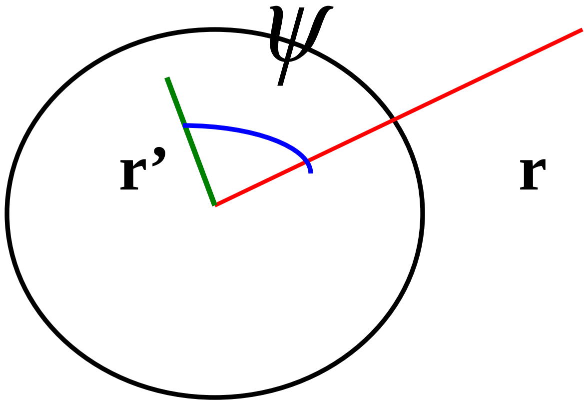
1.5. **Harmonic Analysis,** **(H/M. Section 3.3),** p.69

Start: $$\frac{1}{l} = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} =$$

$$(r^2 + (r')^2 - 2rr'\cos\psi)^{-1/2}$$



r' < r, then power-series development:

$$\frac{1}{l} = \frac{1}{r}\sum_{i=0}^{\infty}\left(\frac{r'}{r}\right)^{i} a_i$$

$$a_i = P_i(\cos(\psi)), \quad \textit{Legendre} - \textit{polynomials}$$

$$P_i(t) = \frac{1}{2^i i!}\frac{\partial^i}{\partial t^i}(t^2 - 1)^i,$$

$$P_0(t) = 1, P_1(t) = t, P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}$$

Harmonic analysis III.

$$\cos\psi = \sin\bar{\varphi}\sin\bar{\varphi}' + \cos\bar{\varphi}\cos\bar{\varphi}'\cos(\lambda - \lambda')$$

$$P_i(\cos\psi) = P_i(\sin\bar{\varphi})P_i(\sin\bar{\varphi}') +$$

$$2\sum_{m=1}^{i}\frac{(i+m)!}{(i-m)!}P_{im}(\sin\bar{\varphi})P_{im}(\sin\bar{\varphi}')$$

$$\left[\cos m\lambda\cos m\lambda' + \sin m\lambda\sin m\lambda'\right]$$

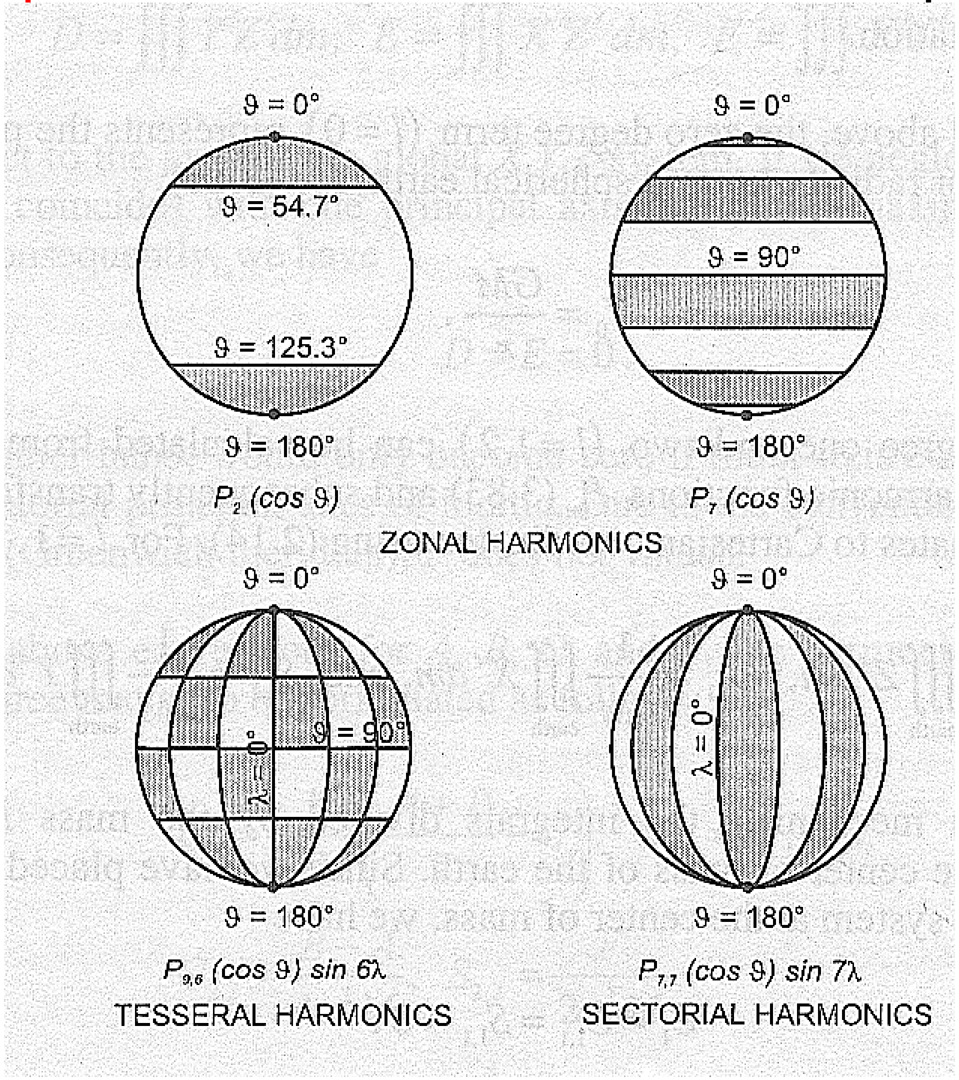
Associated Legendre functions . p.70

$$P_{im}(t) = (1 - t^2)^{m/2} \frac{\partial^m}{\partial t^m} P_i(t)$$

Spherical surface – functions:

$$Y_{im}(\overline{\varphi}, \lambda) = P_{im}(\sin \overline{\varphi}) \begin{Bmatrix} \cos \lambda & m \geq 0 \\ \sin |m| \lambda & m < 0 \end{Bmatrix}$$

Spherical surface functions . p. 74



Fully Normalized Surface Spherical Harmonics. p.71

$$\overline{P}_{im}(\sin\overline{\varphi}) = \sqrt{k(2i+1)\frac{(i-m)!}{(i+m)!}}P_{im}$$

$$\frac{1}{4\pi}\int_{0}^{360}\int_{-90}^{90}\overline{P}_{im}(\sin\overline{\varphi})\overline{P}_{jl}(\sin\overline{\varphi}')\begin{Bmatrix}\cos m\lambda\cos l\lambda' \\ \sin|m|\lambda\sin|l|\lambda' \\ \cos m\lambda\sin|l|\lambda' \\ \sin|m|\lambda\cos m\lambda'\end{Bmatrix}$$

$$= 1\ when\ i = j\ and\ m = l\ otherwise = 0$$

Development in Spherical Harmonic Functions. p.71

$$V(r,\bar{\phi},\lambda) = \frac{G}{r}\sum_{i=0}^{\infty}\sum_{m=-i}^{i} k\frac{(i-m)!}{(i+m)!}\frac{1}{r^i}Y(\bar{\varphi},\lambda)\cdot$$

$$\iiint_{Earth}(r')^i Y_{im}(\bar{\varphi}',\lambda')dm, \quad k = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases}$$

$$= \frac{GM}{r}(1+\sum_{i=0}^{\infty}\sum_{m=-i}^{i}\left(\frac{a}{r}\right)^i C_{im}Y_{im}(\bar{\varphi},\lambda)$$

$$C_{im} = \frac{k}{M}\frac{(i-m)!}{(i+m)!}\iiint_{Earth}(\frac{r'}{a})^i Y_{im}(\bar{\varphi}',\lambda'')dm$$

Orthonormal basis.

We have found a Fourier-development

An orthonormal-system of basis-functions, which all are solutions to the Laplace-equation

$$\Delta\left(\left(\frac{a}{r}\right)^{i+1} Y_{im}(\bar{\varphi},\lambda)\right) = 0$$

C_{im} , Stokes or potential coefficients, spectre or Stokes constants.

Solution of Laplace-equation in Spherical coordinates

$$\Delta(f(r)\cdot g(\cos\theta).h(\lambda)) = 0$$

Laplace in spherical coordinates

$$r^2\frac{\partial^2 V}{\partial r^2} + 2r\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial \theta^2} + \cot\theta\frac{\partial V}{\theta r}$$

$$+\frac{1}{\sin\theta^2}\frac{\partial^2 V}{\partial \lambda^2}, \theta = 90^0 - \overline{\varphi}$$

Solution for r:

$$\mathbf{V}(\mathbf{r},\theta,\lambda) = \mathbf{f}(\mathbf{r})\mathbf{Y}(\theta,\lambda)$$

$$\frac{1}{\mathbf{f}}(\mathbf{r}^2\mathbf{f}''+2\mathbf{rf}') = -\frac{1}{\mathbf{Y}}\left(\frac{\partial^2\mathbf{Y}}{\partial\theta^2} + \cot\frac{\partial\mathbf{Y}}{\partial\theta}\right.$$

$$\left.+\frac{1}{\sin\theta}\frac{\partial^2\mathbf{Y}}{\partial\lambda^2}\right)$$

$$\mathbf{r}^2\mathbf{f}''(\mathbf{r}) + 2\mathbf{rf}'(\mathbf{r}) - \mathbf{n}(\mathbf{n}+1)\mathbf{f}(\mathbf{r}) = 0$$

$$\mathbf{f}(\mathbf{r}) = \mathbf{r}^{\mathbf{n}} \quad \mathbf{or} \quad \mathbf{f}(\mathbf{r}) = \mathbf{r}^{-(\mathbf{n}+1)}$$

Laplace equation

$$\mathbf{V} = \mathbf{r}^{\mathbf{n}} \mathbf{Y}_{\mathbf{n}}(\theta, \lambda) \text{ } \mathbf{eller} \text{ } \mathbf{r}^{-(\mathbf{n}+1)} \mathbf{Y}_{\mathbf{n}}(\theta, \lambda)$$

$$\mathbf{Y}_{\mathbf{n}}(\theta, \lambda) = \mathbf{g}(\theta)\mathbf{h}(\lambda)$$

$$\mathbf{h}''(\lambda) = \mathbf{m}^2 \mathbf{h}(\lambda) \Rightarrow \mathbf{h}(\lambda) = \begin{cases} \cos \mathbf{m}\lambda \\ \sin \mathbf{m}\lambda \end{cases}$$

$$\sin\theta \mathbf{g}''(\theta) + \cos\theta \mathbf{g}'(\theta)$$

$$+ \left[\mathbf{n}(\mathbf{n}+1)\sin\theta - \frac{\mathbf{m}^2}{\sin\theta} \right] \mathbf{g}(\theta) = 0$$

$$\Rightarrow \quad \mathbf{g}(\theta) = \mathbf{P}_{\mathbf{nm}}(\cos\theta)$$

Legendre-polynomials, p.69

Polynomials of degree i with all zero-points between -1 og 1.

Constructed by orthomalizing usual polynomier t^i

$$\left(P_i(t), P_j(t)\right) =$$

$$\int_{-1}^{1} P_i(t) \cdot P_j(t) dt = 0, i \neq j$$

$$P_0(t) = 1, P_1(t) = t = \sin\theta,$$

$$\mathrm{Re}\, cursions - equation: (i > 1)$$

$$P_i(t) = \frac{2i-1}{i} t P_{i-1}(t) - \frac{i-1}{i} P_{i-2}(t),$$

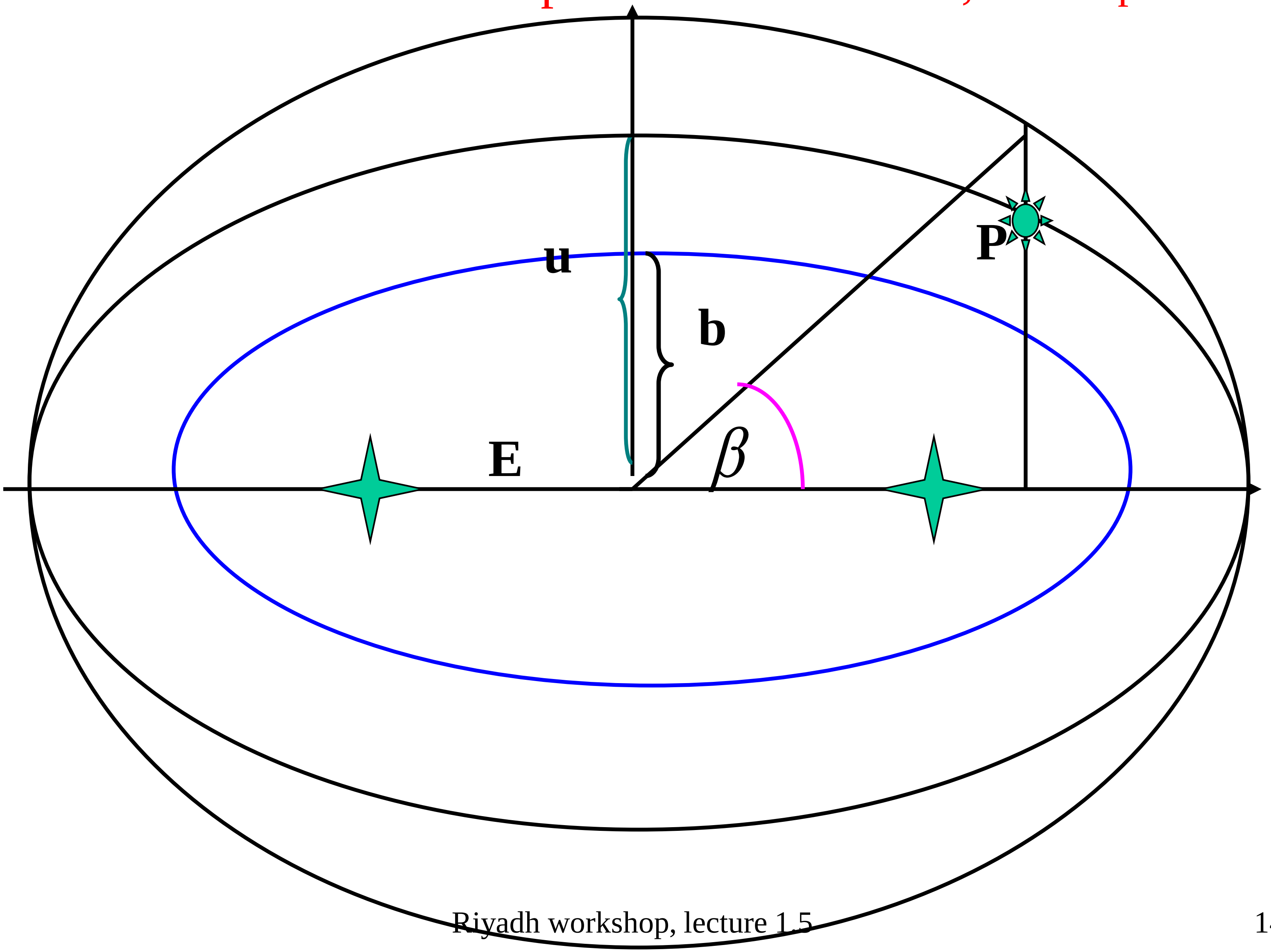
Associated Legendre Functions.

$$Y_{ij}(\overline{\varphi},\lambda) = P_{ij}(\sin\overline{\varphi})\begin{Bmatrix}\cos j\lambda \\ \sin|j|\lambda\end{Bmatrix}$$

$$Zero - po\text{int}\,s \quad j \quad between\left[-\pi,\pi\right]$$

$$i \quad between\left[-\pi/2,\pi/2\right]$$

Ellipsoidal coordinates:, p.101



Solution of Laplace-equation in Ellipsoidal coordinates

$$\Delta(f(u)\cdot g(\cos\beta)\cdot h(\lambda)) = 0$$

Laplace in ellipsoidal coordinates

$$(u^2+E^2)\frac{\partial^2 V}{\partial u^2} + 2u\frac{\partial V}{\partial u} + \frac{\partial^2 V}{\theta^2} + \cot\theta\frac{\partial V}{\partial\theta}$$

$$+\frac{u^2+E^2\cos\theta^2}{(u^2+E^2)\sin\theta^2}\frac{\partial^2 V}{\partial\lambda^2}, \theta = 90^0 - \beta$$

$$h(\lambda) = \cos m\lambda \text{ or } \sin m\lambda, \text{ as before}$$

Solution for u and g:

$$V(u,\theta,\lambda) = f(u)g(\theta)h(\lambda),$$

$$h \text{ eliminates:}$$

$$\frac{1}{f}((u^2 + E^2)f'' + 2uf') + \frac{E^2}{u^2 + E^2}m^2 =$$

$$-\frac{1}{g}(g'' + \cot\theta g' + \frac{m^2}{\sin\theta^2})$$

$$= \text{same constant}: n(n+1)$$

Solution for u and g:

$$New\ \text{var}\,iable: \tau = i\frac{u}{E}, t = \cos\theta$$

Two identical equations with f, τ *and* g, t.

$$(1-\tau^2)f''(\tau) - 2\tau f'(\tau) + n(n+1) - \left[\frac{m^2}{1-\tau^2}\right]f(\tau) = 0$$

$$(1-t^2)g''(t) - 2tg'(t) + n(n+1) - \left[\frac{m^2}{1-\tau^2}\right]g(t) = 0$$

Solution for u and g:

f *fulfill differential equation in the complex plane.*

Solution on interval $[-1,1]$ *or outside.*

$$f(u) = P_{nm}(i\frac{u}{E}), \ or \ Q_{nm}(i\frac{u}{E}),$$

$$Q_{00}(i\frac{u}{E}) = -i\arctan\left(\frac{E}{u}\right)$$

$$g(\theta) = P_{nm}(\cos\theta),$$

Solution for u and g:

Solution outside focii for V.

$$V(u,\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{Q_{nm}(i\frac{u}{E})}{Q_{nm}(i\frac{b}{E})}$$

$$\cdot C_{nm} P_{n|m|}(\cos\theta) \begin{cases} \cos m\lambda, m \geq 0 \\ \sin|m|\lambda, m < 0 \end{cases}$$

Physical meaning of coefficients , p.75

$$\frac{1}{aM}\iiint_{Earth} x_i dm \rightarrow \ Center\ of\ gravity$$

$$\frac{1}{a^2 M}\iiint_{Earth} x_i x_j dm \rightarrow \ Moments,$$

$$i \neq j \rightarrow \ inertia\ products$$

$$A = \iiint_{Earth} (y^2 + z^2) dm, B = \iiint_{Jorden} (x^2 + z^2) dm$$

$$C = \iiint_{Earth} (x^2 + y^2) dm \rightarrow \ moments\ of\ Inertia$$

$$C_{20} = \frac{1}{a^2 M}\left(\frac{A+B}{2} - C\right), C_{22} = \frac{B-A}{4a^2 M}$$