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Basic Mathematics and Statistics for use in Gravity Field Modelling.

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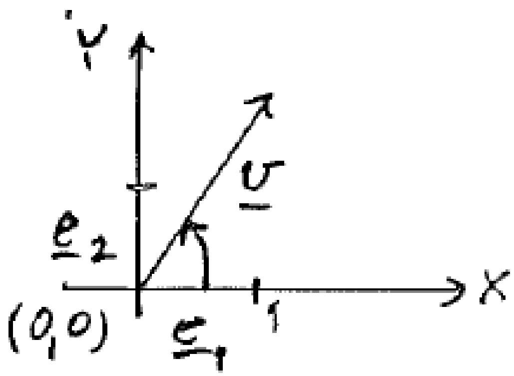
M 1.1 VECTORS

Set of (n) numbers denoted elements. n the Dimension.

n=1: $\{v_1\} = \underline{v},$ a **scalar**, n=2: $\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \underline{v},$ a plane vector

In **Cartesian** coordinate system



$e_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, e_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$, basis-vectors

$\underline{v} = v_1 \cdot \underline{e}_1 + v_2 \cdot \underline{e}_2$, linear combination

Vectors have a size and a direction:

s=$\|\underline{v}\| = \sqrt{{v_1}^2 + {v_2}^2}$, ”Euclidian distance”, or norm,

Unit vector: $\underline{e} = \begin{Bmatrix} v_1/s \\ v_2/s \end{Bmatrix}$, size = 1.

Direction, φ:

Mathematics: Angle between X-axis and vector, positive against the clock. $\varphi = \text{atan}(\frac{v_2}{v_1})$

Nature/geodesy: Angle between Y-axis (North) and vector, positive clockwise = Azimuth.

Example:

$$\underline{v} = \underline{\begin{Bmatrix} \sqrt{2} \\ \sqrt{2} \end{Bmatrix}}, \text{ then } \underline{e} = \begin{Bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{Bmatrix}$$

$$\varphi = \operatorname{atan}(1) = 45^{o} = \pi/4$$

Then $v_1 = s \cdot \cos(\varphi)$,

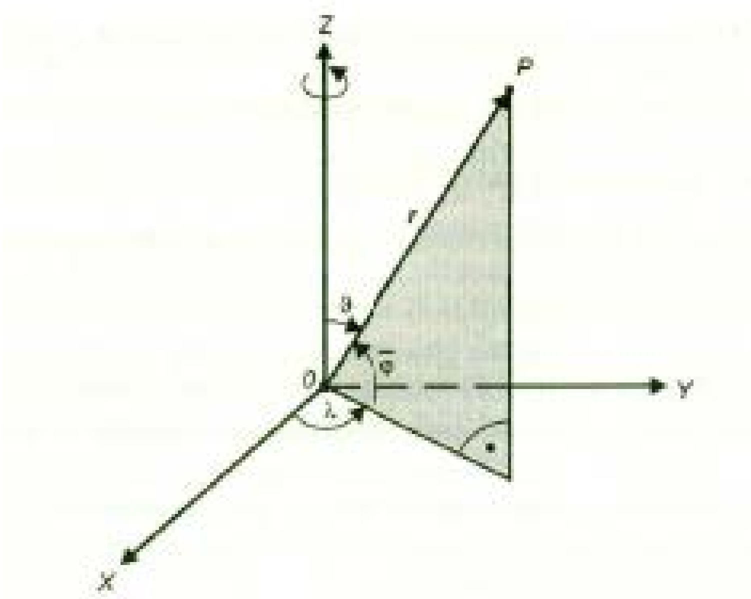
$v_2 = s \cdot sin(\varphi)$, s=r, radius

Polar coordinates: (φ, r)

In maps: Northing = v_2 =N, Easting = v_1=E.

M 1.4 SPACE VECTORS

Example: $\underline{v} = \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$, p= $\sqrt{X^2+Y^2}$, $r = \sqrt{X^2+Y^2+Z^2}$



$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{Bmatrix} r\cos(\overline{\varphi})\cos(\lambda) \\ r\cos(\overline{\varphi})\,sin(\lambda) \\ r\,sin(\overline{\varphi}) \end{Bmatrix}$$

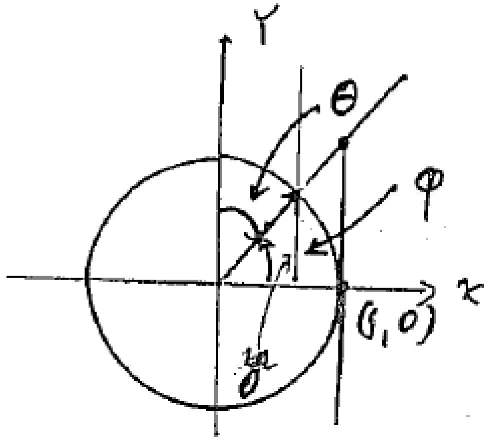
Spherical coordinates: r, $\bar{\varphi}$, λ ,

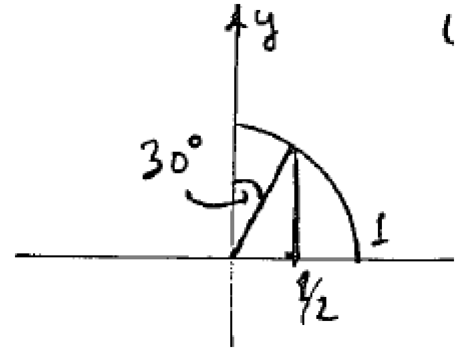
$\overline{\varphi}$ geocentric latitude,

φ geodetic latitude.

M 1.5 UNIT-VECTORS

s=1: φ=60^o





Then $v_1 = \cos(\varphi)$, $v_1 = \frac{1}{2}$

$v_2 = sin(\varphi)$, $v_2 = \frac{\sqrt{3}}{2}$

Azimuth, $\theta = 90^0 - \varphi$

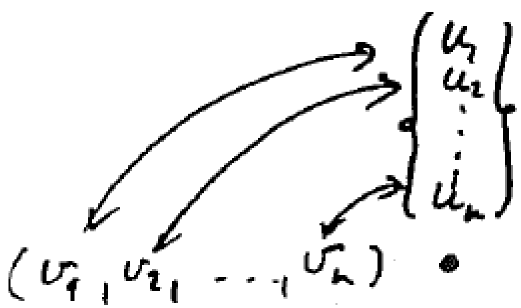
M 1.6 Scalar product, transposed vector

Scalar product

$$\underline{u} \cdot \underline{v} = v_1 \cdot u_1 + v_2 \cdot u_2 + v_3 \cdot u_3 = \sum_{i=1}^{3} v_i \cdot u_i$$

$$\underline{v} = \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}, \qquad \text{transposed: } \underline{v}^T = \{v_1 \quad v_2 \quad v_3\}$$

Scalar or inner product:



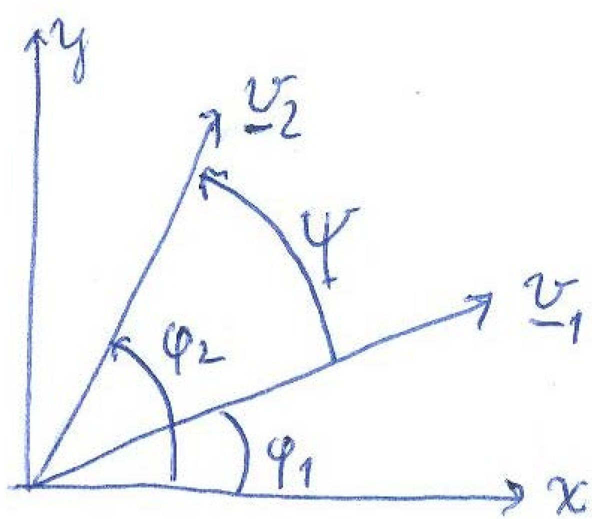
$$= \sum_{i=1}^{3} v_i \cdot u_i$$

Scalar product example

$\underline{v}_1 = \begin{Bmatrix} r_1\cos(\varphi_1) \\ r_1\sin(\varphi_1) \end{Bmatrix}, \underline{v}_2 = \begin{Bmatrix} r_2\cos(\varphi_2) \\ r_2\sin(\varphi_2) \end{Bmatrix},$

$\underline{v}_1\underline{v}_2 = r_1 r_2(\cos(\varphi_1)\cos(\varphi_2) + sin(\varphi_1)\,sin(\varphi_2)) = r_1 r_2\cos(\psi)$

Ψ is the angle between the two vectors. Holds in all diménsions.



If $\underline{v}_1$ is unit-vector: $\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, then

$\underline{v}_1 \cdot \underline{v}_2 = r_2\cos(\varphi_2)$,

Projection of the vector on the x-axis.

Euclidian real n-dimensional vector space,

R^n to R^1 matrix, example n=3, Real, linear mapping.

$$\begin{Bmatrix} 2 & 3 & 4 \end{Bmatrix} \cdot \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} = -1$$

R^n to R^m matrix, example n=3, m=2

$$\begin{Bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \end{Bmatrix} \cdot \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

M 1.10 Identity **mapping**

Euclidian real n-dimensional vector space,

R^n to R^n matrix, example n=3,

$$\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix} \cdot \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$$

M 1.11 Space, subspace, affine subspace

Euclidian real 3-dimensional vector space, coordinates:

x,y,z.

1-D subspace, example straight line through zero,

$y=2x$

2-D subspace, plane through (0,0,0):

$z=2x+3y$

Affine 1-D subspace: Line does no pass through zero !

$y=2x+2$

M 2.1 VECTOR FUNCTIONS

Example: $\underline{v} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$, base-functions (vectors): $\underline{e} = \begin{Bmatrix} 1 \\ x \\ x^2 \end{Bmatrix}$,

Function in 3-dimensional linear vector space of 2.deg. polynomials:

2.degree polynomial = $\underline{v} \cdot \underline{e} =$

$$P(x) = 1 + 2x + 3x^2$$

M 2.2 VECTOR FUNCTIONS – NORM

Many different norms can be used. On Earth integration over whole surface of function squared can be used

$$\|P(x)\|^2 = \frac{1}{2}\int_{-1}^{1} P(x)^2 dx =$$

$$\frac{1}{2}\int_{-1}^{1}(1 + 4x + 10x^2 + 12x^3 + 9x^4)dx = 5$$

$$\|1\| = 1$$

$$\|x\| = \frac{1}{\sqrt{3}}$$

$$\|x^2\| = \frac{1}{\sqrt{5}}$$

$P(x) = 1 + 2x + 3x^2$, Q(x)=3x,

Scalar product:

$(P(x),Q(x)) = \frac{1}{2}\int_{-1}^{1} P(x)Q(x)dx =$

$$\frac{1}{2}\int_{-1}^{1} 3x \cdot (x + 2x^2 + 3x^3)dx = 3$$

$(1, x) = \frac{1}{2}\int_{-1}^{1}(x)dx = 0$, $(1,x^2) = \frac{1}{2}\int_{-1}^{1}(x^2)dx = 1/3$

$(x,x^2) = \frac{1}{2}\int_{-1}^{1}(x^3)dx = 0$

Orthonormal basis of Polynomials:

$\underline{f_0}(x) = 1/\|1\|=1,$

$\underline{f_1}(\mathrm{x})=(\mathrm{x}-(\mathrm{x},1)1)/[\![x]\!] = \mathrm{x}\cdot\sqrt{3},$

$\underline{f_2}(x) = (x^2-(1,x^2)1 - (\underline{f_1},x^2)\underline{f_1}(x))/\|(x^2-(1,x^2)1)\|=$

$(x^2\text{-}1/3)\sqrt{5}\cdot\frac{3}{2} = (\frac{3}{2}x^2\text{-}\frac{1}{2})\sqrt{5}$

Now *P(x)* expressed in this basis:

$$P(x) = (1+\frac{\sqrt{5}}{2})\cdot f_0(x) + 2\sqrt{3}\cdot f_1(x) + \frac{2}{\sqrt{5}}f_2(x)$$

Orthonormal basis in interval $[0,2\pi]$:

$\underline{f_0}(x) = 1/\|1\|=1$,

$\underline{f_1}(x)=\cos(x)$, $\underline{f_2}(x) = \sin(x)$

$f_{2n-1}(x) = \cos(nx)$, $f_{2n}(x)=\sin(nx)$

$f(x) = \sum_{i=0}^{n}(a_i \cos(ix) + b_i(\sin(ix))$, Fourier series.

$$\frac{1}{2\pi}\int_0^{2\pi} f_i(x) f_j(x) dx = \begin{cases} 1, if\ i = j \\ 0, if\ i \neq j \end{cases}$$

M 2.5 LINEAR VECTOR SPACES

Vectors can be added or multiplied by scalar

If scalar or inner product defined:

An orthonormal base can be constructed

A vector can be projected on another

Linear Functional: Mapping from n-dimensional space to real line. Example: derivative of a function in a point

$$\left.\frac{df}{dx}\right|_{x_0}$$

 Linear mapping in Vector Space

Linear vector space of 3D functions:

Example polynomial of maximal degree 2:

$$P(x,y,z) = 1 + z + 2xy + 3z^2$$

Linear functional (L) maps to a real number:

Example: $L_{1,2,3}(P)$=P(1,2,3)=1+3+2*1*2+3*3*3=25

Denoted an *evaluation fuctional* (gives value in a point).

Derivative functional in **point** (1,2,3): $\frac{\partial P}{\partial x}$ =2y=4

Gradient Mapping to m=3, $\nabla P = \begin{Bmatrix} \partial P / \partial x \\ \partial P / \partial y \\ \partial P / \partial z \end{Bmatrix} = \begin{Bmatrix} 2y \\ 2x \\ 1 + 6z \end{Bmatrix}$

Linear vector space of 3D functions:

Example 3D space of Legendre-polynomials.

Reproducing kernel,

$$K(x,y) = \sum_{i=0}^{2} f_i(x) \cdot f_i(y) =$$

$$1+3xy+5(\tfrac{3}{2}x^2 - \tfrac{1}{2})(\tfrac{3}{2}y^2 - \tfrac{1}{2})$$

M 2.8 Riesz representers of linear functionals, L.

Reproducing kernel,

$K(x,y) = \sum_{i=0}^{2} f_i(x) \cdot f_i(y)$ =1+3xy+5$(\frac{3}{2}x^2 - \frac{1}{2})(\frac{3}{2}y^2 - \frac{1}{2})$,

$P(x) = 1 + 2x + 3x^2$

Riesz representer: The functional is applied on the kernel.

$L_{x=1}\left(K(x,y)\right) = K(1,y) =$

1+3y+5 $(\frac{3}{2}y^2 - \frac{1}{2})$

Works as: $L_{x=1}\left(P(x)\right)$=(P(y),K(1,y))=

$\frac{1}{2}\int_{-1}^{1} P(y) \cdot K(1,y) \cdot dx$ =6

M 2.9 Identity **mapping=Reproducing property**

$$(P(x),K(x,y))=\frac{1}{2}\int_{-1}^{1}P(x)K(x,y)dx=P(y)$$

$$\frac{1}{2}\int_{-1}^{1}(1+2x+3x^2)(1+3xy+5(\frac{3}{2}x^2-\frac{1}{2})(\frac{3}{2}y^2-\frac{1}{2}))dx=$$

$$\frac{1}{2}\int_{-1}^{1}(\sum_{i=0}^{2}f_i(x)\cdot f_i(y))\cdot((1+\frac{\sqrt{5}}{2})\cdot f_0(x)+2\sqrt{3}\cdot f_1(x)+\frac{2}{\sqrt{5}}f_2(x))dx$$

$$=\frac{1}{2}\int_{-1}^{1}(f_0(x)\cdot f_0(y)\cdot\left(1+\frac{\sqrt{5}}{2}\right)\cdot f_0(x)+f_1(x)\cdot f_1(y)\cdot 2\sqrt{3}\cdot f_1(x)+$$

$$f_2(x)\cdot f_2(y)\cdot\frac{2}{\sqrt{5}}f_2(x))dx=$$

$$1+2y+3y^2=P(y)$$

Using: $\frac{1}{2}\int_{-1}^{1}(f_i(x)\cdot f_j(y))dx=1$ if i=j, 0 if $i\neq j$

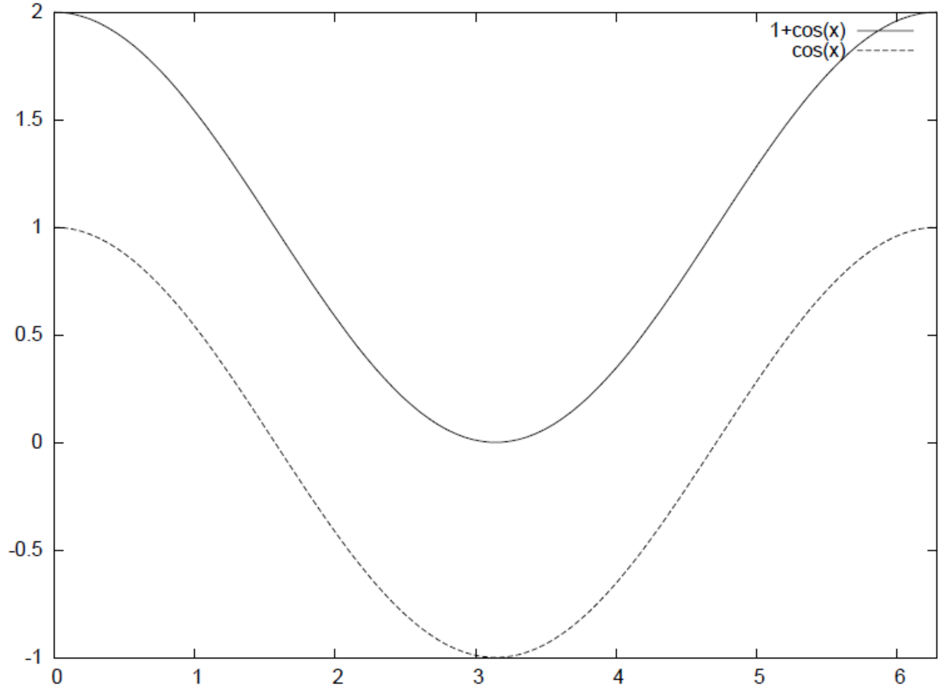
M 2.10 Trigonometric functions, Reproducing kernel.

$$K(x,y) = 1 + \sin(x)\sin(y) + \cos(x)\cos(y) =$$

$$1+\cos(x-y)$$

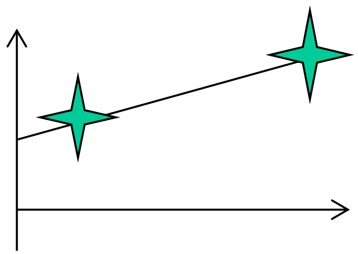
If we remove zero-base-function, then only cos(x-y).

Note that by removing 0-degree term we have crossing



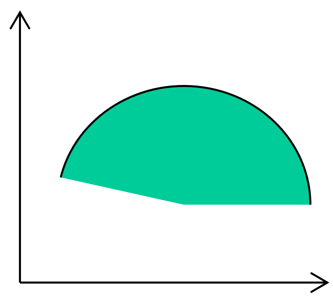
at zero !!

Linear interpolation: given (x_1,y_1), (x_2,y_2),



find: (x_3,y_3),

Polynomial interpolation: given (x_1,y_1), (x_2,y_2), (x_3,y_3),
find: (x_4,y_4),



Example: $P(x) = 1 + 2x + 3x^2$

Two observations:

$L_1(P) = P(x_1 = 0.5) = 2.75$, $L_2(P) = P(x_2 = 0) = 1$

Find best approximation $\tilde{P}$ to P on interval [-1,1], agreeing with data.

Best = $\|P - \tilde{P}\| = minimum$

One can proove: $\tilde{P}(y) = \sum_{i=1}^{2} b_i \cdot L_i(K(\cdot, y))$, then

$$\begin{Bmatrix} \tilde{P}(x_1) \\ \tilde{P}(x_2) \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{2} b_i \cdot L_i(K(x_i, x_1)) \\ \sum_{i=1}^{2} b_i \cdot L_i(K(x_i, x_2)) \end{Bmatrix} = \begin{Bmatrix} 2.75 \\ 1 \end{Bmatrix},$$

Which 2 equations with 2 unknowns !

Example: $P(x) = 1 + 2x + 3x^2$

$$\begin{Bmatrix} \tilde{P}(x_1) \\ \tilde{P}(x_2) \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{2} b_i \cdot L_i(K(x_i, x_1)) \\ \sum_{i=1}^{2} b_i \cdot L_i(K(x_i, x_2)) \end{Bmatrix} = \begin{Bmatrix} 2.75 \\ 1 \end{Bmatrix},$$

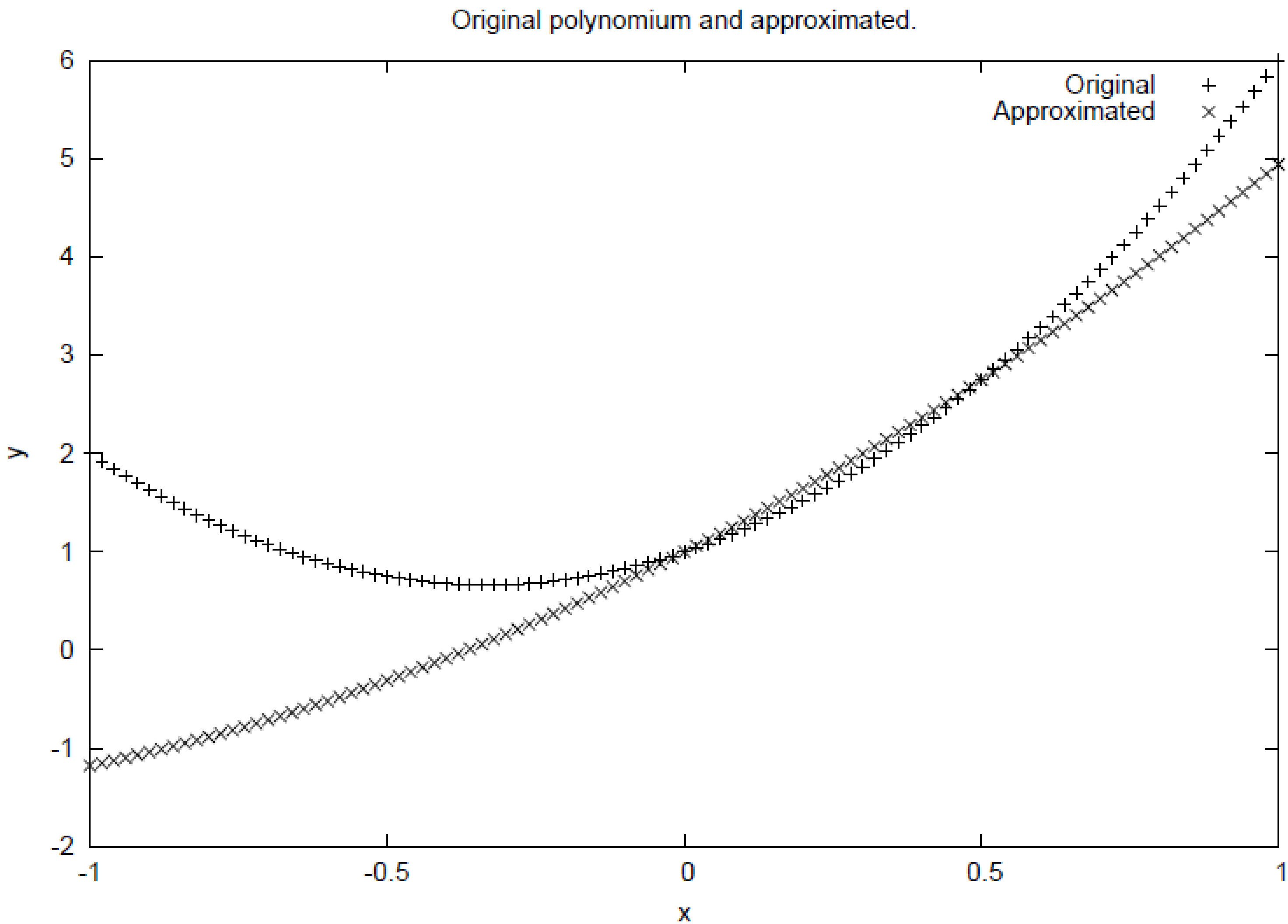
Which 2 equations with 2 unknowns !

$$b_1 \cdot 1.828125 \quad + b_2 \cdot 1.3125 = 2.75$$
$$b_1 \cdot 1.312500 \quad + b_2 \cdot 2.25 \quad = 1.0$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -0.745098 \\ +2.03922 \end{bmatrix}, \text{ so } \begin{Bmatrix} \tilde{P}(x_1) \\ \tilde{P}(x_2) \end{Bmatrix} == \begin{Bmatrix} 2.75 \\ 1 \end{Bmatrix},$$

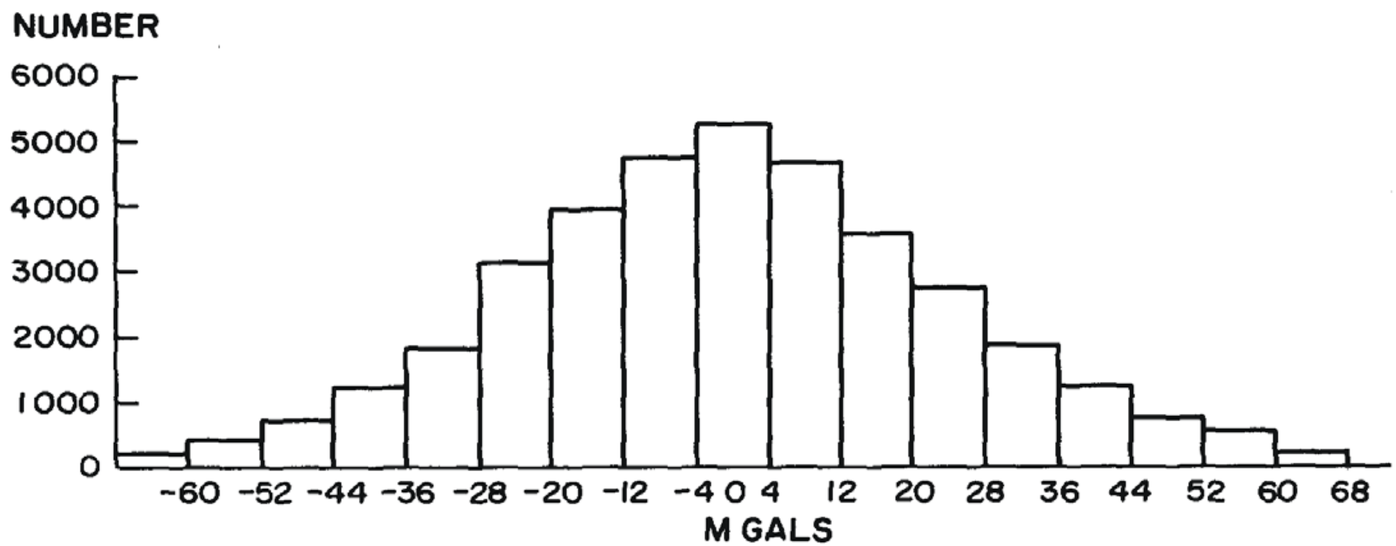
See Figure.

M 3.4. Approximation, Prediction



Histogram: describes distribution of repeated observations. Repeated at different times or locations.

Example: Distribution of Global 1 deg. X 1 deg. Mean gravity anomalies:



Distribution not only of random but also deterministic quantities like gravity

We use statistical terms to describe random AND deterministic quantities.

Deterministic quantities may look as having a Normal (Gaussian) distribution.

The basic quantity is the ”event” or ”observation”.

In Math. The result of a FUNCTIONAL.

In statistics: A stochastic variable.

M 3.3. Mean, standard deviation, Variance, Estimator.

Probability distribution: Probability that the values are in a specific interval:

$$P(a < x < b) = \int_a^b f(x)dx$$

Example: $$f(x) = \frac{1}{\sqrt{2\pi \bullet \sigma_{xx}}} \bullet e^{-(x-E(X))^2/(\sigma_{xx}2)}$$

Mean: $$\bar{x} = \int_{-\infty}^{\infty} x \cdot f(x)dx = E(x)$$

Variance: $$\bar{\sigma}^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 \cdot f(x)dx = \mathrm{E}(\mathrm{x}-\bar{x})^2$$

M 3.4 Mean and variance of VECTOR Y.

X and A_0 n-vectors, Y m-vector,
A= nxm matrix, Σ_Y=Variance-covariance matrix of Y, inverse: weight matrix.

$$Y = A_0 + A \cdot X \rightarrow E(Y) = A_0 + A \cdot E(X)$$

$$\Sigma_Y = E((y - E(Y)) \cdot (Y - E(Y))^T) = A \cdot \Sigma_X \cdot A^T$$

N=2:

Variance-covariance matrix 2x2:

$\Sigma_X = \sigma_X{}^2 = \{\sigma_{ij}\}$, i=1,2, j= 1,2.

$\sigma_{ij} = \iint_{-\infty}^{\infty} (x_1 - \bar{x}_1)(x_2 - \bar{x}_2) f(x_1, x_2) d\, x_1 dx_2$

We use $\sigma_{ii} = \sigma_i{}^2$

$$\sigma_{ij} = covariance$$

Correlation between two quantities:

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii} \bullet \sigma_{jj}}} \in [-1,1]$$

$\rho_{ij} = 0$, means statistical independence.

M 3.7 Estimating mean, standard deviation and correlation.

Best estimate if normal distribution !

Mean: $\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$

Standard deviation: $\sigma_x = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1}}$

Covariance: cov(x,y)=$\sum_{i=1}^{n} (x - x)(y - \bar{y})/n$

Correlation: $\rho = cov(x, y)/(\sigma_x \cdot \sigma_y)$

If x,y varies then

Covariance function, cov(x,y)

If only depending on distance |x-y| on sphere then isotropic.

If only depending on time-difference, then stationary

M 3.8 Covariance example.

Stationary time-series:

$$f(x) = \sum_{i=0}^{n} (a_i \cdot \cos(\frac{2\pi i}{n} \mathrm{x}) + b_i \cdot \sin(\frac{2\pi i}{n} x))$$

Variance of a_i and b_i, identical = σ_i then

Cov(f(x),f(y))=

$$\sum_{i=0}^{n} \sigma_i (cos(\frac{2\pi i}{n} x)\cos(\frac{2\pi i}{n} y) + \sin\left(\frac{2\pi i}{n} x\right) \sin(\frac{2\pi i}{n} y)) =$$

$$\sum_{i=0}^{n} \sigma_i \cos(\frac{2\pi i}{n} (x - y)),\ \ \sigma_i\!: power - spectrum$$

If data are linearily related such as

$y_k = \sum_{i=1}^{n} a_{ki} x_i$ then we can use

The method of least-squares estimation

To find $x_i, i = 1, \ldots, n$

from observations y_k=1,...,m,

m > n and error variance-covariance,$\Sigma_y = W^{-1}$

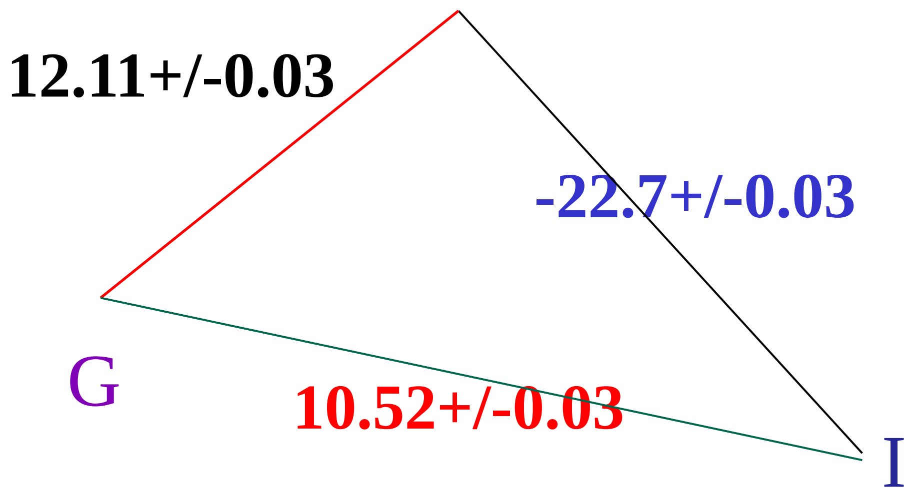
with $\{a_{ik}\}$=A, T is transposed:

$$\{\tilde{x}_i\} = (A^T W A)^{-1} (W A)^T \{y_k\}$$

Gravity measurements:

H with absolute measurement

H, g=981600.15 +/-0.02 mgal



M 3.12 Gravity adjustment: more observations than unknowns.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} G_G \\ G_H \\ G_I \end{pmatrix} = \begin{pmatrix} 981600.15 \\ 12.11 \\ 10.52 \\ -22.70 \end{pmatrix} = A \cdot x = y,$$

$$\Sigma_x = \begin{pmatrix} 0.02^2 & 0 & 0 & 0 \\ 0 & 0.02^2 & 0 & 0 \\ 0 & 0 & 0.03^2 & 0 \\ 0 & 0 & 0 & 0.03^2 \end{pmatrix}$$

Then

$$\tilde{y} = (A^T \Sigma_x{}^{-1} A)^{-1} \Sigma_x{}^{-1} A\mathrm{x}$$

Students: calculate y !

We can NOT use

The method of least-squares estimation

We must use least-squares collocation !

Model is a linear combination of data y_i and it must agree exactly if no noise.

$\widetilde{x_P}=\sum_{k=1}^{n} \alpha_{Pk} y_k,$

Variance of $(x_P-\widetilde{x_P})^2$= minimum

M 3.14. Least-squares prediction.

$$\varepsilon^2{}_P = (x_P - \widetilde{x_P})^2 =$$

$$x^2{}_P - 2\sum_{i=1}^{n} \alpha_{Pi} \cdot x_i \cdot x_P + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{Pi}\, \alpha_{Pj} x_i x_j$$

Now we compute the Variance using

$$E\left(x_i x_j\right) = C_{ij}, E(x_P x_i) = C_{Pi}, C_P = E(x_P)$$

We find the minimum by differentiation wrt. α_{Pi}.

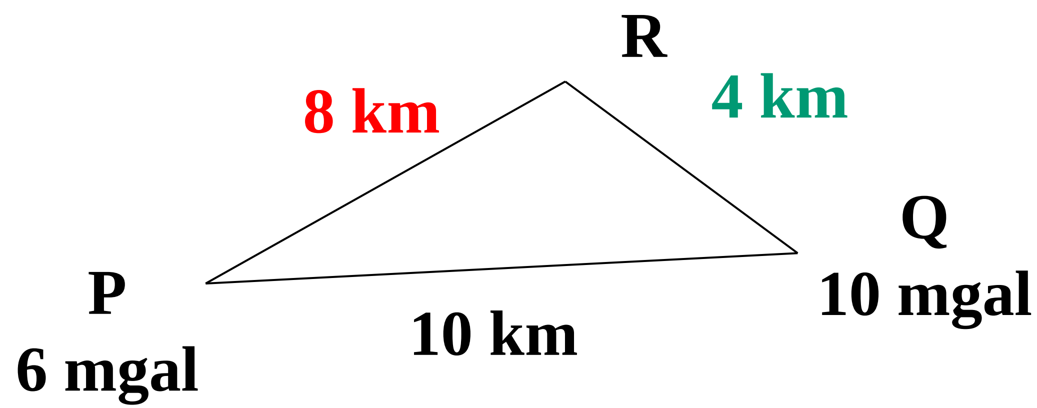
$$\frac{\partial \varepsilon^2{}_P}{\partial \alpha_{Pi}} = -2C_{Pi} - 2\sum_{j=1}^{n} \alpha_{Pj} C_{ij} = 0, \Rightarrow\ \alpha_{Pj} = \left\{C_{ij}\right\}^{-1} \{C_{Pi}\}$$

$$\tilde{x}_P = \{C_{Pi}\}\{C_{ik}\}^{-1}\{x_k\}$$

$\tilde{x}_j = \left\{C_{ij}\right\}\{C_{ik}\}^{-1}\{x_k\} = x_j$ COLLOCATION !

M 3.15. Least-squares prediction of gravity anomaly.

Example: Find Δg *in R*:



Covarianses: COV(0 km)= 100 mgal^2

COV(10 km)= 60 mgal^2

COV(8 km)= 80 mgal^2

COV(4 km)= 90 mgal^2

$$\Delta \tilde{g}_R = \begin{pmatrix} 90 & 80 \end{pmatrix} \begin{pmatrix} 100 & 60 \\ 60 & 100 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 6 \end{pmatrix} = 9\ mgal$$