational blue-shift is greater than the effect of time dilation, whereas for a LEO satellite the velocity is so great that time dilation is the dominant effect. At radius $a = 3GM/2\phi_0 \approx 9545$ km the effects cancel.

4.2 The eccentricity correction

If the satellite orbit is elliptical, then the last term in equation (4.1) will contribute to the elapsed coordinate time. Assuming that $d\tau \approx dt$, we can write the last integral in equation (4.1) as

$$\Delta t_e = \int_{path} \left\{ \frac{2GM}{c^2}\Big(\frac{1}{r} - \frac{1}{a}\Big)\right\} d\tau = \frac{2GM}{c^2}\int \Big(\frac{1}{r} - \frac{1}{a}\Big) dt. \tag{4.3}$$

Using equation (2.11) for the radial distance, leads to

$$\Delta t_e = \frac{2GM}{ac^2}\int \Big(\frac{e\cos E}{1 - e\cos E}\Big) dt. \tag{4.4}$$

Differentiation of Kepler's equation (2.18) with respect to time gives

$$\frac{dE}{dt} = \frac{\sqrt{GM/a^3}}{1 - e\cos E}. \tag{4.5}$$

Substituting (4.5) into (4.4) leads to

$$\Delta t_e = \frac{2\sqrt{GMa}}{c^2} e(\sin E - \sin E_0) = \frac{2\sqrt{GMa}}{c^2} e\sin E + \text{constant}. \tag{4.6}$$

The integration constant can be dropped, because it can be absorbed in the Kalman filter computations along with other constant clock biases. Equation (4.6) is called the *eccentricity correction*, and it must be made by the receiver. It is necessary to incorporate this correction, because the combination of gravitational frequency shift and second order Doppler shift vary due to orbit eccentricity. For Keplerian orbits the position and velocity vectors are given by [15]

$$\mathbf{r} = r(\cos\nu\mathbf{i} + \sin\nu\mathbf{j}) \tag{4.7}$$

and

$$\mathbf{v} = \sqrt{GM/a(1 - e^2)}(-\sin\nu\mathbf{i} + (e + \cos\nu)\mathbf{j}), \tag{4.8}$$

where ν is the true anomaly. By performing the scalar product we get

$$\mathbf{r}\cdot\mathbf{v} = r\sqrt{\frac{GM}{a(1 - e^2)}} e\sin\nu = \sqrt{GMa} e\sin E. \tag{4.9}$$

Thus, the eccentricity correction can without approximation be expressed as

$$\Delta t_e = \frac{2\mathbf{r}\cdot\mathbf{v}}{c^2}, \tag{4.10}$$

where $\mathbf{r}$ and $\mathbf{v}$ is the position and velocity of the satellite at the instant of transmission. Equation (4.10) is probably the most convenient way to express the eccentricity correction, because since $\mathbf{r}\cdot\mathbf{v}$ is a scalar, it can be evaluated in either the ECI frame or the ECEF (WGS-84) frame. On figure 4.2 the eccentricity correction is shown for the GPS-21 satellite.

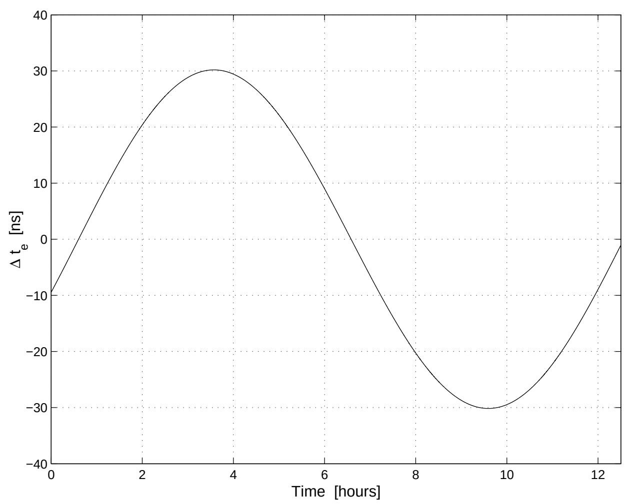

Figure 4.2: Eccentricity correction for a typical GPS satellite, with an amplitude of approximately 30 ns, and a peak to peak variation of 60 ns.

The eccentricity correction is calculated on the basis of simulated data, by equation (4.10). The simulation data was made by the IAG special commission at the University of Bonn Germany, using the EGM96 gravity field model of the Earth to degree and order 36. The ephemeris data is given as radius vector $\mathbf{r}$, and velocity $\mathbf{v}$ in an ECI frame, and can be downloaded from ftp : //geo@atlas.geoid.uni − bonn.de/pub/SC7_SimulationScenarios (password needed). The initial values for the orbit of GPS-21 is given in table 4.1.

We have seen that the eccentricity correction of a typical GPS satellite has an amplitude of approximately 30 ns. The eccentricity correction is by equation (4.6) proportional with the eccentricity. Let us therefore consider a highly elliptic orbit, like the Russian Molniya orbit. The Molniya orbits are of special interests for telecommunication in high northern (or southern) latitudes. In order to achieve an optimal coverage of a country in the northern hemisphere like Russia, the perigee is located at southern latitudes near $\omega = -90°$. Since

Semi-major axis	26560.25169632944 km
Eccentricity	0.01323881349526
Inclination	0.9614884100802 rad
Longitude of the ascending node	-0.4495096737336 rad
Argument of perigee	-3.001488651204 rad
Mean anomaly	-0.3134513508155 rad

Table 4.1: Initial values for the GPS-21 orbit simulation.

the satellite spends most of its time near the apogee of the highly eccentric orbit, it should be visible for at least eight hours each day [25]. Thus three satellites should be sufficient to give full time coverage. To minimize orbital perturbations (see equation (2.39)) the inclination of a Molniya orbit is chosen to be $i \approx 63.4°$. This also gives a good coverage of the northern hemisphere. A ground track of a sample Molniya orbit can be found on figure 4.3.

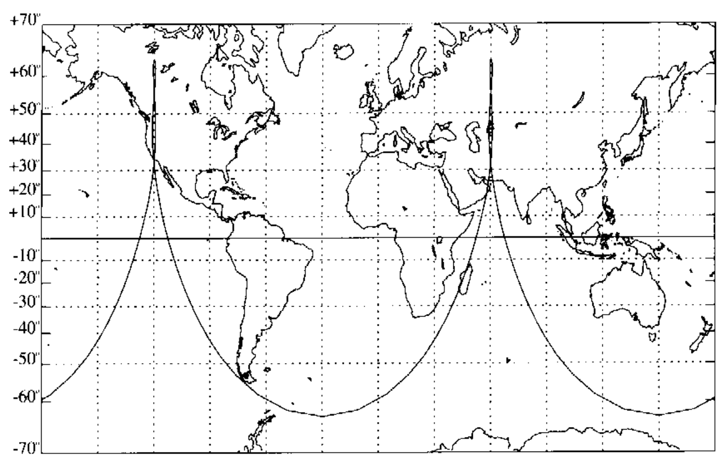

Figure 4.3: Ground track of a sample Molniya orbit, with $a = 26555$ km, $e = 0.7222$, $i = 63.4°$ and $\omega = 270°$. Adopted from [25].

A clock in a Molniya orbit, with same parameters as on figure 4.3 will by equation (4.2) have approximately the same constant offset as a clock in a GPS satellite, because the constant offset is only dependent on a. But the eccentricity correction will by equation (4.6) have a considerably higher amplitude.

In order to calculate the eccentricity correction by equation (4.6), we first have to find the eccentric anomaly E. Due to the high eccentricity ($e = 0.7222$) the series expansion given by equation (2.19) can not be used. Thus

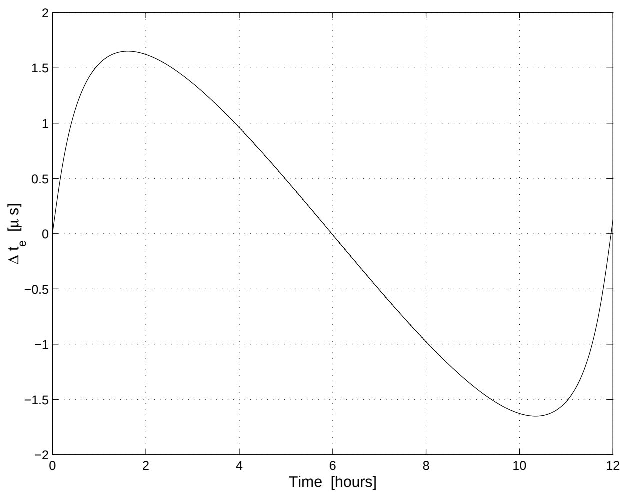

Figure 4.4: Relativistic eccentricity correction for a clock in a Molniya orbit, with $a = 26555$ km, $e = 0.7222$.

I have used the following iteration for every t (from [1]):

$$\begin{aligned} E_1 &= M(t). \\ E_{i+1} &= M(t) + e \sin E_i, \quad i = 1, ...25. \end{aligned} \tag{4.11}$$

On figure 4.4 the eccentricity correction for the sample Molniya satellite is shown. The eccentricity correction for the Molniya orbit is seen to have an amplitude of approximately $1.7 \mu s$, which is 40 times larger than the effect on a GPS satellite. The periodic variation is not symmetric, due to the high ellipticity of the orbit.

4.3 Earth oblateness

In this section we consider the effect of Earth oblateness on a satellite clock. The effect of Earth oblateness is, historically, one of the most celebrated of all orbital perturbations. Numerous books have treated this subject thoroughly (c. f. [38], [39], and [40]). However, the effect of oblateness is rather subtle, because it introduces periodic (as-well as constant) changes in both orbit radius and the velocity, which also affects the monopole and velocity contributions to the frequency shift. This treatment is based on the books cited above and [14].

In the following we only consider the first order oblateness correction, ie., the effect of Earth's quadrupole moment in the limit of small eccentricities.

From conservation of energy, it follows that we can express the fractional frequency shift of an orbiting clock in terms of the potential as

$$\frac{\Delta f}{f} = -\frac{v^2}{2c^2} - \frac{GM}{c^2 r} + \frac{P}{c^2}, \tag{4.12}$$

where the frequency shift is measured with respect to clocks at rest at infinity, and P is the perturbing potential due to the quadrupole moment of the Earth.

4.3.1 The perturbing potential

We have previously seen that P is given by equation (2.26). Since the perturbing potential contains the small factor J_2, we can, to leading order, substitute r with the unperturbed semi-major axis a_0, i with the unperturbed value i_0, and ω with the unperturbed value ω_0, ie.,

$$P = -\frac{GMJ_2R^2}{2a_0^3}\left(1 - \frac{3}{2}\sin^2 i_0\right) - \frac{3GMJ_2R^2\sin^2 i_0}{4a_0^3}\cos(2\omega_0 + 2\nu). \tag{4.13}$$

4.3.2 Perturbation in the square of the velocity

By using the energy conservation condition, it can be shown that the square of the velocity of a Keplerian orbit is given by

$$\frac{v^2}{2} = \frac{GM}{2a}\frac{1 + e\cos E}{1 - e\cos E}. \tag{4.14}$$

In order to find v^2 in the perturbed orbit, expressions of the perturbed elements a, e, and E must be substituted into the right side of the above equation. These expressions can be obtained by Lagrange's perturbation theory. The detailed derivations of the osculating Keplerian elements are lengthy, and can be found in [39] and [40]. Thus I will spare the reader for the cumbersome algebra, and cite the results obtained by N. Ashby, reported in [14].

The *perturbed semi-major axis* is to leading order given by

$$a = K_a + \frac{3J_2R^2}{2a_0^2}\sin^2 i_0\cos(2\omega_0 + 2\nu), \tag{4.15}$$

where K_a is an integration constant.

The *perturbed eccentricity* is to leading order given by

$$e = K_e + \frac{3J_2R^2}{2a_0^2}\Big[\left(1 - \frac{3}{2}\sin^2 i_0\right)\cos\nu + \frac{1}{4}\sin^2 i_0\cos(2\omega_0 + \nu) + \frac{7}{12}\sin^2 i_0\cos(2\omega_0 + 3\nu)\Big], \tag{4.16}$$

where K_e in an integration constant. The eccentric anomaly can be found from Kepler's equation

$$E = M + e \sin E, \tag{4.17}$$

with perturbed values for M and e. Expanding to first order in e and then multiplying by e leads to

$$e \cos E = e \cos M - e^2 \sin M \sin E \approx e \cos M. \tag{4.18}$$

The *perturbed mean anomaly* can be expressed as

$$M = M_0 + \Delta M, \tag{4.19}$$

where ΔM, to leading order, is given by

$$\Delta M = -\frac{3J_2R^2}{2a_0^2}\Big[\Big(1-\frac{3}{2}\sin^2 i_0\Big)\sin\nu - \frac{1}{4}\sin^2 i_0 \sin(2\omega_0+\nu) + \frac{7}{12}\sin^2 i_0 \sin(2\omega_0+3\nu)\Big]. \tag{4.20}$$

Thus, for very small eccentricity

$$\begin{aligned} e\cos E &= e\cos M_0 - \Delta M \sin M_0 = e_0 \cos E_0 \\ &+ \frac{3J_2R^2}{2a_0^2}\Big(1-\frac{3}{2}\sin^2 i_0\Big) + \frac{5J_2R^2}{4a_0^2}\sin^2 i_0 \cos(2\omega_0+2\nu), \end{aligned} \tag{4.21}$$

where the first term is the unperturbed part. We are now in a position to calculate the perturbation in the square of the velocity. Using the above results yield the expression

$$\frac{v^2}{2} = \frac{GM}{2a_0}(1+2e_0\cos E_0) + \frac{3GMJ_2R^2}{a_0^3}\Big(1-\frac{3}{2}\sin^2 i_0\Big) + \frac{GMJ_2R^2}{2a_0^3}\sin^2 i_0 \cos(2\omega_0+2\nu). \tag{4.22}$$

4.3.3 Perturbation of the central term GM/r

Before we can derive an expression for the perturbation of GM/r, we have to consider the perturbation in the radius, which can be written

$$r = a_0(1-e_0\cos E_0) + \Delta a - \Delta(a_0 e \cos E). \tag{4.23}$$

Thus using equation (4.15) and (4.21) yields

$$r = a_0(1-e_0\cos E_0) - \frac{3J_2R^2}{2a_0}\Big(1-\frac{3}{2}\sin^2 i_0\Big) + \frac{J_2R^2}{4a_0}\sin^2 i_0 \cos(2\omega_0+2\nu). \tag{4.24}$$

The above expression for the perturbed r yields the following for the central term

$$\frac{GM}{r} = \frac{GM}{a_0}(1 - e_0 \cos E_0) + \frac{3GMJ_2R^2}{2a_0^3}\Big(1 - \frac{3}{2}\sin^2 i_0\Big) - \frac{GMJ_2R^2}{4a_0^3}\sin^2 i_0 \cos(2\omega_0 + 2\nu). \tag{4.25}$$

4.3.4 The fractional frequency shift

Using the above results, the fractional frequency shift can be calculated by equation (4.12), which leads to

$$\frac{\Delta f}{f} = -\frac{3GM}{2a_0c^2} - \frac{2GM}{a_0c^2}e_0 \cos E_0 - \frac{7GMJ_2R^2}{2a_0^3c^2}\Big(1 - \frac{3}{2}\sin^2 i_0\Big) - \frac{GMJ_2R^2}{a_0^3c^2}\sin^2 i_0 \cos(2\omega_0 + 2\nu). \tag{4.26}$$

This frequency shift is with respect to clocks at rest at infinity. Thus the first term, when combined with the reference potential at the Earth's geoid, is the constant rate correction discussed earlier. The second term gives rise to the eccentricity correction. The third and fourth term is due to Earth's quadrupole moment. Thus we see that the quadrupole moment introduces a secular and a periodic perturbation. For a GPS satellite, the inclination is 55°. The third term in the expression above can thus, as a first approximation, be neglected.

Let us now consider the forth term in the expression above. The additional elapsed coordinate time, which is introduced by this term, on the orbiting clock is given by (using $d\tau \approx dt$ when performing the integral)

$$\Delta t_{perJ_2} = \int_{path} -\frac{GMJ_2R^2}{a_0^3c^2}\sin^2 i_0 \cos(2\omega_0 + 2\nu)dt. \tag{4.27}$$

To a sufficient approximation, we can replace 2ν with $2nt$, where n is the mean motion of the satellite. Thus the periodic effect on the elapsed coordinate time of the satellite clock, due to Earth's quadrupole moment, is given by

$$\Delta t_{perJ_2} = -\sqrt{\frac{GM}{a_0^3}}\frac{J_2R^2}{2c^2}\sin^2 i_0 \sin(2\omega_0 + 2nt). \tag{4.28}$$

The integration constant is dropped, since it is assumed that it can be absorbed in the Kalman filter.

A similar expression has been found by R. A. Nelson, reported in [15]. The amplitude of the expression obtained by R. A. Nelson is - however - slightly different, whereas the periodic dependence is the same. This is due to the fact that he neglects the perturbation in the central term.

For a GPS satellite with semi-major axis $a_0 = 26560$ km, the amplitude of Δt_{perJ_2} is 24 ps. Thus the quadrupole moment of the Earth introduces a peak to peak navigational error of 48ps ~ 1.44 cm. The secular rate correction (the third term in equation(4.26)) gives for the same satellite, an offset of $2.37 \cdot 10^{-16}$, which is equivalent with 20 ps/day. The contribution of the next higher order gravitational harmonics are 1000 times less (see table 2.1). Thus we see that the oblateness of the Earth plays no role for navigation, using the GPS. However, when considering frequency shifts due to orbit changes, the effect of oblateness must be included (c. f. [14]).

For a LEO like CHAMP the quadrupole moment introduces a secular shift of $1.0656 \cdot 10^{-12} = 92$ ns/day, and a periodic peak - to - peak error of 0.54 ns ~ 16.2 cm.

4.4 Tidal effects

The difference between the force of attraction of the other Solar system bodies acting at the center of mass of the Earth, and the force acting at an arbitrary point on Earth's surface, is known as tidal forces. Actually it follows from the equivalence principle, that locally, the net effect of the other Solar system bodies on the artificial Earth satellites, can only come from tidal potentials, because the Satellites and the Earth is in free fall with respect to the gravitational field of the other Solar system bodies, there can be no linear terms in the effective potential from these other bodies.

Let us first come to an understanding of how the Riemann-Christoffel curvature tensor is related to the tidal forces. The center of mass of the Earth moves around the Sun along a Keplerian orbit, which is a geodesic. The motion of a neighboring point is in fact not a geodesic, but I will in the following assume that it can be approximated by a geodesic. Thus, we have to study the problem geodesic deviation[1].

Consider, therefore two neighboring points, $x^\alpha(\tau)$ and $x^\alpha(\tau) + \xi^\alpha(\tau)$ as shown on figure 4.5, where the proper time is assumed to be the same for the two points. From equation (1.14) it follows that the corresponding geodesics are

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad (4.29)$$

[1]This section is based on [33], and [34].

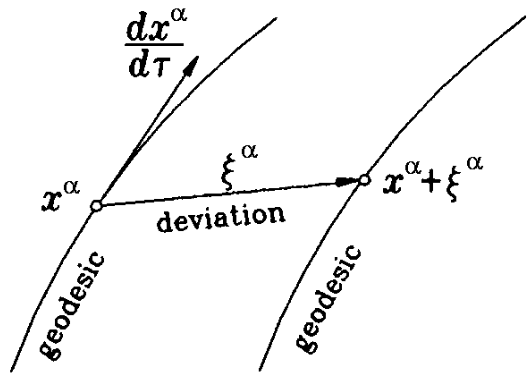

Figure 4.5: Geodesic at two neighboring points. Adopted from [34].

and

$$\frac{d^2}{d\tau^2}(x^\alpha + \xi^\alpha) + \Gamma^\alpha_{\mu\nu}(x+\xi)\Big(\frac{dx^\mu}{d\tau} + \frac{d\xi^\mu}{d\tau}\Big)\Big(\frac{dx^\nu}{d\tau} + \frac{d\xi^\nu}{d\tau}\Big) = 0. \tag{4.30}$$

Above superscripts are omitted when they indicate an argument. The Christoffel symbols can be expanded into a Taylor series [34] as

$$\Gamma^\alpha_{\mu\nu}(x+\xi) = \Gamma^\alpha_{\mu\nu}(x) + \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\lambda}\xi^\lambda. \tag{4.31}$$

Omitting the x argument in the Christoffel symbol, we get after inserting equation (4.31) into (4.30) that

$$\frac{d^2x^\alpha}{d\tau^2} + \frac{d^2\xi^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\xi^\lambda + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\xi^\nu}{d\tau} + \Gamma^\alpha_{\mu\nu}\frac{d\xi^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{4.32}$$

Using the symmetry properties of the Christoffel symbols, and subtracting equation (4.29) from (4.32) leads to

$$\frac{d^2\xi^\alpha}{d\tau^2} + 2\Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\xi^\nu}{d\tau} + \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\xi^\lambda = 0. \tag{4.33}$$

This equation can be rewritten in terms of the covariant derivative[2] and the Riemann-Christoffel curvature tensor as (for details see [33])

$$\frac{D^2\xi^\alpha}{D\tau^2} + R^\alpha_{\mu\nu\lambda}\xi^\nu\frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau} = 0. \tag{4.34}$$

[2]The covariant derivative of a contravariant vector ξ^ν is a mixed tensor defined by

$$\xi^\nu{}_{;\alpha} = \frac{\partial \xi^\nu}{\partial x^\alpha} + \Gamma^\nu_{\alpha\sigma}\xi^\sigma.$$

This is an ordinary second order differential equation for geodesic deviation.

To illustrate the physical meaning of equation (4.34), let us consider the Newton limit, where the components of the metric tensor are $g_{00} = -(1+2V)$, $g_{0j} = 0$ and $g_{ij} = \delta_{ij}$ in natural units. Now calculating the Christoffel symbols by equation (1.16), it is straight forward to show that the only non-vanishing component of the curvature tensor is[3]

$$R^i_{0j0} = -\frac{\partial^2 V}{\partial x^i \partial x^j}, \quad i, j = 1, 2, 3. \tag{4.35}$$

Substituting equation (4.35) into (4.34) gives

$$\frac{d^2\xi^i}{d\tau^2} - \frac{\partial^2 V}{\partial x^i \partial x^j}\xi^j\Big(\frac{dt}{d\tau}\Big)^2 + \frac{\partial^2 V}{\partial x^i \partial x^j}\xi^0 \frac{dt}{d\tau}\frac{dx^j}{d\tau} = 0, \tag{4.36}$$

where it has been used, that the covariant derivative reduces to the ordinary derivative in freely falling coordinates. Assuming that we only have a deviation in space, i. e. $\xi^0 = 0$, and that $d/d\tau \approx d/dt$, we get

$$\frac{d^2\xi^i}{dt^2} = \frac{\partial^2 V}{\partial x^i \partial x^j}\xi^j. \tag{4.37}$$

The left side of this equation is the differential acceleration, and the right side is known as the tidal force per unit mass.

Let us apply our knowledge to the tidal effects of the Moon, which can be considered as a point mass, with potential $V = Gm/r$. It can directly be shown that (for details se [34]) the components of the force are

$$f^i = -Gm\Big(\frac{\xi^i}{r^3} - \frac{3x^i x^j \xi^j}{r^5}\Big). \tag{4.38}$$

Introducing a suitable choice of coordinates, as shown on figure 4.6, we have that $x^1 = x^2 = 0$, and $x^3 = r$ is the distance from Earth's center of mass to

When the motion is along a trajectory $x^\alpha(\tau)$, the differentiation has to be taken along the curve $x^\alpha(\tau)$. Thus, we have to project the covariant derivative on the tangent $dx^\alpha/d\tau$ and the expression above has to be replaced by

$$\frac{D\xi^\nu}{D\tau} = \frac{d\xi^\nu}{d\tau} + \Gamma^\nu_{\alpha\sigma}\xi^\sigma\frac{dx^\alpha}{d\tau},$$

and $D^2\xi^\nu/D\tau^2$ is defined as $D(D\xi^\nu/D\tau)/D\tau$.

[3] As a nice side result we get for a Laplacian potential $\nabla^2 V = 0$, that $R^i_{0i0} = 0$, which automatically leads to the Ricci tensor. In coordinate invariant form we have to generalize this result to $R^\mu_{\alpha\mu\beta} = R_{\alpha\beta} = 0$, as a generalization of Laplace's equation.

the Moon, and introducing polar coordinates for ξ^1, ξ^2, ξ^3 by

$$\xi^1 = a\sin\theta\cos\lambda = \xi, \tag{4.39}$$
$$\xi^2 = a\sin\theta\sin\lambda = \eta, \tag{4.40}$$
$$\xi^3 = a\cos\theta = \zeta, \tag{4.41}$$

where, a is the radial distance to the satellite, θ the polar distance, and λ is the longitude of the point under consideration. The components of the tidal

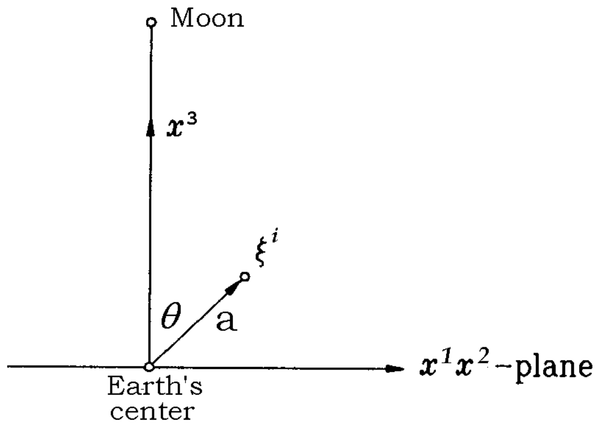

Figure 4.6: Special choice of coordinates for the Moon and Earth.

force are in these new coordinates

$$f^1 = -Gm\frac{\xi}{r^3}. \tag{4.42}$$
$$f^2 = -Gm\frac{\eta}{r^3}. \tag{4.43}$$
$$f^3 = 2\frac{\zeta}{r^3}. \tag{4.44}$$

These components can also be derived form the potential (as a gradient)

$$W = \frac{Gm}{r^3}\Big(-\frac{1}{2}\xi^2 - \frac{1}{2}\eta^2 + \zeta^2\Big). \tag{4.45}$$

This can be rewritten as

$$W = \frac{GM}{r}\Big(\frac{a}{r}\Big)^2 P_2(\cos\theta), \tag{4.46}$$

which is the usual formula for the tidal potential, expressed as a zonal second degree harmonic. The influence of higher degree harmonics are so small, that they can be neglected [34].

Thus, the metric for an ECI frame, including both the potential of the Earth and the tidal potential of the Moon or Sun, is in the weak field limit

$$g_{00} = -\Big(1 + \frac{2V}{c^2} + \frac{W}{c^2}\Big). \quad (4.47)$$

$$g_{0j} = 0. \quad (4.48)$$

$$g_{ij} = \delta_{ij}. \quad (4.49)$$

Thus, the tidal effect from the Moon, on a artificial Earth satellite, introduces a correction to the coordinate time, given by

$$\Delta t_{tidal} = -\int_{path} \frac{W}{2c^2} d\tau. \quad (4.50)$$

Looking at the expression for W, we see that the tidal potential introduces a secular drift rate and a periodic correction. We have from equation (2.23) that $\cos\theta = \sin i \sin(\omega + \nu)$. If the orbital radius a can be approximated with a constant value, then $\omega + \nu \approx n\tau$, where ω is the altitude of perigee, ν the true anomaly and n the mean motion, and the tidal correction to the coordinate time becomes

$$\begin{aligned}\Delta t_{tidal} &= -\frac{Gma^2}{2c^2r^3}\int\Big(-\frac{1}{2} + \frac{3}{2}\cos^2\theta\Big)d\tau \\ &= -\frac{Gma^2}{2c^2r^3}\int\Big(-\frac{1}{2} + \frac{3}{4}\sin^2 i - \frac{3}{4}\sin^2 i\cos(2n\tau)\Big)d\tau \\ &= -\frac{Gma^2}{2c^2r^3}\Big[\Big(-\frac{1}{2} + \frac{3}{4}\sin^2 i\Big)\Delta\tau - \frac{3}{8n}\sin^2 i\sin(2n\tau)\Big], \quad (4.51)\end{aligned}$$

where the trigonometric identity $\sin^2 A = 1/2\cdot(1-\cos 2A)$ has been used. For the Moon $Gm = 4.90\cdot 10^3$ km$^3/s^2$ and $r = 3.8\cdot 10^5$ km. The maximal secular drift rate is found for $i = 0°$ (equatorial orbit). Thus for a GPS satellite the maximal secular drift rate due to the tidal potential is

$$\frac{Gma^2}{4c^2r^3} = 1.7501\cdot 10^{-16} = 15ps/\text{day}. \quad (4.52)$$

The maximal possible amplitude of the periodic correction due to the tidal potential, for a GPS satellite is found to be approximately 1 ps. For the tidal effects of the Sun one finds, that the secular rate drift is 7 ps/day and the periodic correction has an amplitude of 0.5 ps. Thus the effects of the tidal potential of the Sun are roughly one half of those of the Moon. At present time the tidal effects are ignored in the GPS, which makes sense, because a fractional frequency shift of the order 10^{-16} is at the cutting edge of the performance of modern laser cooled atomic clocks. For a low orbit satellite, the tidal effects are even smaller, which can be seen from equation (4.51) and (4.52).

4.5 The Shapiro time delay

In 1967 I. I. Shapiro proposed and, together with some co-workers, worked out some experiments, where they measured the time required for radar signals to travel from the Earth to the inner planets and back again [21]. In this section I will consider the propagation delay to a satellite analyzed in a ECI reference frame. As mentioned before, light travels on a null geodesics, so the propagation equation is given by

$$g_{\alpha\beta}dx^\alpha dx^\beta = g_{00}c^2dt^2 + 2g_{0j}cdtdx^j + g_{ij}dx^idx^j = 0, \tag{4.53}$$

where the coordinate time parameter t is measured with respect to a clock at rest at infinity. Solving for dt gives

$$dt = \frac{1}{c}\frac{1}{-g_{00}}\Big\{g_{0j}dx^j \pm \sqrt{(g_{0i}g_{0j} - g_{ij}g_{00})dx^idx^j}\Big\}. \tag{4.54}$$

Let us first see what happens if there is no gravitational potential. The metric components will then be $g_{ij} = \delta_{ij}$, $g_{0j} = 0$ and $g_{00} = -1$. Substituting into equation (4.54), and integrating along the light path, leads to (choosing the positive sign)

$$\Delta t = \frac{1}{c}\int_{path}\sqrt{\delta_{ij}dx^idx^j} = \frac{1}{c}\int_{path}d\rho = \frac{\rho}{c}, \tag{4.55}$$

where $d\rho$ is the differential distance along the path and ρ is the slant range. The time delay found by equation (4.55) is simply the non-relativistic geometric path delay.

Let us next consider the effect of the gravitational potential. In the weak field limit the metric components are given by [15]

$$g_{00} = -\Big(1 + \frac{2\phi}{c^2}\Big). \tag{4.56}$$

$$g_{0j} = 0. \tag{4.57}$$

$$g_{ij} = \Big(1 - \frac{2\phi}{c^2}\Big)\delta_{ij}. \tag{4.58}$$

Substituting into equation (4.54), choosing the positive sign and integrating along the light path gives

$$\begin{aligned}\Delta t = \frac{1}{c}\int_{path}\sqrt{\frac{g_{ij}}{-g_{00}}dx^idx^j} &= \frac{1}{c^2}\int_{path}\sqrt{\frac{1-2\phi/c^2}{1+2\phi/c^2}\delta_{ij}dx^idx^j} \\ &\approx \frac{1}{c}\int_{path}\Big(1 - \frac{2\phi}{c^2}\Big)d\rho. \end{aligned}\tag{4.59}$$

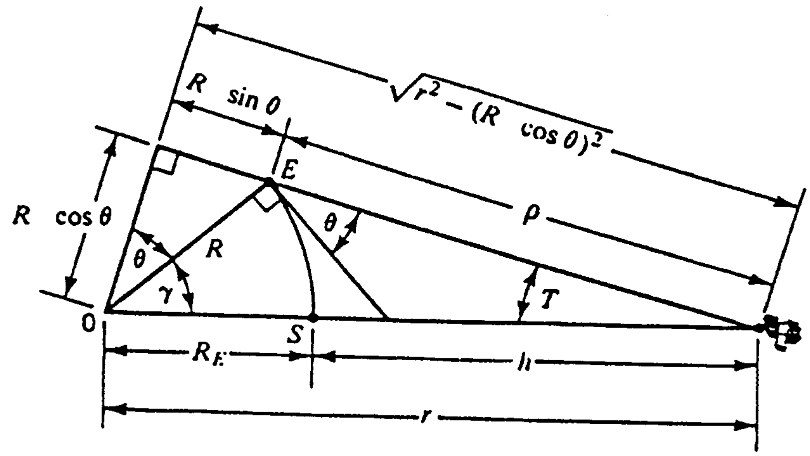

Figure 4.7: Earth satellite geometry, as explained in the text. Adopted from [15].

The Shapiro time delay is so small that quadrupole contributions to the gravitational potential can be neglected. Thus for a straight line path in the radial direction, the time delay is

$$\Delta t = \frac{1}{c}\int_R^r \Big(1 + \frac{2GM}{c^2 r}\Big)dr = \frac{r-R}{c} + \frac{2GM}{c^3}\ln\frac{r}{R}, \tag{4.60}$$

where the second term is called the Shapiro time delay.

Generally the situation is not as simple as described above. The geometry between a satellite and the Earth for the general case is illustrated on figure 4.7, where ρ is the slant range, θ is the elevation angle and r the orbital radius. By inspection of figure 4.7, it is seen that the slant range is related to the elevation angle as

$$\rho = \sqrt{r^2 - (R\cos\theta)^2} - R\sin\theta. \tag{4.61}$$

Thus, the propagation time is given by

$$\begin{aligned}\Delta t &= \frac{1}{c}\int_0^\rho \Big(1 + \frac{2GM}{c^2\sqrt{R^2+\rho^2+2R\rho\sin\theta}}\Big)d\rho \\ &= \frac{\rho}{c} + \frac{2GM}{c^3}\ln\Big(\frac{r+\rho+R\sin\theta}{R(1+\sin\theta)}\Big).\end{aligned} \tag{4.62}$$

Usually the Shapiro time delay is expressed in a way that involves the distances r, R and ρ in a symmetric manner. First note that $R\sin\theta = (r^2 - R^2 - \rho^2)/2\rho$, so the argument of the logarithm is

$$\frac{2r\rho+\rho^2+r^2-R^2}{2R\rho+r^2-R^2-\rho^2} = \frac{(r+\rho)^2-R^2}{r^2-(\rho-R)^2} = \frac{(r+\rho+R)(r+\rho-R)}{(r+\rho-R)(r-\rho+R)}. \tag{4.63}$$

Thus the propagation time delay is

$$\Delta t = \frac{\rho}{c} + \frac{2GM}{c^3} \ln\left(\frac{R + r + \rho}{R + r - \rho}\right). \tag{4.64}$$

If $\rho = r - R$, then equation (4.64) reduces to the time delay for a straight line path (eguation (4.60)). These times are measured with respect to a clock at rest at infinity. In order to use clocks on the Earth's geoid as reference, the scale change given by equation (3.39) must be applied. Thus, with respect to clocks at rest on the Earth's geoid, the propagation delay is:

$$\Delta t'' = \left(1 + \frac{\phi_0}{c^2}\right)\Delta t. \tag{4.65}$$

Inserting equation (4.64) into equation (4.65) and only retaining terms to the order $1/c^3$ gives

$$\Delta t'' = \frac{\rho}{c} + \frac{\rho}{c}\frac{\phi_0}{c^2} + \frac{2GM}{c^3} \ln\left(\frac{R + r + \rho}{R + r - \rho}\right). \tag{4.66}$$

The last two terms in equation (4.66) is the Shapiro time delay as measured by clocks at rest on the Earth's geoid. For a GPS satellite with orbital radius $r = 26562$ km and elevation angle 40°, the Shapiro time delay is approximately -3 ps, with respect to a clock on the geoid. For GPS satellites the two terms (the term containing ϕ_0 and the logarithm term) in the Shapiro time delay tend to cancel each other. For a geostationary satellite with orbital radius $r = 42164$ km, one finds for a signal sent from equator, that the Shapiro time delay is -27 ps with respect to a clock on the geoid. For the CHAMP satellite, with orbital radius 6828 km, the maximum delay is approximately +10 ps and it occurs at the elevation angle $\theta = 0°$. The minimum delay for CHAMP is approximately 1 ps and is found at the elevation angle $\theta = 90°$. Thus for navigational purposes the Shapiro path delay is negligible at present time. One must keep in mind, however, that in the main term ρ/c, ρ is a coordinate distance and further small relativistic corrections are required to convert it into proper distance (proper distance is found by setting $dt = 0$), but spatial curvature effects are however so small that they play no role for navigational purposes[4].

[4] At the level of a few millimeters, spatial curvature effects must be considered [12]. A quick calculation of the proper distance between two points at radius r_1 and r_2 in the weak field limit, gives (needles to say that we can neglect quadrupole contributions to the potential)

$$\int_{r_1}^{r_2} \left(1 + \frac{GM}{c^2 r}\right) dr = r_2 - r_1 + \frac{GM}{c^2} \ln\left(\frac{r_2}{r_1}\right).$$

For a GPS satellite the last term in equation (4.67) is approximately $4.43 \ln(4.2)\text{mm} \approx 6.3$ mm. Thus, those scientists who model systematic errors down to the millimeter level, must carefully specify weather they use standard, isotropic or harmonic coordinates.

4.6 Post Newtonian effects

In this section I will estimate the importance of the post Newtonian effects i. e. the effects that are beyond second order in the metric. The post Newtonian effects are small, so one can neglect the oblateness of the Earth in these calculations. Using the center of energy as origo in our coordinate system, the displacement $\mathbf{D}$ in equation (1.51) can be neglected, and the complete equation of motion becomes (in SI units)

$$\begin{aligned}\ddot{\mathbf{r}} = &-\frac{GM}{r^3}\mathbf{r} - \frac{1}{c^2}\frac{GM}{r^3}\Big\{\Big(\frac{4GM}{r} - (\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})\Big)\mathbf{r} + 4(\mathbf{r}\cdot\dot{\mathbf{r}})\dot{\mathbf{r}}\Big\} \\ &+\frac{1}{c^2}\frac{2GM}{r^3}\Big(\frac{3}{r^2}(\mathbf{r}\cdot\mathbf{J}) - \mathbf{J}\Big)\times\dot{\mathbf{r}} + \frac{1}{c^2}\frac{3GM}{r^3}(\mathbf{r}\times\dot{\mathbf{r}})\times\dot{\mathbf{r}}, \end{aligned} \tag{4.67}$$

where $\mathbf{J}$ is the angular momentum of the Earth per unit mass. The first term in this equation is the usual Newtonian acceleration. The second term is the relativistic correction to the acceleration for the Schwarzschild metric. The third term is known as the Lense Thirring precession, and the last term is called de Sitter precession. It would be interesting to know the post Newtonian correction to the velocity of a satellite, which is used to calculate the clock effect due time dilation.

Following [15], we get upon performing the scalar product of equation (4.67) with the velocity $\dot{\mathbf{r}}$

$$\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} = -\frac{GM}{r^3}\Big\{1 + \frac{1}{c^2}\Big(\frac{4GM}{r} - 3v^2\Big)\Big\}(\mathbf{r}\cdot\dot{\mathbf{r}}). \tag{4.68}$$

The precession terms do not contribute to this expression since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0$. In the first approximation the satellite velocity is given by equation (2.8), which leads to

$$\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} = -\frac{GM}{r^3}\Big\{1 + \frac{GM}{c^2}\Big(\frac{10}{r} - \frac{3}{a}\Big)\Big\}(\mathbf{r}\cdot\dot{\mathbf{r}}). \tag{4.69}$$

Using that $\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}dt = \frac{1}{2}d(v^2)$ and $\mathbf{r}\cdot\dot{\mathbf{r}}dt = \mathbf{r}d\mathbf{r} = rdr$, integration with respect to time gives

$$\frac{1}{2}v^2 = \frac{GM}{r}\Big\{1 + \frac{GM}{c^2}\Big(\frac{5}{r} - \frac{3}{a}\Big)\Big\} + \mathcal{E}, \tag{4.70}$$

where $\mathcal{E}$ is the total energy per unit mass. The semi-major axis is defined by the energy as $\mathcal{E} = -GM/2a$. Thus, the velocity of the satellite to post Newtonian order is give by

$$v^2 = GM\Big\{\frac{2}{r}\Big(1 + \frac{GM}{c^2}\Big(\frac{5}{r} - \frac{3}{a}\Big)\Big) - \frac{1}{a}\Big\}. \tag{4.71}$$

The post Newtonian correction to the square of the velocity is thus given by

$$\frac{G^2M^2}{rc^2}\Big(\frac{5}{r}-\frac{3}{a}\Big). \tag{4.72}$$

a crude estimation of this correction can be done for a circular orbit with $r = a = R$ (equator radius of the Earth), which correspond to NEO satellite. The square of the velocity falls off roughly as $1/r$, so this estimation is a theoretical upper limit

$$\frac{G^2M^2}{rc^2}\Big(\frac{5}{r}-\frac{3}{a}\Big) < 0.17m^2/s^2. \tag{4.73}$$

This would give a correction to the time dilation of the order $v^2/2c^2 = 0.17/(2c^2) \approx 1 \cdot 10^{-19} \approx 0.8 \cdot 10^{-14} s/day$. Fractional frequency shifts of the order 10^{-19} is far beyond the performance of modern atomic clocks, so the post Newtonian correction to the velocity is totally negligible.

Let us next consider the post Newtonian correction to the potential. The post Newtonian potential of a Earth orbiting satellite can be found from equation (1.58):

$$\frac{U}{c^2} = \frac{\phi}{c^2} + \frac{\phi^2}{c^4} + \frac{3\phi v^2}{2c^4} + \frac{v^4}{8c^4}, \tag{4.74}$$

where v is the velocity of the satellite, and ϕ is the Newtonian potential at the satellite's position. The post Newtonian potentials ψ and ζ have been neglected in equation (4.74). The potential has been divided by c^2, because relativistic clock corrections are easily expressed thereby. The post Newtonian vector potential ζ - which is responsible for the Lense Thirring effect - will be treated separately in the next section. The first term in equation (4.74) is the Newtonian potential, whereas the next terms are the post Newtonian corrections to the potential. We can give an estimate of the terms in equation (4.74), by setting $v^2 \approx GM/r$, corresponding to a circular orbit, with radius r. For a NEO satellite , we can set $r \approx R$. Thus the estimates are (neglecting the quadrupole contributions to the potential)

$$\begin{aligned} \text{Newtonian} \qquad &\approx -\frac{GM}{Rc^2} &\approx& \; -6.9 \cdot 10^{-10} \\ \text{Post}-\text{Newtonian} &\approx -\frac{3}{8c^4}\Big(\frac{GM}{R}\Big)^2 &\approx& \; -1.8 \cdot 10^{-19} \end{aligned}$$

From the above estimates we can conclude that the post Newtonian corrections to the potential are negligible.

4.6.1 The Lense Thirring effect

The rotation of the Earth produces a vector potential ζ, which appears in the off diagonal component of the metric c. f. equation (1.47). Thus, for a satellite clock it introduces an increment in coordinate time, given by

$$\Delta t_{\mathrm{LT}} = \frac{1}{c}\int g_{i0}v^i d\tau. \tag{4.75}$$

By equation (1.52), we obtain

$$\Delta t_{\mathrm{LT}} = \frac{2GM}{c^4}\int \frac{\mathbf{r}\times\mathbf{J}}{r^3}\cdot\mathbf{v}d\tau. \tag{4.76}$$

Furthermore, we have that $(\mathbf{r}\times\mathbf{J})\cdot\mathbf{v} = -\mathbf{J}\cdot(\mathbf{r}\times\mathbf{v})$, but $(\mathbf{r}\times\mathbf{v})$ is just the orbital angular momentum per unit mass, given by $rv\cos i$, where i is the inclination of the orbit. Thus we obtain

$$\Delta t_{\mathrm{LT}} = -\frac{2GM}{c^4}\frac{Jv\cos i}{r^3}\Delta\tau. \tag{4.77}$$

This drift rate is rather small, so it will not lead to a big error, if we assume that the orbit is circular, and that the Earth is spherical. In this case $v = \sqrt{GM/r}$ and $J = 2MR^2\omega/5$, and we obtain

$$\Delta t_{\mathrm{LT}} = -\frac{4}{5}\frac{(GM)^{3/2}}{c^4}\left(\frac{R}{r}\right)^2\frac{\omega\cos i}{\sqrt{r}}\Delta\tau. \tag{4.78}$$

The Lense Thirring effect thus gives a constant drift rate. For a GPS satellite $\Delta t_{\mathrm{LT}} = -1.6\cdot 10^{-17}$ s per revolution. The Lense Thirring effect is thus negligible.

4.7 Path delay due to receiver motion

The path delay due to receiver velocity can be a considerable effect, that must be accounted for by the receiver. Let us therefore analyze the signal transmission from a satellite to a moving Earth bound receiver in the ECI reference frame. Following R. A. Nelson [15], we get that the geometric path from a satellite to a receiver on the rotating Earth is given by (see figure 4.8)

$$\mathbf{D} = \mathbf{r}_U(t_R) - \mathbf{r}_S(t_T) = \overrightarrow{\rho} + \mathbf{v}\Delta t, \tag{4.79}$$

where $\mathbf{r}_S(t_T)$ is the satellite position at the time of signal transmission, $\mathbf{r}_U(t_R)$ is the receiver position at the time of signal reception, $\mathbf{v}$ the receiver velocity and $\overrightarrow{\rho}$ is the slant range at the time of signal transmission, given by

$$\overrightarrow{\rho} = \mathbf{r}_U(t_T) - \mathbf{r}_S(t_T). \tag{4.80}$$

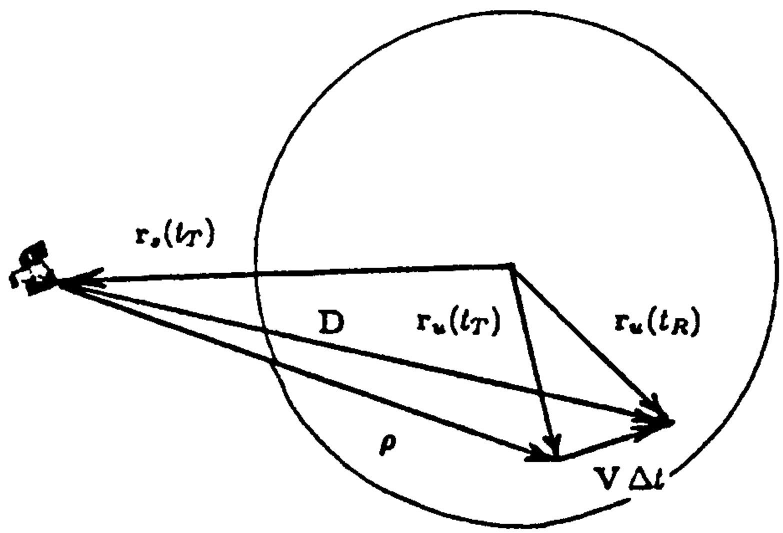

Figure 4.8: Receiver motion on the rotating Earth during time of signal propagation. Adopted from [15].

If we ignore the effect of the gravitational potential on the speed of light in the vicinity of the Earth, Then the signal propagation time is given by

$$\Delta t = t_R - t_T = \frac{D}{c} = \frac{|\mathbf{r}_U(t_R) - \mathbf{r}_S(t_T)|}{c}, \tag{4.81}$$

and the correction for the receiver motion with respect to the ECI reference frame is

$$\Delta t_v = \frac{1}{c^2}\overrightarrow{\rho} \cdot \mathbf{v}. \tag{4.82}$$

For a receiver at rest on the Earth's surface, with $\mathbf{r}_U = \mathbf{R}$, and $\mathbf{v} = \overrightarrow{\omega} \times \mathbf{R}$ we recover the Sagnac effect

$$\Delta t_v = \frac{1}{c^2}\overrightarrow{\omega} \cdot (\mathbf{r}_S \times \mathbf{R}) = \frac{2\omega A}{c^2}, \tag{4.83}$$

where A is the equatorial projection of the triangle consisting of the Earth's center, and the satellite and the receiver as the two vertices at the time of transmission. Thus the correction for receiver motion as analyzed in the ECI reference frame, is equivalent with the Sagnac correction as analyzed in the ECEF reference frame.

In Cartesian coordinates, the slant range at the time of transmission is given by

$$\rho = \sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}, \tag{4.84}$$

where (x,y,z) is the coordinates of the satellite and (X,Y,Z) is the coordinates of the receiver at the time of transmission. Thus, the correction due to the motion of the receiver is

$$\Delta t_v = \frac{1}{c^2}\omega(xY - yX). \tag{4.85}$$

The maximal correction occurs when the slant range is maximum, which is

$$\Delta t_{\max} = \frac{\rho v}{c^2} = \frac{\omega R}{c^2}\sqrt{r^2 - R^2}, \tag{4.86}$$

where r is the radial distance to the satellite. For example, for a GPS satellite with orbital radius 26562 km, we find for a receiver at rest on the equator, that the maximal (Sagnac) effect is 133 ns.

Let us for example estimate the correction for the path delay of a guided missile like the Tomahawk cruise missile, which operates at the speed of approximately 300 m/s in the ECEF frame, and let us for simplicity consider an instant when the cruise missile is flying along the equator in the eastward direction, thus having the inertial velocity 765 m/s. Assume furthermore, that a GPS satellite happens to be at the radial distance $r = 26560$ km above the missile, and the missile is flying 100 m above the ground. Then at this instant the slant range is $\rho = 25782.779$ km. Thus, the path delay is $\Delta t_v \approx 217$ ns, which corresponds to an additional path length of 65 m, that must be accounted for. The cruise missile does not only rely on the GPS, but uses several additional guidance systems.

Another important example is a LEO satellite like CHAMP. CHAMP has a very low eccentricity ($e = 0.004$), so in this example I will set the velocity to be $v \approx \sqrt{GM/r} \approx 7.642$ km/s. For such a satellite the path delay due to receiver motion is considerably higher. Let us assume that the receiver (CHAMP) is crossing the equator, and that a GPS satellite is right above CHAMP at the same altitude as before. Then the slant range is $\rho \approx 25667.334$ km. Thus $\Delta t_v \approx 2.179\mu$s, which corresponds to an additional path length of approximately 654 m, that must be accounted for.

4.8 The Doppler effect

As we have seen sofar, orbiting clocks in the GPS are rate adjusted before launch, so that they apart from a periodic term, are beating coordinate time. It is up to the receiver to correct for the periodic effect due to eccentricity. Therefore one has to be careful when analyzing the Doppler effect for signals transmitted from the GPS satellites, because even though second order

Doppler effects (which are equivalent to time dilation) have been accounted for - for Earth fixed users - one still needs to deal with the first order, longitudinal Doppler effect, which must be accounted for by the receiver.

Let us consider the transmission of signals from a GPS satellite in a ECI reference frame, with position $\mathbf{r}_S$ and velocity $\mathbf{v}_S$. Let the gravitational potential of the satellite be V_S, and let the coordinate time of transmission be t_S. The receiver's position, velocity and gravitational potential is denoted $\mathbf{r}_R$, $\mathbf{v}_R$ and V_R respectively, and let the time the receiver receives the signal be t_R.

We then have according to equation (1.30), that the ratio of the proper frequency f_S of the transmitted signal to the proper frequency f_R of the received signal is given by

$$\frac{f_R}{f_S} = \frac{d\tau_S}{d\tau_R} = \frac{(dt_R/d\tau_R)}{(dt_S/d\tau_S)}\frac{dt_S}{dt_R}. \tag{4.87}$$

The slant range from the transmitter to the receiver is by definition given by

$$\overrightarrow{\rho} = \mathbf{r}_R - \mathbf{r}_S. \tag{4.88}$$

The difference in coordinate times of reception and transmission is

$$t_R - t_S = \frac{\rho}{c}. \tag{4.89}$$

Differentiating both sides of equation (4.89) with respect to t_R, gives

$$1 - \frac{dt_S}{dt_R} = \frac{1}{c}\Big(\frac{\partial\rho}{\partial t_R} + \frac{\partial\rho}{\partial t_S}\frac{dt_S}{dt_R}\Big) = \frac{1}{c}\Big(\mathbf{N}\cdot\mathbf{v}_R - \mathbf{N}\cdot\mathbf{v}_S\frac{dt_S}{dt_R}\Big), \tag{4.90}$$

where $\mathbf{N}$ is a unit vector in the propagation direction of the signal, given by

$$\mathbf{N} = \frac{\overrightarrow{\rho}}{\rho} = \frac{\mathbf{r}_R - \mathbf{r}_S}{|\mathbf{r}_R - \mathbf{r}_S|}. \tag{4.91}$$

Thus, equation (4.90) can be rewritten as

$$\frac{dt_S}{dt_R} = \frac{1 - \mathbf{N}\cdot\mathbf{v}_R/c}{1 - \mathbf{N}\cdot\mathbf{v}_S/c}. \tag{4.92}$$

In the Newton limit the components of the metric tensor is given by $g_{00} = -(1 + 2\phi/c^2)$, $g_{0j} = 0$ and $g_{ij} = \delta_{ij}$, where ϕ is the Newtonian gravitational potential. Thus, we get

$$\frac{dt_R}{d\tau_R} \approx 1 - \frac{V_R}{c^2} + \frac{v_R^2}{2c^2}. \tag{4.93}$$

$$\frac{dt_S}{d\tau_S} \approx 1 - \frac{V_S}{c^2} + \frac{v_S^2}{2c^2}. \tag{4.94}$$

Inserting equations (4.92)-(4.94) into (4.87) leads to

$$f_R = f_S\Big(\frac{1 - V_R/c^2 + v_R^2/2c^2}{1 - V_S/c^2 + v_S^2/2c^2}\Big)\frac{(1 - \mathbf{N}\cdot\mathbf{v}_R/c)}{(1 - \mathbf{N}\cdot\mathbf{v}_S/c)}. \tag{4.95}$$

Since the gravitational field of the Earth is weak, we can expand the denominator in the second factor on the right side of equation (4.95)

$$(1 - V_S/c^2 + v_S^2/2c^2)^{-1} \approx 1 + V_S/c^2 - v_S^2/2c^2 + ... \tag{4.96}$$

Inserting this into equation (4.95), and only retaining terms to the order $1/c^2$ leads to

$$f_R = f_S\Big(1 + \frac{-V_R/c^2 + v_R^2/2 + V_S - v_S^2/2}{c^2}\Big)\frac{(1 - \mathbf{N}\cdot\mathbf{v}_R/c)}{(1 - \mathbf{N}\cdot\mathbf{v}_S/c)}. \tag{4.97}$$

Assuming that the GPS satellite follows a Keplerian orbit, conservation of energy, gives

$$\frac{v_S^2}{2} + V_S = -\frac{GM}{2a} \Rightarrow \frac{v_S^2}{2} = -V_S - \frac{GM}{2a}. \tag{4.98}$$

Thus after insertion of equation (4.98) into (4.97), we get

$$f_R = f_S\Big(1 + \frac{-V_R/c^2 + v_R^2/2 + GM/2a + 2V_S}{c^2}\Big)\frac{(1 - \mathbf{N}\cdot\mathbf{v}_R/c)}{(1 - \mathbf{N}\cdot\mathbf{v}_S/c)}. \tag{4.99}$$

Incorporating the rate adjustment, discussed earlier,

$$f_S = \Big(1 + \frac{3GM}{2a} + \frac{\phi_0}{c^2}\Big)f_0, \tag{4.100}$$

where $f_0 = 10.23$ Mhz is the frequency of the satellite transmission before rate adjustment, we finally find that the received frequency is (here we again only retain terms to the order $1/c^2$)

$$f_R = f_0\Big(1 + \frac{-V_R/c^2 + v_R^2/2 + 2GM/a + \phi_0 + 2V_S}{c^2}\Big)\frac{(1 - \mathbf{N}\cdot\mathbf{v}_R/c)}{(1 - \mathbf{N}\cdot\mathbf{v}_S/c)}. \tag{4.101}$$

For a receiver fixed on the Earth's geoid equation (4.101) reduces to

$$f_R = f_0\Big\{1 + \frac{2GM}{c^2}\Big(\frac{1}{a} - \frac{1}{r}\Big)\Big\}\frac{(1 - \mathbf{N}\cdot\mathbf{v}_R/c)}{(1 - \mathbf{N}\cdot\mathbf{v}_S/c)}. \tag{4.102}$$

The first correction term above gives rise to the eccentricity effect. The first order Doppler shift is of the order 10^{-5} while the eccentricity effect is of the order $e \cdot 10^{-10}$.

4.9 Crosslink ranging

The GPS is going trough a modernization faze at present time. The satellites in the constellation will be replaced by a new type of satellites, maned Block IIR, each carrying a GPS receiver. When the replenishment series of Block IIR satellites are fully operational, it is the intension that they should be able to operate independent of the ground station control for a certain period of days. The satellites shall then determine their position by "listening" to the other satellites in the constellation. This is called "auto navigation".

This calls for a consideration of transferring coordinate time from one satellite clock to another via direct exchange of signals. As we have seen sofar, the clock in the transmitter satellite suffers a rate adjustment plus an eccentricity correction. When the signal arrives at the receiver satellite there is a further rate transformation, and a eccentricity correction to get the time on the receiver clock. Rate adjustment does not introduce any ambiguities. Let us therefore assume that rate adjustment has been done, and denote the coordinate time of rate adjusted satellite clocks by the t^S.

Let us consider a signal being transmitted form a satellite with position $\mathbf{r}_T$ and velocity $\mathbf{v}_T$, with respect to an ECI reference frame, at satellite time t_T^S, and received at a satellite with position $\mathbf{r}_R$ and velocity $\mathbf{v}_R$. The coordinate time of transmission is then, apart from a constant offset, given by (c. f. equation (4.6))

$$t_T = t_T^S + \frac{2\sqrt{GMa_T}}{c^2} e_T \sin E_T, \tag{4.103}$$

where a_T, e_T and E_T is the transmitting satellite's semi-major axis, eccentricity and eccentric anomaly respectively. The coordinate time of propagation must be accounted for: The elapsed coordinate time during propagation is in the first approximation, - neglecting the Shapiro time delay - given by

$$\Delta t = t_R - t_T = \frac{D}{c}. \tag{4.104}$$

The coordinate time of reception is related to the time on the receiver satellite's rate adjusted clock, by

$$t_R^S = t_R - \frac{2\sqrt{GMa_R}}{c^2} e_R \sin E_R. \tag{4.105}$$

Combining equation (4.103)-(4.105), gives

$$\Delta t = t_R^S + \frac{2\sqrt{GMa_R}}{c^2} e_R \sin E_R - t_T^S - \frac{2\sqrt{GMa_T}}{c^2} e_T \sin E_T = \frac{D}{c}. \tag{4.106}$$

Thus, the satellite time of reception is (the time on the receiver satellite's rate adjusted clock)

$$t_R^S = t_T^S + \frac{D}{c} + \frac{2\sqrt{GMa_T}}{c^2} e_T \sin E_T - \frac{2\sqrt{GMa_R}}{c^2} e_R \sin E_R, \tag{4.107}$$

where the distance D must properly account for motion of the receiver satellite during signal propagation, i. e.

$$D = |\mathbf{r}_R(t_T) - \mathbf{r}_T(t_T)| + \frac{1}{c}\mathbf{v}_R \cdot (\mathbf{r}_R(t_T) - \mathbf{r}_T(t_T)), \tag{4.108}$$

where $\mathbf{r}_R(t_T)$ is the position of the receiver satellite at the time of signal transmission, $\mathbf{r}_T(t_T)$ is the position of the transmitting satellite at the time of transmission, and $\mathbf{v}_R$ the velocity of the receiver satellite.

For high precision measurements at the sub-nanosecond level, the Shapiro time delay may also be included.

Crosslink ranging has a lot of benefits, among other there's very little dispersion of the signal, due to atmospheric effects.

Chapter 5

Time scales

Before the invention of atomic clocks, the rotation of the Earth was used to measure time. There are two time scales related to the diurnal rotation of the Earth, these are *siderial time* and *universal time*. These two times are still being used as an angle measure between terrestrial and celestial reference systems. I will not go into a deep discussion of these timescales, but only give the definitions for the sake reference (the reader can consult [1], for more explanations). Two times refer to the *true* vernal equinox:

- LAST (Local Apparent Siderial Time), which is the local hour angle of the true equinox, i. e. the angle between the observers local meridian and the true vernal equinox, corrected for precession and nutation.
- If the angle refers to the Greenwich mean astronomical meridian, then it is called GAST (Greenwich apparent siderial time).

Besides these two times, there are two corresponding times LMST and GMST, which refer to the *mean* vernal equinox, corrected for precession only. The apparent and mean siderial times are related by the *Equation of Equinox* (from [1])

$$GMST - GAST = \Delta\psi \cos\epsilon, \tag{5.1}$$

where $\Delta\psi$ is the nutation in longitude. According to Kepler's second law, the Earth does not revolve around the Sun at a constant angular velocity, therefore a fictious Sun was introduced, which moves with a constant velocity. The hour angle of the fictious Sun is called *Universal Time* UT. The time UT1 is UT corrected for polar motion. UT1 is not uniform, mainly because of a steady increase of the day due to tidal friction, and seasonal periodic variations. A relation between siderial and universal time has been *defined* by the International Astronomical Union (IAU) to be [1]

$$GMST = \text{UT1} + 6^h 41^m 50''.5481 + 8640184''.812866 T_u \\ +0''.093104 T_u^2 - 6''.2 \cdot 10^{-6} T_u^3, \quad (5.2)$$

where T_u is the time interval from the standard epoch J2000, 12^h UT1, counted in Julian centuries of 365.25 days.

5.1 Relativistic time scales

The experimental precision of time measurements are increasing every year. This demands an improvement in the accuracy of the theoretical models including physically meaningful and mathematically rigorous definitions of all the quantities involved in time measurements. Considerable contributions have been done by V. A. Brumberg and S. Kopeikin and is reported in [35]. The theory of relativistic time scales in the Solar system, can be founded on the relativistic theory of reference systems, based on the hierarchy of dynamically non-rotating harmonic reference systems for a four dimensional spacetime of general relativity. A concise treatment of this subject is beyond the scope of this report. I will thus restrict my self to brief summary, for more details the reader can consult [35], [36] and [18].

Atomic time: The fundamental atomic time scale TAI (Temps Atomique International) is based on atomic clocks. TAI is a uniform time scale coinciding with universal time, UT, at midnight January 1, 1958. The fundamental time unit is the SI (Systeme International) second, which is defined as "the duration of 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the Cesium-133 atom" [19]. It is here important to note, that TAI is a time scale defined in a geocentric reference frame with the SI second realized on the rotating geoid [18].

In the Newton limit the proper time is related to coordinate time by

$$\begin{aligned} d\tau^2 &= \left(1 + \frac{2\phi}{c^2}\right) dt^2 - \frac{d\mathbf{x}^2}{c^2} \\ \Rightarrow \frac{d\tau}{dt} &= 1 + \frac{2\phi}{c^2} - \frac{\mathbf{v}^2}{2c^2} \end{aligned} \quad (5.3)$$

The time scale TAI is such that the proper time τ^* of an atomic clock at the Earth's geoid is τ^* =TAI. The proper time τ of an arbitrary Earthbound clock, at rest, is thus related to TAI by

$$\frac{d\tau}{d\tau^*} \approx 1 + \frac{\phi - \phi_{geoid}}{c^2} \approx 1 + \frac{gh}{c^2} \approx 1 + h[\text{km}] \cdot 10^{-13}, \quad (5.4)$$

where the same approximations as in (1.32) and (1.33) has been used. The picture given here is rather simplified, because TAI is defined by some averaging procedure over some 200 different atomic clocks [19], operated by different associations in several countries.

Since TAI is a continuous time scale, it does not maintain synchronization with the solar day (UT). Due to tidal friction, TAI will get more and more ahead of UT. This problem is solved by defining the so called *Universal Coordinated Time* (UTC), which runs at the same rate as TAI, but is periodically incremented by leap seconds.

An other atomic time is the GPS Time (GPST), which runs at the same rate as the atomic clock at the GPS Master control station. Since GPST is not incremented by leap seconds, there is a offset between GPST and TAI.

Further two timescales are introduced: Terrestrial Dynamical Time (TDT), which is the time coordinate in the equation of motion of bodies in the gravitational field of the Earth, according to the general theory of relativity. Equivalently the Barycentric Dynamical Time (TDB), is the dynamical time with respect to the barycenter of the Solar system.

Since TDT is a theoretically uniform time scale based on the relativistic equations of motion, while TAI is a statistically derived (from several atomic clocks) scale, TDT and TAI can not be identical. However within a specified tolerance their difference is a constant [18],

$$\mathrm{TDT} = \mathrm{TAI} + 32.184s. \tag{5.5}$$

The constant offset is due to historical reasons, and is an estimate of the difference between TAI and TU1 at the origin of TAI in 1958. The constant was introduced, so that TDT could maintain continuity with Ephemeris Time (ET), which is the Newtonian equivalent of TDT. The time coordinate t'' introduced in equation (3.39) corresponds to TDT and can be regarded as an ideal form of TAI.

According to the 1976 IAU resolution, the difference in TDT and TDB must only contain periodic terms, never exceeding 2 ms, and is in practice computed by means of a theoretical expression.

Chapter 6

Concluding Scientific Postscript

So where are we now? Well, the quick answer to this question is that relativistic effects play a crucial role for the GPS; it is particularly important to confirm that the basis for synchronization is on a firm conceptual foundation. The main conclusions are given in the beginning of this thesis; so the reader who needs a brief answer to this question can simply flip back to the Summary and introduction section, where the main conclusions are summarized rather succinctly.

In this postscript, I am going to consider a slightly different question, namely: Where will we go from here? First, it is worth mentioning that the work presented in this thesis can easily be modified to cower the Galileo system - which is the European equivalent of the GPS - currently under development by the European Union. At this early development stage it can be a good idea to use the knowledge and experience we have about relativistic corrections in the GPS. Furthermore, when the Galileo system is fully operational it will be possible for receiver manufacturers to integrate both the GPS and the Galileo system in real time positioning.

Second, the GPS is going through a modernization faze at present time. The new replenishment series of Block IIR satellites (as depicted on the front page), will be equipped with new and better atomic clocks than the ones currently in operation, and when "auto navigation" is implemented, it is expected that time transfer at the 100 pico second level should be achievable. This calls for a careful consideration of some of the sub - nanosecond corrections treated in this thesis.

In addition to the civilian and scientific usage of the GPS, the military applications are also of critical importance. These include - among others - navigation, tracking, rescue, targeting and guidance. In modern warfare we have seen an increasing usage of so-called intelligent weapons, and we will probably see more of these weapon systems in the future. An example

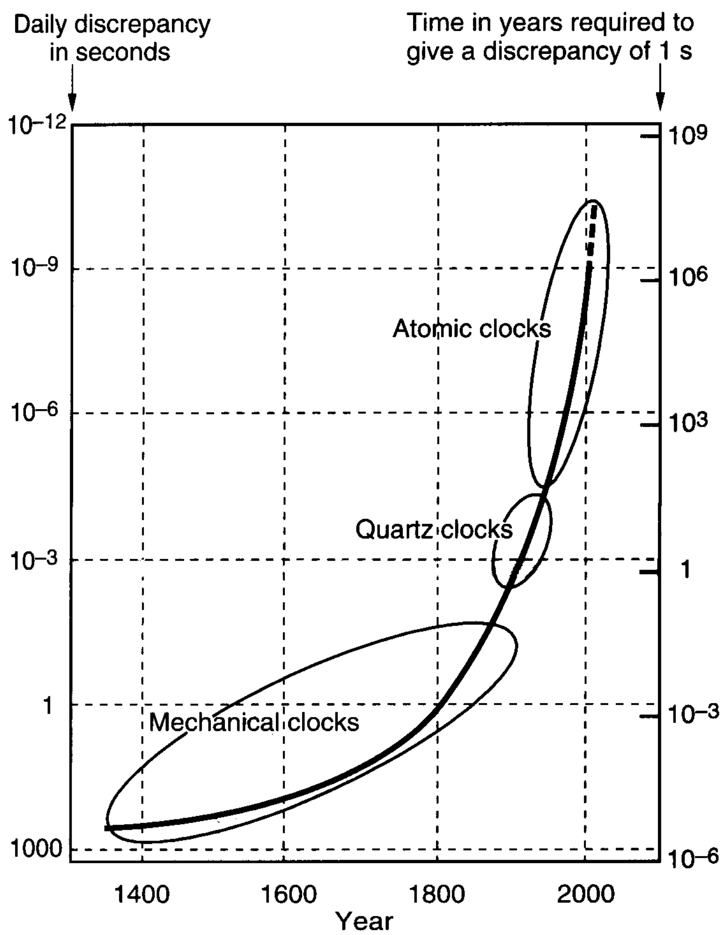

Figure 6.1: Improvement in the quality of artificial clocks. Adopted from [19].

of an intelligent weapon, which relies heavily on GPS for their precision and lethality is the cruise missile. As we previously have seen the relativistic path delay due to receiver motion is of critical importance for this weapon system. The cruise missile is a rather slow missile, essentially a sophisticated airplane. For other missile systems relativistic effects may be even more important.

On figure 6.1 the historical development of clocks is depicted. We see that the invention of the atomic clock has pushed the limit of the science of timekeeping further than any other clock system. Now fractional frequency stabilities on the 10^{-16} level can be obtained with the most sophisticated laser cooled clocks. Plans have been made to put these laser cooled clocks in orbit around the Earth, on the International Space Station. This will open up for further tests of the theory of relativity, possibly leading to further improvements of GPS, and aid in the development of the Galileo system. There is - however - a limit on how stable and accurate atomic clocks can be, which is determined by the lifetime of the atomic levels. Fractional frequency stabilities on the level of 10^{-19} is highly unlikely, but if this level of accuracy can be obtained, gravitational wave detection - using clocks - should (theoretically) be achievable.

Appendix A

Spherical coordinates

The relation between cartesian coordinates (x, y, z) and spherical coordinates (r, θ, φ) is given by

$$\begin{aligned} x &= r\cos\varphi\sin\theta & \text{(A.1)} \\ y &= r\sin\varphi\sin\theta & \text{(A.2)} \\ z &= r\cos\theta & \text{(A.3)} \end{aligned}$$

Where $r \in [0, \infty[$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi[$. Thus the position vector is

$$\mathbf{r} = \begin{bmatrix} r\cos\varphi\sin\theta \\ r\sin\varphi\sin\theta \\ r\cos\theta \end{bmatrix} \quad \text{(A.4)}$$

Hence the unit vectors are

$$\hat{\mathbf{r}} = \begin{bmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{bmatrix}, \qquad \hat{\varphi} = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{bmatrix}, \qquad \hat{\theta} = \begin{bmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ -\sin\theta \end{bmatrix} \quad \text{(A.5)}$$

The time derivative of the position vector is

$$\begin{aligned} \dot{\mathbf{r}} &= \begin{bmatrix} \cos\varphi\sin\theta\dot{r} - r\sin\varphi\sin\theta\dot{\varphi} + r\cos\varphi\cos\theta\dot{\theta} \\ \sin\varphi\sin\theta\dot{r} + r\cos\varphi\sin\theta\dot{\varphi} + r\sin\varphi\cos\theta\dot{\theta} \\ \cos\theta\dot{r} - r\sin\theta\dot{\theta} \end{bmatrix} \\ &= \dot{r}\hat{\mathbf{r}} + r\sin\theta\dot{\varphi}\hat{\varphi} + r\dot{\theta}\hat{\theta} & \text{(A.6)} \end{aligned}$$

The velocity is thus

$$v = |\dot{\mathbf{r}}| = \sqrt{\frac{dr^2 + r^2(\sin^2\theta d\varphi^2 + d\theta^2)}{dt^2}} \quad \text{(A.7)}$$

Appendix B

Calculations

We can write

$$\frac{dM}{r^3} = \frac{1}{r^3}\left(\frac{d\nu}{dM}\right)^{-1} d\nu \tag{B.1}$$

By definition we have

$$M = n(t - t_0) \tag{B.2}$$

so

$$dM = ndt \tag{B.3}$$

Therefore

$$\frac{d\nu}{dM} = \frac{1}{n}\frac{d\nu}{dt} = \frac{\dot{\nu}}{n} \tag{B.4}$$

Inserting equation (B.4) into (B.1), leads to

$$\frac{dM}{r^3} = \frac{nd\nu}{r^3\dot{\nu}} = \frac{nd\nu}{r(r^2\dot{\nu})} = \frac{nd\nu}{rh} \tag{B.5}$$

Where h is the magnitude of the orbital angular momentum. Now using the equation for an ellipse

$$r = \frac{a(1-e^2)}{(1+e\cos\nu)} \tag{B.6}$$

we get

$$\frac{dM}{r^3} = \frac{n}{ah(1-e^2)}(1+e\cos\nu)d\nu \tag{B.7}$$

But

$$\frac{n}{h} = \frac{1}{a^2\sqrt{1-e^2}} \tag{B.8}$$

so

$$\frac{dM}{r^3} = \frac{1}{a^3(1-e^2)^{3/2}}(1+e\cos\nu)d\nu \tag{B.9}$$

Integrating from 0 to 2π gives

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 dM = \frac{1}{(1-e^2)^{3/2}}\frac{1}{2\pi}\int_0^{2\pi}(1+e\cos\nu)d\nu = (1-e^2)^{-3/2} \quad \text{(B.10)}$$

And since $\cos 2\nu$ and $\sin 2\nu$ are ortogonal to both 1 and $\cos\nu$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 \cos(2\nu)dM = \frac{1}{(1-e^2)^{3/2}}\int_0^{2\pi}\cos(2\nu)(1+e\cos\nu)d\nu = 0$$

And by the same argument

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 \sin(2\nu)dM = 0$$

Appendix C

Tensors

In this appendix, I will define a tensor, and list some tensors, which play an important role in the general theory of relativity. This appendix is not thought of as an introduction to tensor analysis, only the defining relationships are stated. I will thus only introduce the reader to, what is absolutely necessary in order to understand the main part of the report. For a more comprehensive treatment I will refer to [21], or some modern textbook on Riemannian geometry.

For simplicity I will restrict my self to tensors with two indices in the defining relations. Let us first look at the transformation rule of a contravariant vector, which can be thought of as a tensor of first order. Under a coordinate transformation $x^\mu \to x'^\mu$ a contravariant vector V^μ is defined by the transformation rule

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu. \tag{C.1}$$

Similarly a covariant vector V_μ is defined by the transformation

$$V'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu. \tag{C.2}$$

The generalizations to tensors is obvious. Thus contravariant, covariant, and mixed tensors are defined by the transformations

$$T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\kappa}\frac{\partial x'^\nu}{\partial x^\lambda} T^{\kappa\lambda} \tag{C.3}$$

$$T'_{\mu\nu} = \frac{\partial x^\kappa}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu} T_{\kappa\lambda} \tag{C.4}$$

$$T'^\mu_\nu = \frac{\partial x'^\mu}{\partial x'^\kappa}\frac{\partial x^\lambda}{\partial x'^\nu} T^\kappa_\lambda \tag{C.5}$$

This way of defining tensors, is what a mathematician will call “clumsy”, because it is not clear from these transformation laws, that a tensor is an autonomous geometrical object independent of the coordinates. I have chosen this way of doing it, first because the transformation laws provide an easy way of determining whether we are dealing with a tensor or not, and secondly because a modern mathematical approach will take up to much space. I must stress that this way of defining tensors leads to **no mathematical ambiguities**, even though it lags the formalistic beauty of modern geometry. In the following I will list some important tensors, which are used - but not defined - in the main text.

C.0.1 Definition of the Curvature-tensor

The definition of the Riemann-Christoffel curvature tensor reads:

$$R^{\mu}{}_{\nu\alpha\beta} = \frac{\partial \Gamma^{\mu}_{\nu\beta}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\mu}_{\nu\alpha}}{\partial x^{\beta}} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha}. \tag{C.6}$$

The covariant form of the curvature tensor is

$$R_{\mu\nu\alpha\beta} = g_{\mu\sigma} R^{\sigma}_{\nu\alpha\beta}. \tag{C.7}$$

C.0.2 Definition of the Ricci-tensor

The Ricci-tensor is defined as

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}. \tag{C.8}$$

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