Lecture 2.2. **Global Gravity field-modeling**.

T/M Section 6.6

Before 1990 Earth's surface not known precisely: now known from radar altimetri, InSAR and GPS

Before 1990: Problem formulated as Boundary-value problem for elliptic partial differential-equation, boundary S, for volume v.

$$\Delta W = 2\omega^2 = \sum_{i=1}^{3} \frac{\partial^2 W}{\partial X_i^{\,2}}$$

Solution, if gravity vector component $\frac{\partial W}{\partial n}\Big|_s$ **vertical to surface, S, is known.**

Gravity field modeling.

Anomalous gravity field:

T=W-U,

Observations are linear functionals:

$$\delta g \approx -\frac{\partial T}{\partial r}\bigg|_P \; or \; -\frac{\partial T}{\partial h}\bigg|_P \; (better\, approximation)$$

$$\Delta g \approx -\frac{\partial T}{\partial r}\bigg|_P + \frac{T}{\gamma}\left(\frac{\partial \gamma}{\partial h}\right) \approx -\frac{\partial T}{\partial r}\bigg|_P - \frac{T(P)}{r}$$

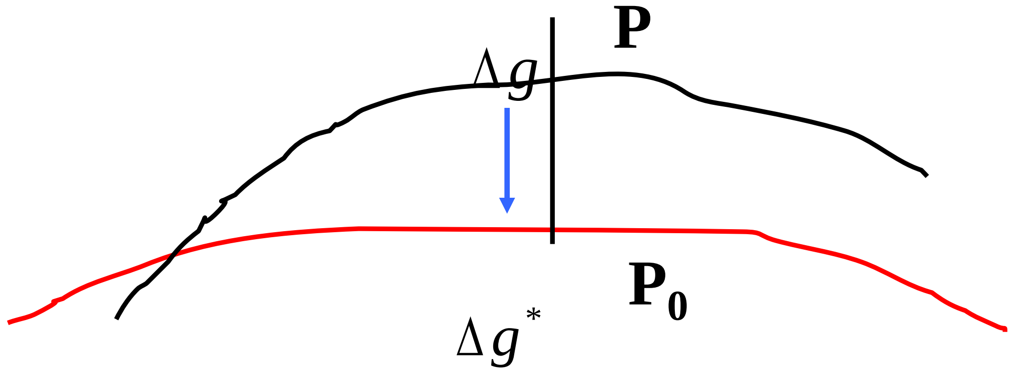
$$\zeta = \frac{T(P)}{\gamma(Q)}$$

Gravity field modeling on Sphere or Ellipsoid.

If Earth may be regarded as spherical or ellipsoidal, then Laplace-equation may be solved using different methods.

Data must be moved from Earth's surface to the Sphere or Ellipsoid and

Masses between surface and Sphere/Ellipsoid must be removed or data corresponding to harmonic function outside sphere/ellipsoid must be determined.



Coefficient-comparison.

Example: polynomium:

$$p(t) = a_0 + a_1 t + a_2 t^2$$

$$p'(t) = \quad a_1 + 2a_2 t$$

If we know that a_0=0, then p(t) can be found !

For anomalous potential are 4 first coefficients = 0, because we know GM and Earth-center.

Gravity field modeling, Sphere., p. 273.

$$T(r,\varphi,\lambda) = \sum_{i=2}^{\infty} \frac{GM}{r}\left(\frac{R}{r}\right)^{i} \sum_{j=-i}^{i} \overline{P}_{ij}(\sin\varphi) \cdot \begin{Bmatrix} \cos j\lambda \\ \sin|j|\lambda \end{Bmatrix} \Delta\overline{C}_{ij}$$

a put equal to R (mean radius), $r = R + h, \quad \overline{\varphi} = \varphi$

then

$$\Delta g(R,\varphi,\lambda) = \sum_{i=2}^{\infty} \frac{GM}{R}\frac{i-1}{R} \sum_{j=-i}^{i} \overline{P}_{ij}(\sin\varphi) \cdot \begin{Bmatrix} \cos j\lambda \\ \sin|j|\lambda \end{Bmatrix} \Delta\overline{C}_{ij}$$

$$= \sum_{i=2}^{\infty} \frac{GM}{R} \sum_{j=-i}^{i} \overline{P}_{ij}(\sin\varphi) \cdot \begin{Bmatrix} \cos j\lambda \\ \sin|j|\lambda \end{Bmatrix} H_{ij}$$

$$\Rightarrow \Delta\overline{C}_{ij} = H_{ij}\frac{R}{i-1}$$

Gravity field-modeleing.

Spherical harmonic analysis gives H_{ij} and thereby $\overline{C}_{ij}$

Normalized Legendre-functions used, so that we get orthonormal-system Y_{ij} in space of square-integrable functions $L_2(S)$

$$\frac{1}{4\pi}\iint_S \overline{P}_{ij}(\sin\varphi)^2 \begin{Bmatrix} \cos^2 j\lambda \\ \sin^2 j\lambda \end{Bmatrix} \cos\varphi d\varphi d\lambda = 1$$

$$Y_{ij}(\varphi,\lambda) = \overline{P}_{ij}(\sin\varphi)\begin{Bmatrix} \cos j\lambda \\ \sin|j|\lambda \end{Bmatrix} \begin{cases} j \geq 0 \\ j < 0 \end{cases}$$

Gravity field modeling., p. 273.

We use that the coefficients in the development in the space may be determined by calculating the inner product (integral over sphere) of basis-functions with the functions:

$$\frac{GM}{R}\Delta\overline{C}_{ij} = \frac{1}{4\pi}\iint_S T(\varphi',\lambda',R)\cdot Y_{ij}(\varphi',\lambda')dS$$

$$(i-1)\frac{GM}{R^2}\Delta\overline{C}_{ij} = \frac{1}{4\pi}\iint_S \Delta g(\varphi',\lambda',R)\cdot Y_{ij}(\varphi',\lambda')dS$$

Gravity field modeling, Ellipsoid. (EGM2008 !)

$$T(r,\varphi,\lambda) = \sum_{i=2}^{\infty} \frac{GM}{r}\left(\frac{a}{r}\right)^{i} \sum_{j=-i}^{i} \overline{P}_{ij}(\sin\varphi) \cdot \begin{Bmatrix} \cos j\lambda \\ \sin|j|\lambda \end{Bmatrix} \Delta\overline{C}_{ij}$$

Then integration over ellipsoid:

$$\Delta\overline{C}_{ij} = \frac{1}{4\pi a\gamma} \sum_{n=0}^{N-1} r_n^e \sum_{k=0}^{s'} \frac{L_{ijk}}{\overline{S}_{n-2k,|j|}(b/E)} \frac{\overline{IP}^{\,n}_{n-2k,|j|}}{(i-2k-1)q^{i}_{i-2k}}$$

$$\sum_{n=0}^{2N-1} \overline{\Delta\overline{g}}^{\,e}_{nm} \begin{Bmatrix} IC \\ IS \end{Bmatrix}^{n}_{j} \quad \begin{matrix} j \geq 0 \\ j < 0 \end{matrix}$$

$$\overline{IP}^{\,n}_{i,|j|} = \int_{\delta_n}^{\delta_{n+1}} \overline{P}_{i|j|}(\cos\delta)\sin\delta\, d\delta$$

$$\begin{Bmatrix} IC \\ IS \end{Bmatrix}^{n}_{j} = \int_{\lambda_m}^{\lambda_{m+1}} \begin{Bmatrix} \cos m\lambda \\ \sin|m|\lambda \end{Bmatrix} d\lambda$$

Gravity field –modeling - sphere., p.273

We use the development for T and integrate over sphere:

$$T(\varphi,\lambda,r)=\sum_{i=2}^{\infty}\frac{GM}{r}\left(\frac{R}{r}\right)^{i}\sum_{j=-i}^{i}Y_{ij}(\varphi,\lambda)\frac{i-1}{i-1}\left(\frac{R^2}{R^2}\right)\Delta\overline{C}_{ij}$$

$$=\sum_{i=2}^{\infty}\frac{1}{r}\left(\frac{R}{r}\right)^{i}\sum_{j=-i}^{i}Y_{ij}(\varphi,\lambda)\frac{R^2}{i-1}\frac{1}{4\pi}\iint\Delta g(\varphi',\lambda',R)Y_{ij}(\varphi',\lambda')dS$$

$$=\frac{R}{4\pi}\iint\left[\Delta g(\varphi',\lambda',R)\sum_{i=2}^{\infty}\frac{1}{i-1}\left(\frac{R}{r}\right)^{i+1}\sum_{j=-i}^{i}Y_{ij}(\varphi,\lambda)Y_{ij}(\varphi',\lambda')\right]dS$$

Stokes formula, p. 284.

We get:

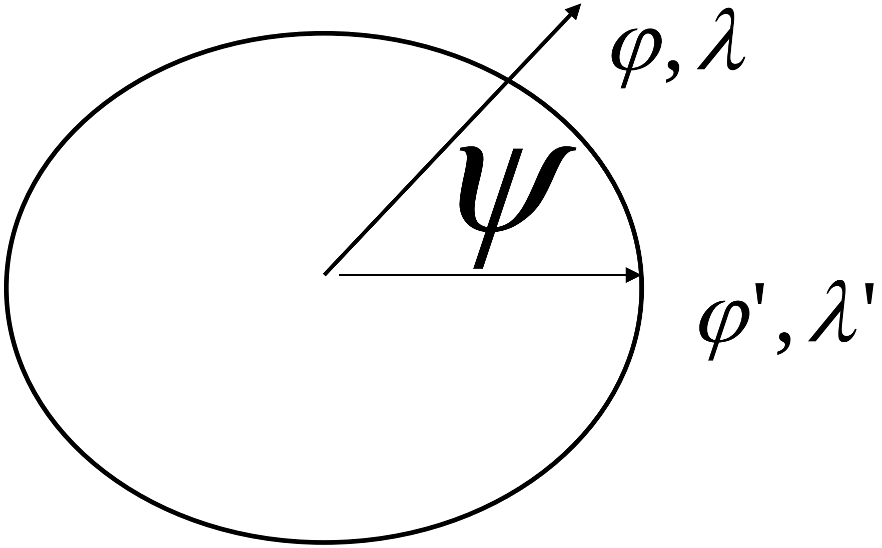
$$T(\varphi, \lambda, r) =$$

$$\frac{R}{4\pi} \iint_S \Delta g \cdot \left(\frac{R}{r} \right)^{i+1} \sum_{i=2}^{\infty} \frac{2i+1}{i-1} P_i(\cos\psi) d\sigma$$

Gravity field-modeling.

We use the development in Legendre-polynomials

$$P_i(\cos\psi) = \frac{1}{2i+1}\sum_{j=-i}^{i} Y_{ij}(\varphi,\lambda)Y_{ij}(\varphi',\lambda')$$



Stokes formula, 1849, Pizetti, 1911. p. 283.

We may obtain closed expression because:

$$\frac{1}{l} = \sum_{i=0}^{\infty} \frac{1}{r} \left(\frac{R}{r} \right)^{i} P_i(\cos\psi)$$

$$S(\psi, r) = \left(\frac{R}{r} \right)^{i+1} \frac{2i+1}{i-1} P_i(\cos\psi),$$

so on the sphere : $R/r = 1$

$$S(\psi, R) = \frac{1}{\sin(\psi/2)} + 15\cos\psi - 6\sin(\psi/2)$$

$$-3\cos\psi \cdot \ln(\sin(\psi/2) + \sin^2(\psi/2))$$

Stokes formula, p. 284

We have solved a boundary value problem for an elliptic partial differential equation, but

Note singularity:

$$\frac{1}{\sin(\psi/2)} \rightarrow \infty$$

$$when$$

$$\psi \rightarrow 0$$

Integration must be done with special method.

Use of Stokes formulae., p. 284.

Height-anomaly/geoid height:

$$\zeta(\varphi,\lambda,r)=\frac{R}{\gamma\cdot 4\pi}\iint_S \Delta g\cdot S(\psi,r)dS$$

Deflections of the vertical, Vening-Meinesz, 1928:

$$\xi(\varphi,\lambda,r)=\frac{1}{\gamma\cdot 4\pi}\iint_S \Delta g\cdot\frac{\partial}{\partial\varphi}S(\psi,r)dS$$

$$\eta(\varphi,\lambda,r)=\frac{1}{\gamma\cdot 4\pi\cos\varphi}\iint_S \Delta g\cdot\frac{\partial}{\partial\lambda}S(\psi,r)dS$$

Stokes formula, implementation. p. 287.

Due to the singularity most of the contribution comes from data with a short distance from the computational-point.

A subtraction of a spherical harmonic model (EGM) to N=360, then only very small area is needed.

$$\zeta(\varphi,\lambda,r) =$$

$$\zeta_{EGM}(\varphi,\lambda,r) + \frac{R}{\gamma \cdot 4\pi}\iint_{S_0}(\Delta g - \Delta g_{EGM}) \cdot S(\psi,r)\,dS$$

Note: EGM must be added back later !

Stokes formula, plane Earth. p. 288.

If we want to compute the integral locally, we may regard the Earth as plane.

Integral may be computed using Fourier-transformation (convolution) in the plane, where the spectral relation (i-1)/r is used.

$$\zeta(x, y, R) = \zeta(z, R)_{EGM} + \int \Delta g(t) S(z - t) dt$$

Developed by R.Forsberg and M.Sideris in 1986.

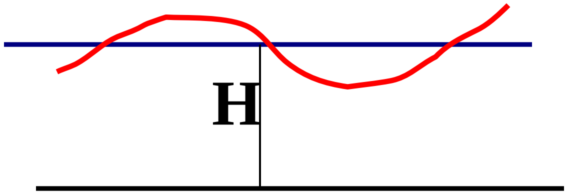
R.Forsberg/CCT: subtract (and add) also local topography

Stokes on the sphere.

Remove masses T_M over altitude 0 (adder back later)

Gravity anomali must then be computed in altitude zero 0 ("downward continuation")

Simpel metod to remove: Bouger-plate removed in each point:

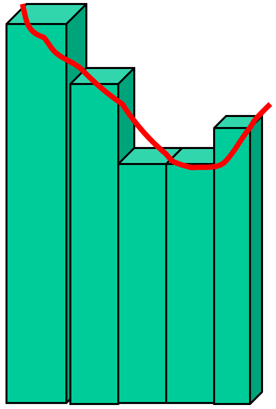


$$2\pi G\rho H \approx 0.11 \quad mgal/m$$

Not good because $T-T_M$ not harmonic anymore.

Removal of residual topography, (lecture 3.1) p. 288.

Rectangular prisms better:



May be refined with sloping boundaries – more details in R.Forsberg's lecture 3.1.

Geodetic Earth-models, see ICGEM.

If only finite number of coefficients used (N) for T and M other data,

then we get traditional least-squares solution withN+M unknowns. Typical N=240 or 2196. M=300*3+100 tide-parametre+GM, a, f +datum-shift.

WGS84: NIMA (US DoD),

GEM: Goddard (NASA) Earth Model serie

GRIM: French/German model series

EIGEN: German/French/US, latest EIGEN6_C2

ITG-Grace: Uni. Bonn.

GOCE: GO_CONS_GCF_2_DIR/TIM_R3/4.

Only gravity: OSU91, EGM96, EGM2008.