From CIS to CTS

- **We must transform from Conventional Inertial System to Conventional Terrestrial System using siderial time, θ:**
- **Rotation Matrix**

$$R_3(-\theta)\vec{u} = \begin{Bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

From q-system to CIS

- **3 rotations. R_i with integer i subscript is rotation about i-axis. R_{xu} is rotation from u to x.**

$$R_{qx} = R_3(\omega)R_1(i)R_3(\Omega), \quad R_{qu} = R_{qx}R_{xu}$$

$$\mathbf{R}_{xq} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega, & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega, & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega, & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega, & -\cos\Omega\sin i \\ \sin i\sin\omega, & \sin i\cos\omega, & \cos i \end{bmatrix}$$

Elliptic orbit

$$\ddot{\vec{x}} = -k\vec{x}/r^3 \quad \vec{x}/r \ \text{ is unit - vector}$$

- **We use spherical coordinates r,λ in (q_1,q_2)-plane**

$$x = r\cos\lambda,$$

$$y = r\sin\lambda,$$

$$\dot{x} = \dot{r}\cos\lambda - r\dot{\lambda}\sin\lambda,$$

$$\dot{y} = \dot{r}\sin\lambda + r\dot{\lambda}\cos\lambda,$$

$$\ddot{x} = \ddot{r}\cos\lambda - 2\dot{r}\dot{\lambda}\sin\lambda - r\ddot{\lambda}\sin\lambda - r(\dot{\lambda})^2\cos\lambda = -\mu\cos\lambda/r^2,$$

$$\ddot{y} = \ddot{r}\sin\lambda + 2\dot{r}\dot{\lambda}\cos\lambda + r\ddot{\lambda}\cos\lambda - r(\dot{\lambda})^2\sin\lambda = -\mu\sin\lambda/r^2.$$

Angular momentum

- **λ is arbitrary := 0 !**

$$\ddot{r} - r(\dot{\lambda})^2 = \mu / r^2$$

$$r\ddot{\lambda} + 2\dot{r}\dot{\lambda} = 0, \quad \text{multiplying with r:}$$

$$r^2\dot{\lambda} = h \qquad \text{angular momentum conserved!}$$

$$\lambda = \tan^{-1}(y/x), \text{ so}$$

$$\dot{\lambda} = \frac{1}{1 + y^2/x^2} \cdot \frac{x\dot{y} - y\dot{x}}{x^2} \quad \text{and } h = x\dot{y} - y\dot{x}$$

Integration

- **With u=1/r**

$$\frac{du}{dr} = -\frac{1}{r^2}.$$

$$\frac{dt}{d\lambda} = \frac{r^2}{h},$$

$$\frac{du}{d\lambda} = \frac{du}{dr} \cdot \frac{dr}{dt} \cdot \frac{dt}{d\lambda} = -\frac{1}{r^2}\dot{r}\frac{r^2}{h} = -\frac{\dot{r}}{h}.$$

$$\frac{d^2u}{d\lambda^2} = \frac{d}{dt}\left(-\frac{\dot{r}}{h}\right)\frac{dt}{d\lambda} = -\frac{\ddot{r}}{h}\frac{r^2}{h} = -\frac{\ddot{r}}{u^2h^2}$$

$$\ddot{r} = -h^2u^2\frac{d^2u}{d\lambda^2}.$$

Integration

Use the result in $\ddot{r} - r(\dot{\lambda})^2$, so

$$-h^2u^2\frac{d^2u}{d\lambda^2} = \frac{1}{u}h^2u^4 = -\mu / r^2 \Rightarrow$$

$$\frac{d^2u}{d\lambda^2} + u = \mu / h^2$$

$$\frac{1}{r} = u = A\cos(\lambda - \lambda_0) + \mu / h^2$$

Ellipse as solution

- **If ellipse with center in (0,0)**

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}=1, \quad \text{and we use spherical coordinates (f, r)}$$

$$\xi = ae + e\cos(f)$$

$$\mu = \quad r\sin(f), \qquad \text{with } b^2 = a^2(1-e^2),$$

$$\text{equation with polynomium of 2.-degree}$$

$$\mathrm{r} = \frac{\mathrm{a(1\text{-}e^{2)}}}{(1-e\cos(f))} \quad \text{or} \quad \frac{1}{\mathrm{r}} = \frac{1}{a(1-e^2)} + \frac{e\cos(f)}{a(1-e^2)}$$

Expressed in orbital plane

$$f = \lambda - \lambda_0, \ A = \frac{e}{a(1-e^2)}, \quad h = \sqrt{\mu a(1-e^2)}$$

expresed using excentric anomaly, E :

$$q_1 = \xi - ae = a(\cos E - e)$$

$$q_2 = \eta = a\sqrt{1-e^2}\,\sin E,$$

$$r = \sqrt{q_1^2 + q_2^2} = a(1 - e\cos E)$$

Parameter change

$\dot{\lambda} = \dot{f}$ then $r^2 \dot{f} = h$, and we substitute f with E:

$$\frac{dr}{df} = -r^2 \frac{d(1/r)}{df} = \frac{r^2 e}{a(1-e^2)} \sin f$$

$$dr = \frac{r e q_2}{a(1-e^2)} df, \quad \text{then} \quad r = \sqrt{q_1^2 + q_2^2}$$

$$\frac{dr}{dE} = (2q_1 a(-\sin E) + 2q_2 a\sqrt{1-e^2} \cos E)/(2r) =$$

$eq_2 / \sqrt{1-e^2}$ (after a long derivation)

Further substitution

$$a^2(1-e\cos E)^2 \cdot \frac{a(1-e^2)}{req_2} \cdot \frac{e}{\sqrt{1-e^2)}} q_2 \frac{dE}{dt} =$$

$$\sqrt{\mu a(1-e^2)} \quad \Rightarrow$$

$$(1-e\cos E) = M = n(t-t_0),$$

with t time for passage of perigaeum and

$$n = \mu^{1/2} a^{-3/2}$$

Transformation to CIS

$$\mathbf{x} = \mathbf{R}_{xq}\{\Omega,\ i,\ \omega\}\,\mathbf{q}\{a,\ e,\ M\},$$

$$\dot{\mathbf{x}} = \mathbf{R}_{xq}\{\Omega,\ i,\ \omega\}\,\dot{\mathbf{q}}\{a,\ e,\ M\},$$

$$\mathbf{q} = \begin{bmatrix} a(\cos E - e) \\ a\sqrt{1 - e^2}\sin E \\ 0 \end{bmatrix} = \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix};$$

$$\dot{\mathbf{q}} = \begin{bmatrix} -\sin E \\ \sqrt{1 - e^2}\cos E \\ 0 \end{bmatrix} \frac{na}{1 - e\cos E} = \begin{bmatrix} -\sin f \\ e + \cos f \\ 0 \end{bmatrix} \frac{na}{\sqrt{1 - e^2}}.$$

Velocity

$$
\begin{aligned}
v^2 &= \dot{q}_1^2 + \dot{q}_2^2 \\
&= \frac{n^2 a^2}{(1 - e^2)} (\sin^2 f + e^2 + 2e \cos f + \cos^2 f) \\
&= \frac{\mu}{a(1 - e^2)} [(2 + 2e \cos f) - (1 - e^2)] \\
&= \mu \left(\frac{2}{r} - \frac{1}{a} \right).
\end{aligned}
$$

- $$T - V = \frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a}$$

From orbital plane to CIS

$$\vec{h} = (h_1, h_2, h_3) \quad \text{orthogonal to } (q_1, q_2, 0)$$

$$\Omega = \tan^{-1}(h_1 / (-h_2))$$

$$i = \tan^{-1}((h_1^2 + h_2^2)^{1/2} / h_3), \quad \text{we put}$$

$$\vec{p} = R_1(i) R_2(\Omega) \vec{x}, \text{ then}$$

$$\omega + f = \tan^{-1}(p_2 / p_1) \rightarrow \omega \text{ known if we can find f.}$$

- .

Determination of f

$$\dot{r} = (v^2 - h^2 / r^2)^{1/2}$$

$$a = \mu r / (2\mu - r v^2), \quad e = (1 - h^2 / (\mu a))^{1/2}$$

$$\cos E = (a - r)/(ae) \quad (\text{from } r = \sqrt{q_1^2 + q_2^2})$$

$$\sin E = r \cdot \dot{r} / (e(\mu a))^{1/2} \Rightarrow$$

$$f = \tan^{-1}(q_2 / q_1) = \tan^{-1}(\sqrt{1 - e^2} \sin E / (\cos E - e))$$

- .

General equations of motion (Kaula 3.2)I2.1a

$(x_i, \dot{x}_i)$ coordinates in Tangent-space $T_* R^3$ (6-dim)

$(\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{M}, \omega, \Omega)$ also coordinates in same space.

Express equations of motion in the new variables:

Maybe the orbits will be straight lines!

We use chain-rule to change variables!

Change of variables

$$\frac{d}{dt}x_i = \dot{x}_i \qquad \frac{d}{dt}\dot{x}_i = \frac{\partial V}{\partial x_i}$$

s_k any of the new variables:

$$(*) \quad \sum_k \frac{\partial x_i}{\partial s_k}\frac{ds_k}{dt} = \dot{x}_i \qquad (**) \sum_k \frac{\partial \dot{x}_i}{\partial s_k}\frac{ds_k}{dt} = \frac{\partial V}{\partial x_i}$$

(*) multiplies with $\frac{\partial x_i}{\partial s_k}$ (**) with $-\frac{\partial \dot{x}_i}{\partial s_k}$

and are added pair by pair

- $$-\frac{\partial \dot{x}_i}{\partial s_k}\sum_k \frac{\partial x_i}{\partial s_k}\frac{ds_k}{dt} + \frac{\partial x_i}{\partial s_k}\sum_k \frac{\partial \dot{x}_i}{\partial s_k}\frac{ds_k}{dt} = -\frac{\partial \dot{x}_i}{\partial s_k}\dot{x}_i + \frac{\partial x_i}{\partial s_k}\frac{\partial V}{\partial x_i}$$

Kaula (3.38)

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial F}{\partial M},$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e}\frac{\partial F}{\partial M} - \frac{(1-e^2)^{1/2}}{na^2 e}\cdot\frac{\partial F}{\partial \omega},$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial F}{\partial i} + \frac{(1-e^2)^{1/2}}{na^2 e}\cdot\frac{\partial F}{\partial e},$$

$$\frac{di}{dt} = \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial F}{\partial \omega} - \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial F}{\partial \Omega}, \qquad (3.38)$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial F}{\partial i}, \qquad \Leftarrow$$

$$\frac{dM}{dt} = -\frac{1-e^2}{na^2 e}\frac{\partial F}{\partial e} - \frac{2}{na}\frac{\partial F}{\partial a}.$$

Force Function

- **We take the zero term out:**

$$F = \frac{\mu}{r} + R - T = \frac{\mu}{2a} + R$$

$$\text{derivatives of constant term} = \text{zero, F} \rightarrow \text{R,}$$

$$\text{except}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e} \cdot \frac{\partial R}{\partial e} - \frac{2}{na} \cdot \frac{\partial R}{\partial a}$$

Conversion of spherical harmonics (Kaula, 3.3)I2.2a

- **We want to express the terms in the expansion in Kepler variables:**

$$V(r,\overline{\varphi},\lambda)=$$

$$\frac{GM}{r}\sum_{n=0}^{\infty}\left(\frac{a_e}{r}\right)^n\sum_{m=0}^{n}P_{nm}(\sin\overline{\varphi})\left[C_{nm}\cos m\lambda+S_{nm}\sin m\lambda\right]$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{GM}{r}\left(\frac{a_e}{r}\right)^n P_{nm}(\sin\overline{\varphi})\left[C_{nm}\cos m\lambda+S_{nm}\sin m\lambda\right]$$

- .

Kaula 3.72, 3.73.

$$V_{lm} = \frac{\mu a_e^l}{a^{l+1}} \sum_{p=0}^{l} F_{lmp}(i) \sum_{q=-\infty}^{\infty} G_{lpq}(e) S_{lmpq}(\omega, M, \Omega, \theta), \tag{3.70}$$

where

$$S_{lmpq} = \begin{bmatrix} C_{lm} \\ -S_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \cos[(l-2p)\omega + (l-2p+q)M + m(\Omega-\theta)] + \begin{bmatrix} S_{lm} \\ C_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \sin[(l-2p)\omega + (l-2p+q)M + m(\Omega-\theta)]. \tag{3.71}$$

Kaula 3.74.

$$\frac{da}{dt} = 0,$$

$$\frac{de}{dt} = 0,$$

$$\frac{d\omega}{dt} = \frac{\mu C_{20} a_e^2}{n(1-e^2)^{1/2} a^5}\left[-\cot i \frac{\partial F_{201}}{\partial i} G_{210} + \frac{(1-e^2)}{e} F_{201} \frac{\partial G_{210}}{\partial e}\right]$$

$$= \frac{3nC_{20}a_e^2}{4(1-e^2)^2 a^2}[1 - 5\cos^2 i],$$

$$\frac{di}{dt} = 0, \qquad (3$$

$$\frac{d\Omega}{dt} = \frac{\mu C_{20} a_e^2 G_{210}}{n(1-e^2)^{1/2} a^5 \sin i} \cdot \frac{\partial F_{201}}{\partial i}$$

$$= \frac{3nC_{20}a_e^2}{2(1-e^2)^2 a^2} \cos i,$$

$$\frac{dM}{dt} = n + \frac{\mu C_{20} a_e^2 F_{201}}{na^5}\left[-\frac{1-e^2}{e} \cdot \frac{\partial G_{210}}{\partial e} + 6G_{210}\right]$$

$$= n - \frac{3nC_{20}a_e^2}{4(1-e^2)^{3/2} a^2}(3\cos^2 i - 1).$$

Kaula 3.75.

With C_{20}=-0.00010827, e=0.001, a=1.2a_e

$$\frac{d\omega}{dt} \approx +3.55(5\cos^2 i - 1) \text{ degrees/day},$$

$$\frac{d\Omega}{dt} \approx -6.70 \cos i \text{ degrees/day},$$

$$\frac{dM}{dt} \approx 14.37 + 0.0093(3\cos^2 i - 1) \text{ revolutions/day}.$$

Applications

- **Orbit with repeating ground track**
- **Orbit which gives resonance with specific term(s)**
- **Orbit which is sun-syncroneous**
- **Orbit which enables close ”encounter” with an object, such as the poles.**

Sol-synkron bane

Så må vi have:

$$\frac{d\Omega}{dt} = 0^{o}.9863 / \text{dag.} \qquad \text{Så :}$$

$$\frac{d\Omega}{dt} = C_{20}\,\frac{3na_e^2}{2a^2(1-e^2)^2}, \;\text{med}\; n = \sqrt{\frac{GM}{a^3}},$$

$$GM = 3.986\times 10^5\, km^3 / s^2, a = 63780\acute{}\,\text{km}, C_{20} = -1.08\times 10^{-3}$$

$$\frac{d\Omega}{dt} = -\frac{2.0598\times 10^{14}\times \cos(i)}{\sqrt{a^7}(1-e^2)^2}[\text{grader/dag}]$$

Geostationær

$$n = 2\pi / 86160$$

$$a = \left(\frac{GM}{n^2} \right)^{\frac{1}{3}}$$