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ESTIMATION AND MODELLING OF THE LOCAL EMPIRICAL COVARIANCE FUNCTION USING GRAVITY AND SATELLITE ALTIMETER DATA

Abstract

The estimation of a local empirical covariance function from a set of observations was done in the Faeroe Islands region. Gravity and adjusted Seasat altimeter data relative to the GPM2 *spherical harmonic approximation were selected holding one value in celles of* $1/8° \times 1/4°$ *covering the area. In order to center the observations they were transformed into a locally best fitting reference system having a semimajor axis* 1.8 m *smaller than the one of* GRS80. *The variance of the data then was* 273 mgal^2 *and* 0.12 m^2 *respectively. In the calculations both the space domain method and the frequency domain method were used. Using the space domain method the auto-covariances for gravity anomalies and geoid heights and the cross-covariances between the quantities were estimated. Furthermore an empirical error estimate was derived. Using the frequency domain method the auto-covariances of gridded gravity anomalies was estimated. The gridding procedure was found to have a considerable smoothing effect, but a deconvolution made the results of the two methods to agree.*

The local covariance function model was represented by a Tscherning/Rapp degree-variance model, $A/((i-1)(i-2)(i+24))(R_B/R_E)^{2i+2}$, *and the error degree-variances related to the potential coefficient set* GPM2. *This covariance function was adjusted to fit the empirical values using an iterative least squares inversion procedure adjusting the factor* A, *the depth to the Bjerhammar sphere* $(R_E - R_B)$, *and a scale factor associated with the error degree-variances. Three different combinations of the empirical covariance values were used. The scale factor was not well determined from the gravity anomaly covariance values, and the depth to the Bjerhammar sphere was not well determined from geoid height covariance values only. A combination of the two types of auto-covariance values resulted in a well determined model.*

1. Introduction

The method of least squares collocation (Moritz, 1980) is widely used for interpolation and estimation of gravity field related quantities such as gravity anomalies and geoid heights. The quantities are related to the anomalous potential through linear functionals and the value of the covariance between two quantities is then obtained by applying the linear functionals on the covariance function of the anomalous potential. The best least squares approximation to the true potential field is obtained when an

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empirical covariance function is used in the computations. In local areas one first removes most of the long-wavelength field and the effect of the topography. Then an expression representing the local empirical covariance function is applied to the residual data. The removed effects are finally restored to the estimated quantities.

The determination of a local empirical covariance function was discussed by Goad et al. (1984). They arrived at the following definition of a local covariance function : "A local covariance function is a special case of a global covariance function where the information content of wavelengths longer than the extent of the local area has been removed, and the information outside, but nearby, the area is assumed to vary in a manner similar to the information within the area." They suggest that the Tscherning/ Rapp models (Tscherning and Rapp, 1974) are used in order to adjust an expression to empirical values in a similar way as in the global case. Such covariance models have been investigated and related to the density distribution in the earth by Forsberg (1984).

In this paper the estimation and the modelling of the local empirical covariance function is discussed and results from the Faeroe Islands region are given. The spectral density is calculated from the empirical covariance function and problems in using Discrete Fourier Transform techniques for spectral estimations are discussed. Furthermore a Tscherning/Rapp model is fitted to empirical values of the local covariance function for free-air anomalies and geoid heights simultaneously.

2. The local covariance function

In the following T' is the anomalous potential T where the information content of wavelengths longer than the extent of a local area $(\varphi_1, \varphi_2, \lambda_1, \lambda_2)$ is subtracted. L and L' are two linear functionals associated with the observations $y = L(T')$ and $y' = L'(T')$ located at $(\varphi . \lambda)$ and (φ', λ'). If the averages of y and y' over the area are zero, then the local autocovariance of y (if $L = L'$) or the local crosscovariance between y and y' is given by

$$C(\psi) = \frac{1}{A} \int_{\varphi_1}^{\varphi_2} \int_{\lambda_1}^{\lambda_2} \frac{1}{2\pi} \int_0^{2\pi} yy' \, d\alpha \cos(\varphi) \, d\varphi \, d\lambda \tag{2.1}$$

where $\cos\psi = \sin\varphi \sin\varphi' + \cos\varphi \cos\varphi' \cos(\lambda - \lambda')$ (2.2)

A is the size of the area on a unit sphere,

α is the azimuth.

This rotational invariant representation of the covariance function is calculated as an average of the products yy' over the area (the two outer integrations) and an average over the azimuth (the inner integration). (Corresponding to homogeneity and isotropy respectively in a plane.) The integration over the local area is restricted to the observations y. When ψ goes from zero to π the observations y' are used located over the whole sphere. (See also Tscherning, 1985, and Sansò, 1986.)

In practice the observations are given in discrete points in the area and the calculation of the covariance function is then done by numerical integration. If each observation y_i represents a small area A_i and y_j' represents an area A_j then

$$C_k = \frac{\Sigma A_i A_j y_i y_j'}{\Sigma A_i A_j} \tag{2.3}$$

with $\psi_{k-1} < \psi_{ij} \leq \psi_k$ (2.4)

If the area is subdivided into small cells holding one observation each and A_i and A_j are assumed to be equal then equation (2.3) reduces to

$$C_k = \frac{\Sigma\, y_i y_j'}{N_k} \tag{2.5}$$

where N_k is the number of products, $y_i y_j'$, in the k'th interval.

Suppose T is expanded into spherical harmonics, and a global gravity potential approximation up to degree and order N is subtracted in order to obtain T'. Then the covariance function, $K(\psi)$, associated with T' is expressed by a sum of a series of Legendre polynomials of order i, P_i, (Heiskanen and Moritz, 1967)

$$K(\psi) = \sum_{i=0}^{N} \epsilon_i (T, T) \left(\frac{R_E^2}{rr'}\right)^{i+1} P_i (\cos\psi)$$

$$+ \sum_{i=N+1}^{\infty} \sigma_i (T, T) \left(\frac{R_B^2}{rr'}\right)^{i+1} P_i (\cos\psi) \tag{2.6}$$

where $\epsilon_i(T, T)$ are the error degree-variances related to the potential coefficient set,

$\sigma_i(T, T)$ are the potential degree-variances,

r, r' are the radial distances of y and y',

R_E is the mean earth radius,

R_B is the radius of a Bjerhammar sphere ($R_B < R_E$).

The integer N, relative to the size of the local area, is supposed to fulfil the condition that $2\pi/N$ is smaller than the extention of the area where the covariance function is determined. The degree-variances are positive numbers and are related to the spherical power spectrum of the earth gravity field. It is well known that the degree-variances tend to zero somewhat faster than i^{-3} and that the Tscherning/Rapp (1974) model 4 is a reasonable choice for a degree-variance model,

$$\sigma_i(T, T) = A/((i-1)(i-2)(i+24)) \tag{2.7}$$

where A is a constant in units of $(m/s)^4$. For geoid heights, ζ, and gravity anomalies, Δg, we then have the degree variances associated with the respective auto– and cross-covariance functions (γ is the normal gravity)

$$\sigma_i(\zeta, \zeta) = 1/(\gamma\gamma')\, \sigma_i(T, T) \tag{2.8}$$

$$\sigma_i(\Delta g, \Delta g) = (i-1)^2 / (rr')\, \sigma_i(T, T) \tag{2.9}$$

$$\sigma_i(\zeta, \Delta g) = (i-1)/(\gamma r')\, \sigma_i(T, T) \tag{2.10}$$

Using these expressions (2.6–10) it is possible in an easy way to compute the covariance values (see also Tscherning, 1976). In many cases it is possible to fit the expression (2.6) to covariance values related to linear functionals and then obtain the covariances associated with the potential field by applying the inverse linear functionals.

In a flat earth approximation the sphere is replaced by a plane and (φ, λ, ψ) is replaced by (x, y, s) where $s^2 = x^2 + y^2$. The power spectral density, or, more loosely, the power spectrum of T', $\phi(u, v)$, is then evaluated using a 2-dimensional Fourier transform. Let $\tilde{y}$ (and $\tilde{y}'$) be the Fourier transform of y (and y'), then

$$\tilde{y}(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y(x, y)\, e^{-i(ux + vy)}\, dx\, dy \tag{2.11}$$

and $\phi_{LL'}(u, v)$ the power spectrum associated with y and y'

$$\phi_{LL'}(u, v) = \tilde{y}(u, v)\tilde{y}'(u, v)^* \tag{2.12}$$

The power spectrum of T' is then obtained from $\phi_{LL'}(u, v)$ in a similar way, as in the spherical case, by applying the inverse linear functionals related to y and y' on the spectrum. Then the 2-dimensional covariance function, $K(x, y)$, is obtained using the inverse Fourier transform.

$$K(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(u, v)\, e^{i(ux + vy)}\, du\, dv \tag{2.13}$$

and the isotropic covariance function, $K(s)$, by averaging over the azimuth between origo and (x, y)

$$K(s) = \frac{1}{2\pi} \int_0^{2\pi} K(x, y)\, d\alpha \tag{2.14}$$

Using this procedure the integrations in eq. (2.1) are calculated as a convolution first and then an average over the azimuth.

If the power spectrum, $\phi(u, v)$, is isotropic (or had become isotropic by averaging over the azimuth in the frequency domain) the isotropic covariance function, $K(s)$, is obtained from eq. (2.13) where the Fourier transform is reduced to an inverse Hankel transform. Consequently the isotropic power spectrum is obtained from the isotropic covariance function using a Hankel transform (and not a 1-dimensional Fourier transform). That is

$$\phi(\omega) = \int_0^{\infty} K(s)\, J_0(s\omega)\, s\, ds \tag{2.15}$$

and

$$K(s) = \int_0^{\infty} \phi(\omega)\, J_0(\omega s)\, \omega\, d\omega \tag{2.16}$$

where $\omega^2 = u^2 + v^2$ and J_0 is the Bessel function of order zero.

For details cf. Nash and Jordan (1978) and Forsberg (1984).

These formulas (2.11–2.16) are given in an infinite plane and the spectrum, given in eq. (2.12), is continuous, but it becomes discrete in the local case if periodicity is assumed. Then the integration in eq. (2.11) becomes finite and the integration in eq. (2.13) and (2.16) reduces to a summation. The discrete values of the spectrum appear for wavelengths $(x_2 - x_1)/j$ and $(y_2 - y_1)/k$ in each direction, when j and k are positive integers. On a sphere this corresponds to harmonic degree $2\pi * j/(x_2 - x_1)$ and $2\pi * k/(y_2 - y_1)$ respectively. With a discrete data distribution the Discrete Fourier transform is used. When data are arranged in a regular grid, the integrations in eq. (2.11) are calculated as sums and the power spectrum becomes finite. The highest frequency which may be estimated depends on the spacing of the data, Δx and Δy, since the smallest wavelength is equal to two times the spacing, which on a sphere corresponds to harmonic degree $\pi/\Delta x$ and $\pi/\Delta y$,

For details on the application of the Fourier transform cf. Bracewell (1983).

Eq. (2.6) and eq. (2.16) express the covariance function in a spherical and a plane approximation respectively. In a local area these approximations converge to each other and there exists a link between the degree variances and the power spectrum (Forsberg, 1984)

$$\sigma_i \, (R_B/R)^{2i+2} = (i + 1/2) \; 1/(2\pi R) \; \phi\left((i + 1/2)/R\right) \tag{2.17}$$

where R is the distance from the center of the earth to the plane.

Normalized potential degree variances are obtained as dimensionless quantities through a division by $(GM/R)^2$.

The local covariance function can be determined in two ways. The first method is to evaluate eq. (2.1) using eq. (2.3) or eq. (2.5). The other method is to evaluate eq. (2.1) (given in planar coordinates) using the Discrete Fourier transform and an azimuth-average. The advantage of the second method is that the amount of computation is much smaller than in method one, and that the power spectrum is obtained during the computations. The disadvantage is that the data have to be arrenged in a regular grid.

3. The determination of a local covariance function

In this section it is described how the empirical covariance function is estimated from gravity and SEASAT altimeter data in the Faeroe Islands region. The preprocessing of the observations is discussed and results obtained using both methods (the space domain method using eq. (2.3) or (2.5) and the frequency domain method using eq. (2.11–14)) are presented. Furthermore the spectral properties of the results are evaluated. (See also Schwarz and Lachapelle (1980), Schwarz (1985), Rapp (1985), and Forsberg (1986)).

In the Faeroe Islands region $(60° < \varphi < 65°, \; 345° < \lambda < 360°)$ the amount of available data is on the order of 30000 gravity anomalies and 6100 altimeter data. The altimeter data had been adjusted using a local cross-over analysis. Then the altimeter data are assumed to be free from non-gravimetric signals and they may be treated like geoid heights. First a part of the two data sets was selected holding one value in each cell of $1/8° \times 1/4°$ covering the area (squares at $60°$ latitude). This resulted in 1854 gravity anomalies and 1740 altimeter values. The removal of the longwavelength part of the potential field was carried out by subtracting the contribution from GPM2 coefficient set (Wenzel, 1985) truncated at degree 180. This corresponds to wavelengths of 2 degrees which is far below the extent of the area and the residual data are expected to be centered.

The mean value of the gravity anomalies was close to zero, but the mean value of the altimetry was far from zero (-1.8 m). This means that the observations are not centered and equation (2.1) should not be used for the estimation of the covariance function. In order to fulfil the criterion of zero average the observations have to be centered. Therefore a locally best fitting reference system was defined and both kinds of observations were transformed into it. This ensures that different kinds of observations refer to the same reference system after the centering. The semimajor axis of this reference system is 6378135.2 m which is 1.8 m smaller relative to the GRS-80 (the flattening is the same). The differences between this local reference system and the global reference system do not necessarily mean that the global reference system is wrong, because errors in the lower degree coefficients of the spherical harmonic approximation will have the same effect. Mean values different from zero may also be caused by bias in the observations. Such effects have to be treated in the preprocessing of the observations.

The empirical covariance function was calculated using eq. (2.5) since the observations were representing areas of equal size. Only the observations were used and no values were estimated in the empty cells. From a distance of zero, calculating the variance (or crossvariance) of the observations, steps of 4 arc min were used as interval for the spherical distances, see eq. (2.4), and the values were assigned to the middle of each interval. This resulted in values of the auto-covariances for geoid heights and gravity anomalies and the cross-covariances between the two quantities. The estimated variances were 273 mgal^2 and 0.12 m^2 for the gravity anomalies and the altimetry respectively.

A kind of error, err_k, associated with each estimated covariance value may be obtained. The accuracy of the numerical integration (eq. (2.5)) depends on the variance and the number of observations. The spacing between the observations, located in cells of $\Delta\varphi \times \Delta\lambda$, is assumed to be small (relative to the second and higher order derivatives) so the function which is observed, is described by the data. The standard error which describes the error of a mean value, is equal to the standard deviation divided by the squareroot of the number of data : $\text{Std.err.} = \text{std.dev.}/n^{1/2}$ (Arley and Buck, 1950). In this case where the variance of the observations $(y = y')$ is estimated (eq. (2.5)), this quantity, C_0, divided by $N_0^{1/2}$ is used as an error estimate. If the crossvariance is estimated, C_0 is not used since it may be zero. Here the geometric mean, $C = (C_0 C'_0)^{1/2}$, of the variance of y, C_0, and the variance of y', C'_0, is used. This error estimate is then used as an error of C_k when one observation is located in each and every cell : $Err_k = C/n_0^{1/2}$ (first term of eq. (3.1)). n_0 is the number of cells covering the area. If there are empty cells the number of products in eq. (2.5) is decreased. Hence the errors are divided by the actual number of products, N_k, relative to the expected number of products, n_k, (second term of eq. (3.1)). n_k depends on the size of the cells the size of the area, the spherical distance, and the size of the interval. That is

$$\mathrm{err}_k = C/n_0^{1/2}\, n_k/N_k \tag{3.1}$$

where

$$C = (C_0 C_0')^{1/2}$$

$$n_k = \begin{cases} (\varphi_2 - \varphi_1)/\Delta\varphi\,(\lambda_2 - \lambda_1)/\Delta\lambda & k = 0 \\ n_0\, 2\pi\,(\psi_k + \psi_{k-1})/2\,(\psi_k - \psi_{k-1})/(\Delta\varphi\,\Delta\lambda) & k > 0 \end{cases}$$

Experiments with data in a square area have shown that the actual number of products is decreased to 75 % and 50 % of the expected number at distances of a quarter and a half of the length of the square respectively. The error of the values (eq. (3.1)) is therefore increased by 1/3 and 1/1 at those distances when only data inside the area are used in the computations.

The use of a Fourier transform requires that the data are arranged in a regular grid. Therefore it is necessary to predict the values in some specified points from the observations. Least squares collocation is a method for such an interpolation where the noise of the observations are taken into account. The amount of computations is rather large since a system of equations with a number of unknowns corresponding to the number of observations has to be solved. In order to save time approximative procedures have been developed. Weighted means and collocation using only the closest observations are some possibilities. The set of gravity observations selected in cells of $1/8° \times 1/4°$ covering the Faeroa Islands region were gridded using both least squares collocation and weighted means (balanced in azimuth through a quadrant search scheme). The mesh was $1/8°$ which corresponds to a harmonic degree of 1440 in a spherical harmonic expansion.

During the computations of the Fourier transform a 10 % cosine-tapered window was applied in order to avoid spectral leakage. The loss of power was subsequently re-established in the frequency domain through a multiplication factor and agreement with the variance of the data is obtained. The 2-dimensional covariance function was calculated from the power spectrum, using eq. (2.13), and made isotropic by an azimuth-averaging.

Figure 1 shows the empirical covariance function estimated from the gravity anomalies. The solid curve represents the result obtained in the space domain and the dashed curves represent the results obtained using the Fourier transform and data gridded by least squares collocation and weighted means respectively. The variances are : 273 , 175 , and 153 mgal^2 and the correlation lengths are : 0.17° , 0.21° , and 0.22° respectively. (The correlation length is the distance at which the covariance is 1/2 of the variance). This means that the results obtained using the Fourier transform have smaller variances and larger correlation lengths. This is not caused by the Fourier transform in it self, since it just is another method of calculating the integrals in eq. (2.1). Therefore it must be an effect of what has been done to the observations : The gridding, and the windowing.

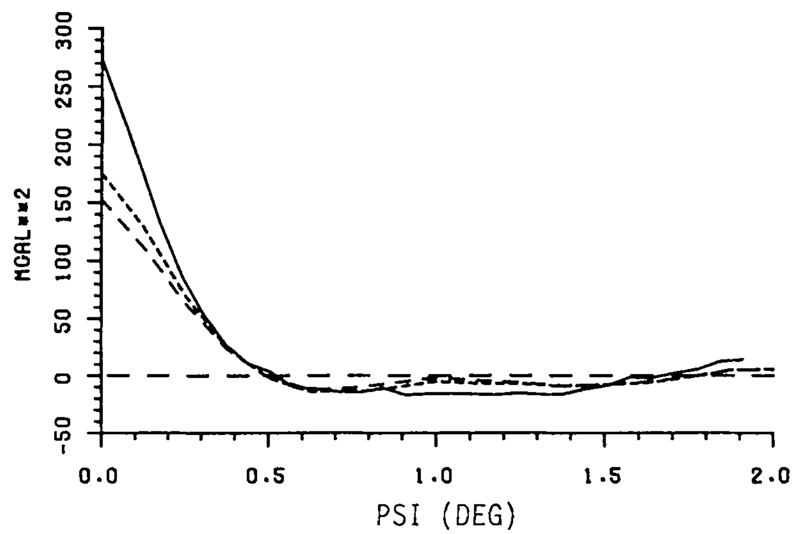

Fig. 1 — Local empirical covariance function estimated from gravity anomalies. The space domain method using original observations (——) and the frequency domain method using observations gridded by collocation (----) and weighted means (— —).

A comparison of the estimated covariance functions was done by a comparison of their power spectra. In the frequency domain different spectral contents are easily detected. The isotropic power spectra of the gridded data sets are obtained from the 2-dimensional spectra by an azimuth averaging in the frequency domain. The isotropic power spectrum of the observations are obtained from the covariance function calculated in the space domain using a Hankel transform (eq. (2.15)). From these spectra the degree variances are calculated using eq. (2.17) and degree variances associated with the potential field are obtained using eq. (2.9). *Figure 2* shows the results. (The result of the Hankel transform is influenced by some numerical noise which causes the fluctuations above harmonic degree 1000 .)

The effect of the windowing was tested by using different widths. A change of width from 5 % to 15 % was not found to change the content at the higher frequencies, but a small loss in the lower degree variances could be an effect of this windowing. However small deviations from the assumed periodicity could play a role. The effect of the gridding is remarkable (see *fig. 2*). The degree variances at harmonic degree 400 , 600 , 1000 , and 1400 of the collocation gridded quantities are decreased to approximately 0.79 , 0.56 , 0.31 , and 0.13 relative to the degree variances calculated from the original observations. The power of the decay in this interval is thereby decreased from approximately -3.8 to -5.0 . This means that the gridded observations are smoother than the original observations.

The differences between the original and the gridded observations show further how important it is not to mix original and gridded observations when the covariance function is calculated using eq. (2.5). The result will then be a mixture of the results from the two types of observations. It is therefore more correct to compute the

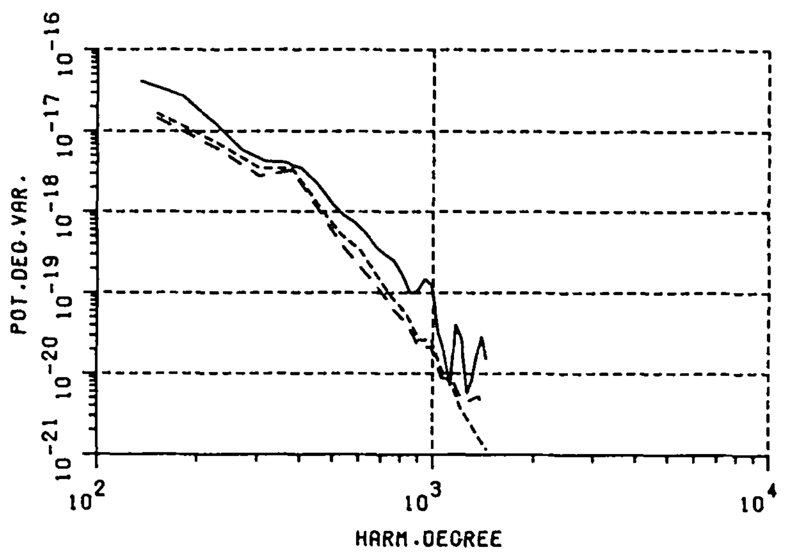

Fig. 2 — Normalized degree variances corresponding to the covariance functions in figure 1. The space domain method using original observations (——) and the frequency domain method using observations gridded by collocation (----) and weighted means (— —).

covariances without filling in estimated values in empty cells, and then evaluate the quality of the result using the error estimates (eq. (3.1)).

A complete interpretation of the smoothing is not possible. However one source of the smoothing is obvious. An averaging takes place when the observations are gridded. When no noise is taken into account both gridding procedures will reproduce the observations, but estimates in points between the data points will result in some weighted mean. The characteristics of this phenomena depend on the sampling density and the weight function. An increasing sample spacing will result in a more smooth function in the grid points. A broader weight function will have the same effect. Another source of the smoothing is the noise of the observations when it is taken into account in the gridding. Then as much signal as possible is interpretated implicitly as noise, and the result becomes as smooth as possible. The corresponding filter is the covariance function of the quantities related to the true gravity field deconvoluted with the observations (including the noise) (Hofmann-Wellenhof and Moritz, 1986). An example of another type of gridded observations is mean values in cells forming a grid. A moving average with a rectangular weight function corresponds to a multiplication of the spectrum by a sinc function $(\mathrm{sinc}(u) = \sin(u)/u)$.

The spacing of the gravity observations was approximately $1/8°$. This data set was used in both gridding procedures, but the noise, 5–10 mgal , was only taken into account in the collocation procedure. The weight function used in this procedure was a covariance function of the type in equation (2.1) having a correlation length of $0.17°$. In the procedure using weighted means, the five closest values in each quadrant were weighted by the reciprocal of the square of the distance. The results *(fig. 2)* indicate that the spacing between the observations is important, since the two grids are smoothend to approximately the same degree, or the spacing in combination with the weighting and the treatment of noise results in comparable smoothing effects. A filter representing the actual smoothing in the collocation procedure may be modelled as a Gaussian function with its argument divided by $0.15°$ as a compromise between the spacing and the correlation length. This filter explains about 90 % of the decrease in the power of the decay. The remaining 10% could be an effect of the treatment of the noise. The actual smoothing in the weighted means procedure has been a little more effective because the weight function is wider. A representation of the weight function is not possible because it is truncated at a distance depending on the position of the observations.

In this section the empirical covariance function has been estimated from observations in The Faeroe Islands Region. The removal of the information content of wavelengths longer than the extent of the area was done using the GPM2 potential coefficient set and the observations were centered by a transformation into a locally best reference system. Then the covariance function was determined using both the space domain and the frequency domain method. In order to use eq. (2.5) in the space domain method the observations were selected in square cells in order to obtain equal weights. The quality of the covariance values may then be evaluated using eq. (3.1). Then the observations were gridded. A comparison of the results obtained using the frequency domain and the space domain methods has shown that the gridding procedure has smoothed the observations to some degree. A compensation for this smoothing is therefore necessary if information of the original observations is evaluated. This is possible to a certain extent using a deconvolution with a filter representing the smoothing caused by the spacing between the observations and the weight function. The results showed that most of the decreasing power of the decay of potential degree variances was described by such a filter. Still many problems are left in the representation of the actual smoothing, since the noise of the observations interferes with the results.

4. The adjustment of a covariance function model

The determination of a covariance function expressed as a sum of an infinite series of Legendre polynomials is equivalent to a determination of the infinite number of degree-variances, see eq. (2.6). This is (of course) not possible without making some assumptions on the behaviour of the degree-variances. Therefore a model is chosen to express these coefficients. The choice of model is discussed in Moritz (1980) where several expressions are presented. The Tscherning/Rapp (1974) models are widely used in collocation because the covariance values are quickly determined using a closed form (and not a sum) for most of the gravity field related quantities. In this section a method of adjusting such a model in order to fit empirical values is shown. As empirical values the values obtained from the data in the Faeroe Islands region applying eq. (2.5) are used. As degree-variance model eq. (2.7) is used.

The adjustment of this model is done by an adjustment of the parameters : R_B, and A. Furthermore a factor, "a", is introduced as a third parameter. With this factor the error degree variances of GPM2 is scaled so they represent the quality of the approximation of the potential coefficient set in the local area and in the local reference system.

Let an example illustrate the behaviour of this covariance model when the three parameters are adjusted. For a fixed variance of the gravity anomalies, C_0 , the covariance function is described by the correlation length, ψ_1 , for gravity anomalies, and the variance, C_{hh} , for geoid heights. The adjustment of the model is carried out by an adjustment of R_B and "a" . The constant A is then determined from the value of C_0 (Tscherning, 1976). With $C_0 = 200\,\text{mgal}^2$ values of the covariance are determined for different values of R_B and a . *Figure 3* shows the correlation length for gravity anomalies as a function of R_B and a . The major part of the variation is caused by the variation of R_B while "a" is of minor importance. *Figure 4* shows the variance of geoid heights in a similar way. Here it is the factor "a" that causes the largest variations. This means that the factor is very important when information about the covariance function for geoid heights is taken into account. It is possible to fit covariance values associated with Δg only by the adjustment of R_B , but a simultaneous fit of covariance values associated with ζ is only obtained when the factor "a" also is adjusted.

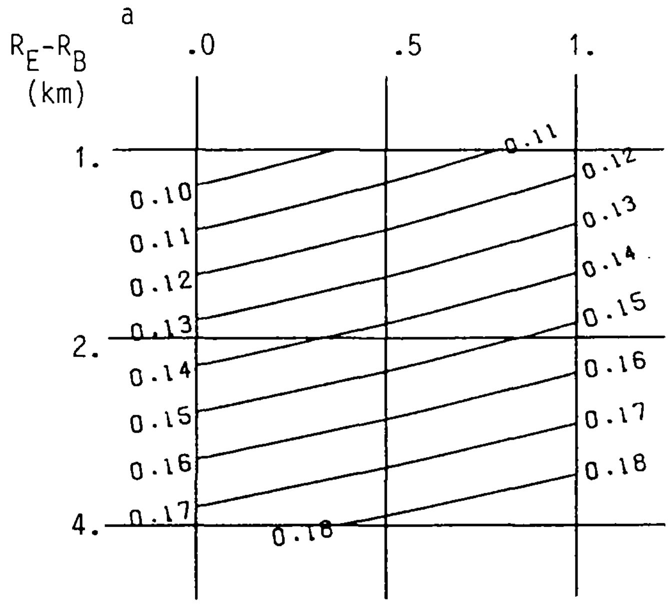

Fig. 3 — Correlation length in degrees of a covariance function for gravity anomalies with a variance of $200\,\text{mgal}^2$ *as a function of* $(R_E - R_B)$ *and* a .

The determination of the parameters from a number of empirical values is done using a least squares inversion procedure. The non-linear relationship between model and R_B is linearized and a solution may be found within some iterations. In each iteration the adjustment of the parameters, x_0 , is calculated using eq. (25) in Jackson (1979) :

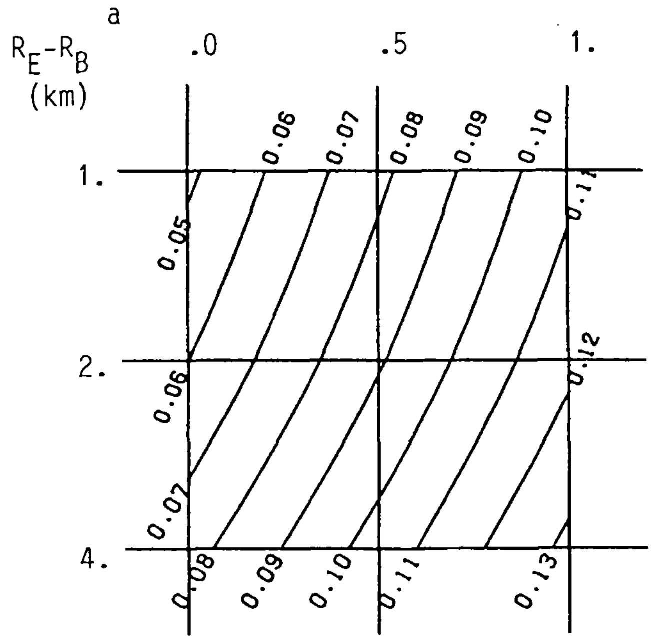

Fig. 4 — Variance in m^2 *of the covariance function for geoid heights corresponding to gravity anomaly covariance function in figure 3.*

$$x - x_0 = (A^T C_y^{-1} A + C_x^{-1})^{-1} A^T C_y^{-1} (y - y_0) \qquad (4.1)$$

where x is the adjusted model parameters from x_0

y is the empirical covariance values

y_0 is the values from the model using x_0

A is the Jacobian matrix $\partial y_i / \partial x_j$

C_y is error matrix of y

C_x is the variance matrix of $(x - x_0)$

In practice this equation is modified by multiplying each column in matrix A with the associated parameter. Then the adjusting values are calculated relatively as dimensionless quantities and computational problems due to the dimension are avoided. It is also appropriate to avoid a parameter like R_B since the adjustment of it is very small (e.g., 0.2 km per 6368 km). Therefore this parameter is substituted by the depth to the Bjerhammar sphere $(R_E - R_B)$ so the improvements are better represented (e.g., 0.2 km per 3 km).

The error matrix of y, C_y, is assumed to be diagonal and contains the square of the error estimates in eq. (3.1). The variance matrix, C_x, is also assumed to be diagonal.

Values different from zero in the diagonal of C_x^{-1} will stabilize the solution. The initial model, x_0, have to be selected (based on experience). $a = 0.5$, $A = 200\ \text{mgal}^2$, and $R_E - R_B = 2.5$ km and apriori variances of the relative adjustment, $(x - x_0)/x_0$, equal to 1.0 were chosen.

Results were calculated using three different combinations of the empirical covariance values as input : The gravity anomaly values (A), the geoid height values (B), and both kinds simultaneously (C). Other combinations and the use of the cross-covariance values are also possible. The three parameters and their estimated errors are shown in *Table 1*. The square of the estimated errors are obtained from the diagonal of the inverse matrix in eq. (4.1) without adding the C_x^{-1} matrix. The ability of the model to describe the empirical values, or the fitness, is measured by the Q value, where

$$Q^2 = 1/(n-m)\ (y - y_0)^T\ C_y^{-1}\ (y - y_0) \tag{4.2}$$

and n is the number of data and m is the number of parameters (=3). (A normalized version of the expression in Jackson, 1976.)

Table 1

Results of the estimation of a covariance function model from gravity covariance values (A), geoid covariance values (B), and both simultaneously (C).

	a		A (mgal²)		$R_E - R_B$ (km)		Q
A	0.77	+/− 0.28	208	+/− 21	3.15	+/− 0.48	0.45
B	0.22	0.04	164	35	0.10	2.49	0.73
C	0.21	0.04	219	11	3.17	0.34	0.75

In all of the three situations the model fits the empirical values within the level of the noise of these values. The Q values are less than but in the region of 1.0 which also means that the error estimates are realistical with a tendency to be overestimated. A solution was found after one or two iterations having a Q value that was less than 1 % larger than the value in *Table 1.* Then it was necessary to stabilize the adjustments by decreasing the apriori variances in the C_x matrix in order to obtain convergence. Never the less the oscillations were much smaller than the estimated errors and the Q values were almost constant, which means that one solution is just as good as another. Changes in the initial model was not found to have influence on the results.

The results show that the constant "a" is not well determined when input A is used. When input B is used it is the depth to the Bjerhammar sphere that is badly determined, but when the data are used simultaneously (input C), the parameters are determined accurately. A comparison of the three solutions, see also *Figure 5,* shows

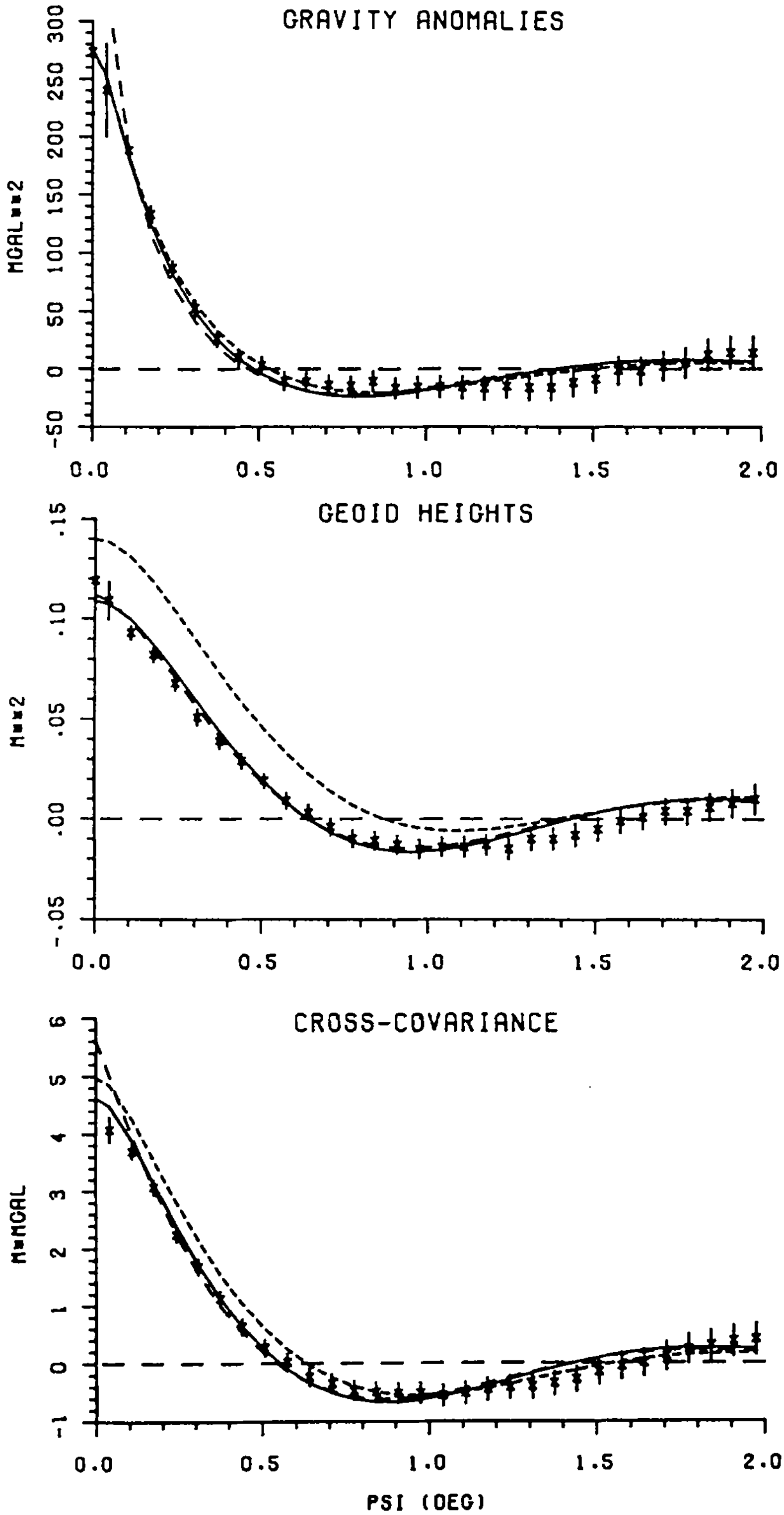

Fig. 5 — Results of the modelling of the empirical values (x) *with errorbars. Result from input* A *(- - - -), input* B *(— —), and input* C *(——) (see text).*

that solution A agrees with solution C in the determination of $(R_E - R_B)$ and the covariance values for gravity anomalies do also agree. A similar agreement exists between solution B and C in the determination of the factor "a" and the covariance values for geoid heights. For the other quantities are the covariance values different and solution C is the solution that fits the empirical cross-covariance values best $(Q = 0.65)$. There is no doubt that this solution is the most reliable model representing the local empirical covariance function.

A description of the covariance function using the parameters from solution C (see *Table 1* and *Figure 5*) is given through the following characteristic values : The variance, the correlation length, and the distance to the first zeropoint. For the gravity anomaly covariance function : $276\,\mathrm{mgal}^2$, $0.16°$, and $0.48°$. For the geoid height covariance function : $0.11\,\mathrm{m}^2$, $0.33°$, and $0.64°$. The cross-variance is $4.62\,\mathrm{m}*\mathrm{mgal}$ which corresponds to a correlation coefficient of 0.84.

5. Conclusion

In this paper some of the problems are treated which may occur in the determination of a local empirical covariance function. Only an isotropic covariance function have been considered, but the results are also of interest for an evaluation of a 2-dimensional covariance function. Using centered observations the computations were carried out in two ways : The method of numerical integration in the space domain and the method of using the Fourier transform based on gridded observations. The results showed that the gridding procedure had smoothed the observations considerably and that it was necessary to compensate for this smoothing. This was done by a deconvolution with a filter representing the smoothing. Many (unsolved) problems are associated with this filter representation, since the actual smoothing depends on the spacing and the noise of the observations and the weight function.

The determination of a covariance model representing the covariance function is important in least squares collocation. Here a Tscherning/Rapp model describing the degree variances by two parameters is used. Furthermore the error degree variances of the used reference potential coefficient set are multiplied by a factor so that they represent the error of the approximation in the local area. This model was successfully adjusted using an iterative least squares inversion procedure. As weights the empirically determined errors of the covariance values were used. Using both the covariance values for gravity anomalies and geoid heights a solution was found within 2 or 3 iterations having small errors of the estimated parameters.

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REFERENCES

N. ARLEY and K.R. BUCK : Introduction to the theory of probability and statistics. John Wiley, New York, 1950.

R.N. BRACEWELL : The Fourier transform and its applications. Second edition, McGraw-Hill, 1978.

R. FORSBERG : Local covariance functions and density distributions. Report No. 356, Department of Geodetic Science and Surveying, The Ohio State University, Columbus, 1984.

R. FORSBERG : Spectral properties of the gravity field in the nordic countries. Bull. Geod. Sci. Aff., in print, 1986.

B. HOFMANN-WELLENHOF and H. MORITZ : Introduction to spectral analysis. In : Suenkel, H. : Mathematical and Numerical Techniques in Physical Geodesy. Lecture Notes in Earth Sciences, Vol. 7, pp. 157–259, Springer-Verlag, 1986.

C.C. GOAD, C.C. TSCHERNING and M.M. CHIN : Gravity empirical covariance values for the continental United States. J. Geophys. Res., Vol. 89, No. B9, pp. 7962–7968, 1984.

W.A. HEISKANEN and H. MORITZ : Physical Geodesy, W.H. Freeman, San Francisco, 1967.

D.D. JACKSON : Most-squares inversion. J. Geophys. Res., Vol. 81, pp. 1027–1030, 1976.

D.D. JACKSON : The use of "a priori" data to resolve non-uniqueness in linear inversion. Geophys. J. R. astr. Soc., 57, pp. 137–157, 1979.

H. MORITZ . Advanced Physical Geodesy. Herbert Wichmann Verlag, Karlsruhe, 1980.

R.A. NASH and S.K. JORDAN : Statistical Geodesy – An engineering perspective. Proceed. of the IEEE, Vol. 66, No. 5, pp. 532–550, 1978.

R.H. RAPP : Detailed gravity anomalies and sea surface heights derived from Geos-3/Seasat altimeter data. Report No. 365, Department of Geodetic Science and Surveying, The Ohio State University, Columbus, 1985.

F. SANSÒ . Statistical methods in physical geodesy. In : Suenkel, H. : Mathematical and Numerical Techniques in Physical Geodesy. Lecture Notes in Earth Sciences, Vol. 7, pp. 49–155, Springer-Verlag, 1986.

K.P. SCHWARZ and G. LACHAPELLE : Local characteristics of the gravity anomaly covariance function. Bull. Géod., 54, pp. 21–36, 1980.

K.P. SCHWARZ : Data types and their spectrale properties. In : K.P. Schwarz : Local gravity field approximation, The University of Calgary, Alberta, No. 60003, pp. 1–66, 1985.

C.C. TSCHERNING and R.H. RAPP : Closed covariance expressions for gravity anomalies, geoid undulations, and deflections of the vertical implied by anomaly degree variance models. Report No. 208, Department of Geodetic Science and Surveying, The Ohio State University, Columbus, 1974.

C.C. TSCHERNING : Covariance expressions for second and lower order derivatives of the anomalous potential. Report No. 225, Department of Geodetic Science and Surveying, The Ohio State University, Columbus, 1976.

C.C. TSCHERNING : Local approximation of the gravity potential by least squares collocation. In : Schwarz, K.P. : Local gravity field approximation. The University of Calgary, Alberta, No. 60003, pp. 277–361, 1985.

H.–G. WENZEL : Hochaufloessende, Kugelfunktionsmodelle fuer das Gravitationspotential der Erde. Wiss. Arb. Fachrichtung Vermessungswesen der Universitaet Hannover, (in print), 1985.

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