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The Inverse Gravimetric Problem in Gravity Modelling

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ABSTRACT

The importance of a simultaneous modelling of mass anomalies and gravity anomalies in the high accuracy approximation of the gravity field has been recognized since several years: in fact after the blossoming of the collocation technique and related procedures, a new era of "topographic corrections" has started.

This calls for the necessity of deeper study of the inverse gravimetric problem. In this paper the essential role of gravity data in modelling lateral density variations is illustrated.

The dependence of the solutions from different types of mass anomalies is also discussed. In this framework in particular it is shown that the use of statistical concepts, modelling of exterior covariance functions depending on the mass modelling, is an important tool to analyze the likelyhood of these models: several proposal advanced in recent years are considered and discussed.

1. INTRODUCTION

One of the main purposes of geodesy is to determine the gravity field of the earth in the space outside its physical surface [13]. This purpose can be pursued without any particular knowledge of the internal density even if we don't know the exact shape of the physical surface of the earth, though this seems to entangle the two domains, as it was in the old Stoke's theory before the appearance of Molodensky's approach [10].

Nevertheless even when large, dense and homogeneous data sets are available, it has always been recognized that subtracting from the gravity field the effect of the outer layer of the masses (topographic effect) yields a much smoother field, which allows for computations with a much lower approximation error [16].

This is obviously more important when we have a sparse data set, so that any smoothing of the gravity field helps in interpolating between the data without raising the modelling error [3]: this approach is nowadays generally followed also because it has become very cheap in terms of computing times since the appearance of spectral techniques [14].

It is from this point of view that geodesists have become interested in the simultaneous estimation of the gravity field and of the internal density. Not to be said the determination of the density is in itself a very interesting target both on a local scale, for the purposes of exploration geophysics and on a global one, for the general purposes of the geophysics of the solid earth, i.e. the

physical modelling of the earth's interior.
In this sense the inverse gravimetric problem is a natural interface between geodesy and geophysics.
In this paper the authors review the state of their knowledge on this problem and summarize the many discussions they had over a time span of several years. Most of the material is therefore already known but for a few examples and discussions in §3 and §4 and for a new proposal in §6.

2. THE MAIN MATHEMATICAL FEATURES OF THE INVERSE GRAVIMETRIC PROBLEM

This paragraph summarizes some technical results on the i.g.p.; the less interested reader can skip most of it and read only the initial definition and the conclusions.
We like first of all to dwell a little on the precise definition of the inverse gravimetric problem (i.g.p.).

Definition 1. The inverse gravimetric problem is to determine the set of all the measures $\{\mu\}$ with support(*) on a given closed body B generating a given external gravity field, with potential u, through the Newton's operator

$$u(P) = \int_B \frac{1}{r_{PQ}}\, d\mu(Q) \qquad P \in \Omega = B^c \quad . \tag{2.1}$$

In this definition we have implicitely made a choice on two important points:

a) we split the problem of interpolating the exterior gravity field from a set of discrete data, which at least in principle can be

(*) We remind that the support of continuous function f is the closure of the set $\{P;\ f(P)\neq 0\}$: a measure μ has support in B if $\int f\, d\mu = 0$, $\forall f$ with support in $\Omega = B^c$.

solved by itself, from the problem of recovering the mass distribution (the measure) generating it;

b) we speak of the exterior gravity field, as given e.g. by the newtonian potential in Ω and we know that this is univocally determined for instance by the set of the values of some functional of u on the boundary $S=\partial B=\partial\Omega$; so we could give $u|_S$ or $\partial u/\partial\nu|_S$ or some combination of the two on different parts of the surface S or some oblique derivative $\partial u/\partial r|_S$; so if the real set of measurements is, for instance, $\partial u/\partial r|_S$ and we want to use in the i.g.p. $\partial^2 u/\partial r^2|_S$, we leave the transition from the two as an "external" (improperly posed) problem in the sense that from the first set of data it is possible in principle to derive (after suitable smoothing or regularization) the second set, knowing nothing of the internal density.

Obviously as any mathematical problem, it needs to be more specified in the sense that, looking for a solution means also to specify the space in which the solution is sought.

In our case it is unnecessary to work with such a large space as a general space of measures: so we shall pick up our examples mainly from cases where there is a body density ρ,

$$d\mu(P) = \rho(P)\, dB(P) \qquad (dB = \text{euclidean volume element}) \quad , \qquad (2.2)$$

or a surface density (single layer)

$$d\mu(P) = \delta(P)\, dS(P) \qquad (2.3)$$

Moreover we shall require our densities to be at least square integrable on their domains, $\rho \in L^2(B)$, $\delta \in L^2(S)$.

Concerning the inversion for a body density on B, we can report the following results [9].

Theorem 2.1. The equation

$$u(P) = \int_B \frac{\rho(Q)}{r_{PQ}}\, dB_Q \qquad (P \in \Omega) \qquad (2.4)$$

has at least one solution in $L^2(B)$ if the trace of u on the boundary S belongs to the hilbert space H_K with reproducing kernel

$$K(P,Q) = \int_B \frac{1}{r_{PM}} \frac{1}{r_{MQ}} dB_M \qquad (P,Q \in S) \tag{2.5}$$

Remark 2.1. It is possible to show that if S is suitably smooth, e.g. if it satisfies a cone condition, the space H_K coincides with the usual space of "traces" $H^{3/2}(S)$ used in the Hilbert theory of boundary value problems [7].

Theorem 2.2. The set of solutions of (2.4) in $L^2(B)$, when non empty, can be described as

$$\rho = \bar{\rho} + h \tag{2.5}$$

where

$$\begin{cases} \bar{\rho} \in L^2(B) \\ \Delta\bar{\rho} = 0 \quad \text{in } B \end{cases} , \tag{2.6}$$

i.e. $\bar{\rho}$ is the harmonic component of ρ and is uniquely determined from u; moreover h is the general solution of the corresponding homogeneous problem, i.e. the general square integrable distribution generating a null external field.

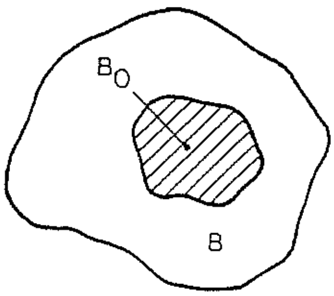
Remark 2.2. It is easy to show that $\bar{\rho}$ is characterized as being the element of minimum L^2 norm in the whole class of solutions.
Furthermore, the correspondence between $u|_S$ and $\bar{\rho}$ is an isometry, so that

$$||u|_S||^2_{H_K} = ||\bar{\rho}||^2_{L^2(B)} \tag{2.7}$$

Remark 2.3. The characterization of h is obviously that this function should be L^2 orthogonal to $\{1/r_{PQ}\}$ $\forall P \in \Omega$, and this, via a Runge-Krarup theorem, means that h is orthogonal to all L^2 harmonic functions in B. Another way to express the same thing, when S satisfies a cone condition, is the following:

$$h = \Delta w \quad ; \quad \{w \in \bar{H}^{2,2}(B),\ w|_S = \frac{\partial w}{\partial \nu}\Big|_S = 0\} \quad . \tag{2.8}$$

We can also put the question whether a certain gravity field u can be inverted to a density ρ_o with support in B_o, a body strictly contained in B. As it is obvious, this is possible first of all if the field u can be harmonically continued inside B, down to $S_o = \partial B_o$; second if the field thus continued satisfies some regularity condition. Again the condition for this to happen can be expressed in terms of a Hilbert space H_{Ko} with reproducing kernel K_o,



$$K_o(P,Q) = \int_{B_o} \frac{1}{r_{PM}} \frac{1}{r_{MQ}}\, dB_M \tag{2.9}$$

In fact we have theorem 2.3.

Theorem 2.3. The equation (2.4) admits at least a square integrable solution ρ_o with support in B_o, if

$$u|_S \in H_{Ko} \quad . \tag{2.10}$$

Remark 2.4. Since u can be continued to S_o, if (2.10) is satisfied, then the non-uniqueness of the solution has the same characteristics described in Theorem 2.2 and in particular the class of all the solutions is represented by

$$\begin{cases} \rho_o = \bar{\rho}_o + h_o \\ \Delta\bar{\rho}_o = 0 \qquad \text{in } B_o \qquad (\bar{\rho}_o \in L^2(B_o)) \\ h_o = \Delta w_o \qquad w_o \in H_o^{2,2}(B_o) \end{cases} \tag{2.11}$$

$$\left(w_o\big|_{S_o} = \frac{\partial w_o}{\partial \nu}\Big|_{S_o} = 0\right)$$

Remark 2.5. Beyond the problems of existence and uniqueness or rather non-uniqueness of the solution, we are mostly interested in the stability of the solution itself. In particular it is interesting to know what happens to the solution if we add to the data an error sufficiently wild, like an L^2 function.

In the first instance examined (inversion on all of B) we see by dint of Remark 2.2 that the solution ρ is regular (i.e. $\in L^2(B)$) if the data $u|_S \in H_K$: if S is regular this means essentially that $u|_S \in H^{3/2}(S)$ i.e. it is "one and half times differentiable".

So, if we add an $L^2(S)$ perturbation to $u|_S$ we can expect ρ to blow up. Whence the inversion is not properly posed with respect to L^2 perturbations and a certain amount of smoothing of the data is always necessary, when these are affected by errors.

Much worse is the situation with the inversion on an inner body B_o: in fact this implies the harmonic continuation to the surface S_o which is a highly unstable operation, the more unstable the deeper the field has to be continued in B.

Remark 2.6. The non uniqueness of the solution of the i.g.p. has to be traced back to that phenomenon which is known as the "sweeping out property": loosely speaking it stems for the fact that we can move (sweep) the masses from the body B on its boundary without changing the exterior field. The homogeneous ball generating an external radial field, exactly as a suitable homogeneous single layer on its surface, is a fitting example.

Now we consider the inversion in terms of single layers, i.e. we seek to solve an equation of the form

$$u(P) = \int_S \frac{\delta(Q)}{r_{PQ}} \, dS_Q \qquad (P \in \Omega, \ S \subset \bar{B}) \qquad (2.12)$$

with a density $\delta \in L^2(S)$.

We recall that for a single layer it is classical the use of normal derivatives, so that it seems mostly reasonable to represent $u(P)$ in terms of $\partial_\nu u|_S$. When u is so regular that at least

$$\partial_\nu u|_S \in L^2(S) \quad , \qquad (2.13)$$

then the following classical theorem holds.

Theorem 2.4. When S is suitably regular, there is one and only one density $\delta \in L^2(S)$ such that the external normal derivative of (2.12) coincides with the given $\partial_\nu u|_S$. The dependence of δ on $\partial_\nu u|_S$ is continuous in $L^2(S)$.

Different are things when we want to invert a given field in terms of a single layer internal to B, i.e. lying on a surface $S_0 \subset B$.

When S is a simple regular surface, with no contact with the boundary S, it is easy to see that a solution, if it exists, is unique. Moreover, like in theorem 2.3 we have:

Theorem 2.5. A solution of (2.12) with an internal surface S_0 exists if $u|_S$ belongs to the Hilbert space H_{Ko} with reproducing kernel

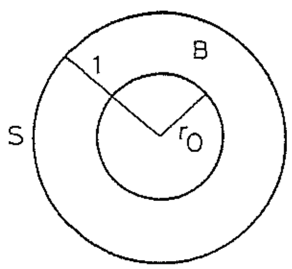
$$K_o(P,Q) = \int_{S_o} \frac{1}{r_{PM}} \frac{1}{r_{MQ}} \, dS_M \qquad (P,Q \in S) \quad . \tag{2.14}$$

To conclude the paragraph we can say that the i.g.p. is dominated by a typical non uniqueness of the solution which is eliminated when we fix in one way or another the "radial" behaviour of the solution so that the information coming from the external field is used to determine the lateral variations of the density. This happens when we look for a minimum L^2-norm solution, giving rise to an internal harmonic density, which then depends only on its "boundary values"; the same reasoning holds for a single layer where there is no radial variation at all of the density, outside the support of the layer.
Another point that comes out from our discussion is that each time that we want to invert a given field on a support deep inside B, if there is any solution it must display a substantial instability with respect to the data because of the analytic continuation of the exterior field inside B.

3. EXAMPLES

In this paragraph we want to present some examples where the general properties discussed in §2 happen to be very clear. All examples are taken in a form relevant for a global inversion of the gravity field: for each of them however a local version could be elaborated.

Example 3.1. (<u>Harmonic density in the sphere</u>). We take a body B, constituted by a sphere whose radius we assume to be unitary for the sake of simplicity. The gravity field u is given in Ω and by virtue of the sphericity of the domain, it can be represented in terms of a spherical harmonic series as



$$u(r,\sigma) = \sum_{n,m} \frac{u_{nm}}{r^n} Y_{nm}(\sigma) \quad , \tag{3.1}$$

the series being convergent for $r \geq 1$.
We assume that $\{Y_{nm}\}$ are the fully normalized, real spherical harmonics,

$$\frac{1}{4\pi} \int Y^2_{nm} \, d\sigma = 1 \quad .$$

We want to invert (3.1) with a density $\bar{\rho}_o$, harmonic in the sphere B_o. From the representation of $\bar{\rho}_o$ in terms of

$$\bar{\rho}_o = \Sigma \, \rho_{nm} \, r^n \, Y_{nm}(\sigma) \qquad\qquad r \leq r_o \tag{3.2}$$

and the well known formula

$$\frac{1}{r_{PQ}} = \Sigma_n \frac{r_Q^n}{r_P^{n+1}} P_n(\cos \theta_{PQ}) = \\ = \sum_{n,m} \frac{r_Q^n}{r_P^{n+1}} \frac{1}{2n+1} Y_{nm}(\sigma_P) \, Y_{nm}(\sigma_Q) \tag{3.3}$$

$$(r_P \geq 1, \quad r_Q \leq 1) \quad ,$$

it is easy to compute the Newton's integral

$$\int_{B_o} \frac{\bar{\rho}_o(Q)}{r_{PQ}} \, dB = 4\pi \sum_{n,m} \frac{\rho_{nm}}{(2n+1)(2n+3)} \frac{r_o^{2n+3}}{r_P^{n+1}} Y_{nm}(\sigma_P) \quad . \tag{3.4}$$

Comparing with (3.1) we derive the relation of inversion

$$\rho_{nm} = \frac{(2n+1)(2n+3)}{4\pi} \frac{u_{nm}}{r_o^{2n+3}} \quad . \tag{3.5}$$

As we see the correspondence (3.5) is one to one, because the radial variation of $\bar{\rho}_o$ has been fixed by the harmonic hypothesis [cfr. (3.2)].

On the other hand, if we want the coefficients ρ_{nm} to represent a function through (3.2), we must impose regularity conditions on u. For instance, if we want $\bar{\rho}_o \in L^2(B_o)$, since

$$\int_{B_o} \bar{\rho}_o^2 \, dB = 4\pi \sum_{n,m} \frac{\rho_{nm}^2 \, r_o^{2n+3}}{(2n+3)} \quad ,$$

we see that u must fulfill the condition

$$\frac{1}{4\pi} \sum_{n,m} (2n+1)^2(2n+3) \frac{u_{nm}^2}{r_o^{2n+3}} < +\infty \quad . \tag{3.6}$$

We can observe that (3.6) is particularly stringent when $r_o < 1$ (B_o strictly interior to B), and only when $r_o = 1$, the condition is exactly equivalent to $u|_S \in H^{3/2}$, if we accept the thumb-rule, one derivative of u is equivalent to multiplication of u_{nm} by a $O(n)$ coefficient.

Example 3.2. (<u>Single layer for the sphere</u>). We continue the case of the previous example, but looking for the inversion in terms of a single layer on S_o (radius r_o). We can put

$$\delta(\sigma) = \sum_{n,m} \delta_{nm} Y_{nm}(\sigma) \tag{3.7}$$

and compute ($r_P \geq 1$, $r_Q \leq 1$)

$$\int_{S_o} \frac{\delta(Q)}{r_{PQ}} dS = 4\pi \sum_{n,m} \frac{\delta_{nm}}{2n+1} \frac{r_o^{n+2}}{r_P^{n+1}} Y_{nm}(\sigma_P) \tag{3.8}$$

Comparing with (3.1) we find the inversion equation

$$\delta_{nm} = \frac{(2n+1)}{4\pi \, r_o^{n+2}} u_{nm} \quad . \tag{3.9}$$

Once again (3.9) shows the uniqueness of the solution. As for the regularity, if we want to impose

$$\int_{S_o} \delta^2 \, dS = 4\pi r_o^2 \sum_{n,m} \delta_{nm}^2 < +\infty \quad ,$$

we find the condition

$$\frac{1}{4\pi} \sum_{n,m} \frac{(2n+1)^2}{r_o^{2n+2}} u_{nm}^2 < +\infty \quad . \tag{3.10}$$

As we see this condition is strong for $r_o < 1$ (single layer inside B), and it is equivalent to the requirement $\partial_r u|_S \in L^2(S)$ when $r_o = 1$.

Remark 3.1. In spite of its triviality this example has a relevant geophysical interpretation since it comes out in the interpretation of B as a layered body. In fact, assume that the field B is known to be generated by a body B with external spherical shape and only two constant densities inside, ρ_1, ρ_o (cfr. Fig. 3.1) separated by a surface S_o: assume further that S_o is a perturbation of a sphere so that

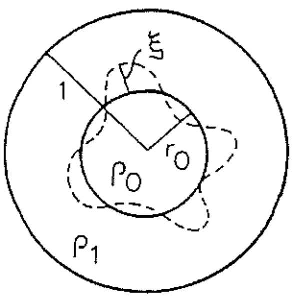


Fig. 3.1

$$r|_{S_o} = r_o + \zeta \tag{3.11}$$

with $\zeta = \zeta(\sigma)$ infinitesimal.

In the corresponding Newton integral we have a purely spherical term M/r constant on S, where M corresponds to the unperturbed spherical layers

$$M = \rho_o \cdot \frac{4}{3} \pi r_o^3 + \rho_1 \cdot \frac{4}{3} \pi (1 - r_o^3) \quad , \tag{3.12}$$

while the lateral variations of the field are due to the undulations according to the formula(*)

(*) If r_o is chosen in such a way that u_o equals the average of u on S_o, then also the average of ζ has to be zero, i.e. r_o is the radius of S_o.

$$u(P) - u_o = \int d\sigma \int_{r_o}^{r_o+\zeta} \frac{(\rho_o-\rho_1)}{r_{PQ}} r_Q^2 \, dr_Q \cong$$

$$\cong (\rho_o-\rho_1) \int d\sigma \frac{r_o^2 \zeta(\sigma)}{r_{PQ}} \quad . \tag{3.13}$$

This represents a single layer on the internal sphere of radius r_o, with density

$$\delta = (\rho_o - \rho_1) \zeta(\sigma) \quad . \tag{3.14}$$

Whence solving the corresponding (linearized) inverse problem, namely deriving δ from the lateral variations of u, amounts to determining the undulation ζ , as soon as the density contrast $\rho_o-\rho_1$ is known. This is the situation, for instance, with the core-mantle topography.

Example 3.3. (Lateral variations in a thick layer). This example is somewhat more general in that it assumes no special shape for the external body. We assume again that we have a two layered model with separation surface S_o (cfr. Fig. 3.2): we suppose further that S and S_o are sufficiently smooth, star shaped surfaces, i.e. they meet only once each radius issuing from the origin O, under an angle strictly smaller than $\pi/2$. The internal density ρ_o is assumed to be known, the density ρ of the external layer is unknown, but we stipulate that it has to be constant along the radius

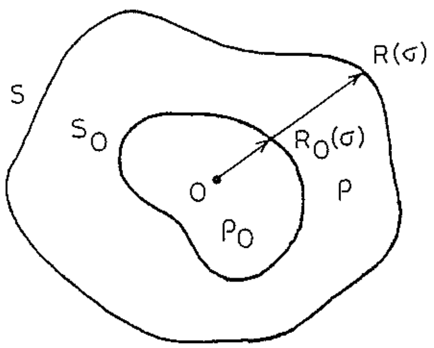


Fig. 3.2

$$\rho(P) = \rho(\sigma_P) \quad . \tag{3.15}$$

We want to recover the unknown $\rho(\sigma_P)$ from the known external potential $u_1(P)$.
Considering that $u_1(P)$, $P \in \Omega$, is composed by the sum of the potential $u_o(P)$ of the known internal body B_o and the potential $u(P)$ of the external layer we see that our problem reduces to look for the solution of

$$u(P) = u_1(P) - u_o(P) = \int_{B\setminus B_o} \frac{\rho(Q)}{r_{PQ}} \, dB = \tag{3.16}$$
$$= \int d\sigma_Q \, \rho(\sigma_Q) \int_{R_o(\sigma)}^{R(\sigma)} dr_Q \, r_Q^2 \frac{1}{r_{PQ}} \quad ,$$

where $r = R_o(\sigma)$, $r = R(\sigma)$ are the equations in polar coordinates of S_o and S.
Written in this way (3.16) is essentially an integral Fredholm equation of the 1^{st} kind and thus improperly posed, however we want to show that it can be turned to a properly posed one if we assume $u(P)$ to be so regular as to admit second radial derivatives(*) at the boundary S.
In fact let us apply the differential $-r_P(\partial/\partial r_P)$ to (3.16): taking into account that

$$-r_P \frac{\partial}{\partial r_P} \frac{1}{r_{PQ}} = (r_Q \frac{\partial}{\partial r_Q} + 1) \frac{1}{r_{PQ}}$$

we get $(P \in \Omega)$

$$\int d\sigma_Q \, \rho(\sigma_Q) \frac{R^3(\sigma_Q)}{r_{PQ}} - \int d\sigma_Q \, \rho(\sigma_Q) \frac{R_o^3(\sigma_Q)}{r_{PQ}} - 2 \int_{B\setminus B_o} \frac{\rho(\sigma_Q)}{r_{PQ}} \, dB = \tag{3.17}$$
$$= -r_P \frac{\partial}{\partial r_P} u(P) \quad .$$

To avoid anbiguities we specify that in (3.17) Q in the first integral

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(*) This happens for instance if $u(P)$ has locally square integrable third derivatives in Ω.

runs over S, in the second integral it runs over S_o and in the third one throughout the layer $B\backslash B_o$; σ_Q is the projection of r_Q on the unit sphere, while P is any point in Ω down to the boundary S.

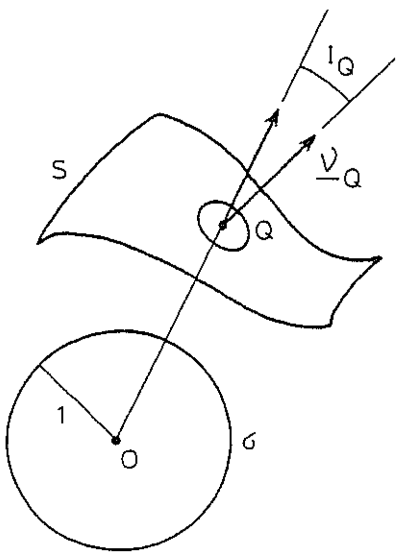

Fig. 3.3

Clearly (3.17) is still improperly posed. However, if we now apply the external normal derivatives $\partial/\partial\nu_P$ to (3.17) with P on S, we derive a regular Fredholm equation of the second kind. In order to do that exploiting the usual properties of the normal derivative of a single layer we note that (I_Q being the angle between ν_Q and r_Q)

$$dS_Q = R^2(\sigma_Q) \cos I_Q \, d\sigma_Q$$

so that we can substitute $\sec I_Q dS_Q$ to $R^2 d\sigma_Q$ in the first term of (3.17) thus obtaining the classical expression of a single layer on S with density

$\rho(\sigma_Q)\, R(\sigma_Q) \sec I_Q$.

Whence we derive now from (3.17)

$$- \frac{1}{2}\, \rho(\sigma_P)\, R(\sigma_P) \sec I_P + \frac{1}{2} \int_S dS_Q\, \rho(\sigma_Q)\, R(\sigma_Q) \sec I_Q \frac{\partial}{\partial\nu_P} \frac{1}{r_{PQ}} +$$

$$- \overset{(*)}{\int d\sigma_Q\, \rho(\sigma_Q)\, R_o^3(\sigma_Q) \frac{\partial}{\partial\nu_Q} \frac{1}{r_{PQ}}} - \int d\sigma_Q\, \rho(\sigma_Q)\, K(P,Q) = \qquad (3.18)$$

$$= \frac{\partial}{\partial\nu_P} \{- r_P \frac{\partial}{\partial\nu_P} u\}\Big|_S \quad ,$$

(*) Note that if S_o has no contact with S, what we suppose, the kernel of this integral is absolutely regular.

where

$$K(P,Q) = 2 \int_{R_o(\sigma_Q)}^{R(\sigma_Q)} dr_Q \; r_Q^2 \frac{\partial}{\partial \nu_P} \frac{1}{r_{PQ}} \quad . \tag{3.19}$$

If we consider (3.18) as an integral equation in $L^2(\sigma)$ we see that it is a regular Fredholm equation, on condition that $I_P > \pi/2$ everywhere on σ: in fact the first kernel

$$- \frac{\partial}{\partial \nu_P} \frac{1}{r_{PQ}} = \frac{\underline{\nu}_P \cdot \underline{r}_{PQ}}{r_{PQ}^3}$$

is well known to be compact in $L^2(\sigma)$, the second is even analytical on S, the third is known to transform functions in $L^2(B \backslash B_o)$ into functions in $H^{1/2}(S)$ so that the same is true for functions radially constant as

$$||\rho||_{L^2(B\backslash B_o)} = \int d\sigma \; \rho^2(\sigma) \int_{R_o}^{R} r^2 dr \leq c \; ||\rho(\sigma)||_{L^2(\sigma)} \quad . \tag{3.20}$$

As for every Fredholm equation, the existence of a solution of (3.18) derives from its uniqueness. This, in turn, can be proved along the following line: let ρ be a solution of the homogeneous equation (3.18), then considering the potential u generated by this ρ [cfr. (3.16)] we see that it must satisfy a boundary relation of the type(*)

$$\frac{\partial}{\partial \nu_P} \{-r_P \frac{\partial}{\partial r_P} u\}\Big|_S = 0 \quad ,$$

so that we must have identically $u=0$ in Ω.
But then ρ is generating a zero external field, what implies that ρ has to be $L^2(B)$ orthogonal to all L^2-functions harmonic in B (cfr. Remark 2.3).
Now if we choose this harmonic function to be of the form $r \frac{\partial}{\partial r} h$, with h again an arbitrary harmonic function in B, we see that

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(*) Note that if $\rho \in L^2(\sigma)$ we can follow the derivation (3.16)->(3.17) and from (3.17) the existence of $\frac{\partial}{\partial \nu_P} \{-r \frac{\partial u}{\partial r}\}$ is guaranteed by classical theorems on single layers.

$$0 = \int_B \left(r \frac{\partial}{\partial r} h\right) \rho \, dB = \int_{B \setminus B_o} \left(r \frac{\partial}{\partial r} h\right) \rho \, dB =$$

$$= \int_S d\sigma_Q \, r^3(\sigma_Q) \, \rho(\sigma_Q) \, h(Q) - \int_{S_o} d\sigma_Q \, r_o^3(\sigma_Q) \, \rho(\sigma_Q) \, h(Q) +$$

$$+ 3 \int_{B \setminus B_o} h \, \rho \, dB \quad .$$

On the other hand the last term is again an L^2 product with a harmonic function and therefore it is zero by itself: whence we derive the identity

$$\int_S d\sigma_Q \, r^3(\sigma_Q) \, \rho(\sigma_Q) \, h(Q) - \int_{S_o} d\sigma_Q \, r_o^3(\sigma_Q) \, \rho(\sigma_Q) \, h(Q) = 0 \qquad (3.21)$$

$$(\forall h, \text{ harmonic in } B)$$

Now if we choose h on the boundary in such a way that

$$h(Q)\big|_S = \begin{cases} +1 & \rho(\sigma_Q) \geq 0 \\ -1 & \rho(\sigma_Q) < 0 \end{cases} \quad , \qquad (3.22)$$

due to the maximum property of harmonic functions, we know that

$$1 > \text{Max } h\big|_{S_o} > \min h\big|_{S_o} > -1 \quad ,$$

so that

$$r^3(\sigma) \, h(Q)\big|_S \, \rho(\sigma) = r^3(\sigma) \, |\rho(\sigma)| \geq r_o^3(\sigma) \, |h(Q)|\big|_{S_o} \, \rho(\sigma) \quad ,$$

the equality sign being valid only when $\rho(\sigma) = 0$.
Whence we see that (3.21) can be verified for h as in (3.22) only if $\rho = 0$, i.e. unicity is proved.
The line of the proof follows strictly the approach of [1]. The authors have never met this example, at least in the present form, in the literature: that's why it is presented in a more lengthy fashion.

Remark 3.2. We note that after arriving at (3.17) we could apply again a radial derivative operator $\partial/\partial r_P$, still obtaining an integral equation, however with a strongly singular kernel, due to the behaviour of an oblique derivative of a single layer.

Example 3.4. (Thick spherical layer). We solve explicitely the problem of the Example 3.3 when S and S_o are spheres of radius R, R_o respectively.
If ρ is function of σ only it admits the series development

$$\rho(\sigma) = \Sigma_{n,m}\, \rho_{nm}\, Y_{nm}(\sigma) \quad , \tag{3.23}$$

independent of r.
Recalling (3.3) we have then

$$\begin{aligned} u(P) &= \int_{B\setminus B_o} \frac{\rho(Q)}{r_{PQ}}\, dB = \\ &= 4\pi \sum_{n,m} \rho_{nm} \frac{R^{n+3} - R_o^{n+3}}{r_P^{n+1}} \frac{Y_{nm}(\sigma_P)}{(2n+1)(n+3)} \quad . \end{aligned} \tag{3.24}$$

Since in this case the normal ν_P is directed radially, we can consider as boundary values

$$w(\sigma_P) = -\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} u\right)\Big|_S$$

so that the known function w has spherical coefficients

$$w_{nm} = -\frac{(n+1)^2}{R}\, u_{nm} \quad . \tag{3.25}$$

Comparing with (3.24), after setting $r_P = R$, we get

$$w_{nm} = -4\pi R \left[1 - \left(\frac{R_o}{R}\right)^{n+3}\right] \frac{(n+1)^2}{(2n+1)(n+3)}\, \rho_{nm} \tag{3.26}$$

which is easily solved for ρ_{nm}.
An interesting point to underline is that if we consider $-\frac{\partial}{\partial r} r \frac{\partial u}{\partial r}$ as

observable, to determine ρ becomes a well posed problem, as predicted in Example 3.3.
Another point is that it is clear in this example that we have found a unique solution exactly because we have fixed the radial variation of ρ.

4. THE RELATION BETWEEN THE STATISTICAL BEHAVIOUR OF ρ AND u

As we know the principle of minimum norm estimation applied to the external potential in a reproducing kernel Hilbert space is equivalent to the statistical Kolmogorof principle of minimizing the mean square estimation error, when the reproducing kernel is considered as the covariance function of the field [12].
Via the solution of the i.g.p. this sheds some light on the statistical behaviour of the internal density ρ.
Obviously, due to the non uniqueness of the solution of the i.g.p., there will be no uniquely defined covariance function for the density ρ.
As the radial behaviour of ρ remains unrevealed by the outer gravity field, accordingly there is no means to know the correlation of ρ at different depths, unless some further constraint is imposed.
To simplify matters we shall assume our body B to be a ball of radius 1.
Now we summarize the relations between ρ and u (more precisely $u|_S$) in the following form:

$$u = \Sigma\, u_{nm}\, Y_{nm}(\sigma) \quad , \quad \rho = \sum_{n,m} \rho_{nm}(r)\, Y_{nm}(\sigma)$$

$$u_{nm} = \frac{1}{2n+1} \int_o^1 \rho_{nm}(r)\, r^{n+2}\, dr \tag{4.1}$$

Since the covariance function for a gravity potential u in a spherical domain is usually defined by averaging over all rotated configurations with equal probability,

$$C_{uu}(\psi_{PQ}) = E\{u(P)\, u(Q)\}^{(*)} =$$
$$= E_\omega\{u(R_\omega \underline{r}_P)\, u(R_\omega \underline{r}_Q)\} \quad , \quad (R_\omega \text{ rotation operator}) \tag{4.2}$$

the only chance to gain statistical information on ρ is to apply the <u>same</u> operator. Namely we must have

$$C_{\rho\rho}(r_P, r_Q, \psi_{PQ}) = E_\omega\{\rho(R_\omega \underline{r}_P)\, \rho(R_\omega \underline{r}_Q)\} \quad . \tag{4.3}$$

Now let us recall the well known averaging property (cfr. [12], formulas (6.23),(6.27))

$$E\{Y_{nm}(R_\omega \sigma_P)\, Y_{n'm'}(R_\omega \sigma_Q)\} = \delta_{nn'}\, \delta_{mm'}\, P_n(\cos \psi_{PQ}) \tag{4.4}$$

From (4.1),(4.2),(4.3),(4.4) we derive then

$$\begin{cases} C_{uu}(\psi_{PQ}) = \Sigma\, \sigma^2{}_{u,n}(2n+1)\, P_n(\cos \psi_{PQ}) \\ \sigma^2{}_{u,n} = \frac{1}{2n+1} \Sigma_m\, u^2{}_{nm} \quad , \end{cases} \tag{4.5}$$

$$\begin{cases} C_{\rho\rho}(r_P, r_Q, {}_{PQ}) = \Sigma\, \sigma_{\rho,n}(r_P, r_Q)(2n+1)\, P_n(\cos \psi_{PQ}) \\ \sigma_{\rho,n}(r_P, r_Q) = \frac{1}{2n+1} \Sigma_m\, \rho_{nm}(r_P)\, \rho_{nm}(r_Q) \end{cases} \tag{4.6}$$

where $\sigma^2_{u,n}$, $\sigma_{\rho,n}(r_P, r_Q)$ are the degree variances of u, and the "degree covariances" of ρ respectively.
From (4.1) we see that

$$\sigma^2_{u,n} = \frac{1}{(2n+1)^2} \int_0^1 dr_1 \int_0^1 dr_2\, \sigma_{\rho,n}(r_1, r_2)\, r_1^{n+2}\, r_2^{n+2} \tag{4.7}$$

(*) It is important to realize that (4.2) needs to be defined only on the boundary S as the external radial variation of $C(r_P, r_Q, \psi_{PQ})$ is fixed by the harmonic continuation, which derives from the same property of u.

Therefore all what we can know on the degree covariances from the external field, is contained in (4.7).
Naturally (4.7) allows us to identify $\sigma_{\rho,n}$ only for particular models.

Example 4.1. (Harmonic densities). If we assume

$$\rho_{nn}(r) = \rho_{nm} r^n$$

we find from (4.7)

$$\begin{cases} \sigma^2_{u,n} = \dfrac{\sigma^2_{\rho,n}}{(2n+1)^2(2n+3)^2} \\ \\ \sigma^2_{\rho,n} = \dfrac{1}{2n+1} \Sigma_m \rho^2_{nm} \quad , \end{cases} \tag{4.8}$$

which, solved for $\sigma^2{}_{\rho,n}$, gives for the degree covariances

$$\sigma_{\rho,n}(r_1,r_2) = \sigma^2_{\rho,n} \cdot (r_1 \cdot r_2)^{n+1} \quad . \tag{4.9}$$

Example 4.2. (Single layer). If we take a single layer distribution on an internal sphere of radius R, we have

$$\rho_{nm}(r) = 4\pi \, \delta_{nm} \, \delta(r-R) \quad ,$$

so that

$$\begin{cases} \sigma^2_{u,n} = (4\pi)^2 \dfrac{\sigma^2_{\delta,n}}{(2n+1)^2} R^{2n+4} \\ \\ \sigma^2_{\delta,n} = \dfrac{1}{2n+1} \Sigma_m \delta^2_{n,m} \end{cases} \tag{4.10}$$

Remark 4.1. Beyond the covariance of the single layer density, also the cross-covariance between u and δ, $C_{u,\delta} = E\{u(\sigma_P)\ \delta(\sigma_Q)\}$, with "degree cross-covariances"

$$\sigma_{u\delta,n} = \sigma^2_{u,n} \frac{2n+1}{4\pi\ R^{n+2}} \quad , \tag{4.11}$$

can be derived from $\sigma^2_{u,n}$, known from exterior observations.
This opens the way to an approach of the type "collocation" to the problem of estimating δ, i.e. according to Remark 3.1, from observations on the gravity field.
This approach is undergoing a very positive testing on a local scale inversion problem.

Example 4.3. (<u>Radially constant density in a layer R_1, R_2</u>). Assume ρ to be radially constant in the layer $0 \leq R_1 \leq r \leq R_2 \leq 1$,

$$\rho = \Sigma\ \rho_{nm}\ Y_{nm}(\sigma) \quad ;$$

than we find straightforwardly from (4.1),(4.7)

$$\sigma^2_{u,n} = \frac{\sigma^2_{\rho,n}}{(2n+1)^2(n+3)^2}\ [R_2^{n+3} - R_1^{n+3}]^2 \quad . \tag{4.12}$$

From (4.12) $\sigma^2_{\rho,n}$ is derived and

$$\sigma_{\rho,n}(r_1,r_2) = \begin{cases} \sigma^2_{\rho,n} & R_1 \leq r_1 \leq r_2 \leq R_o \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

Although (4.7) is a constraint which becomes sufficient to determine the internal covariance only in the presence of further modelling of the radial behaviour of ρ, still it can be used as it is to judge on an apparent paradox, which appears to be interesting to the authors.
Namely, there seems to be incompatibility between the requirement that $\rho \in L^2(B)$, which is physically justified by the fact that ρ is

bounded, and the way the gravity spectrum $\sigma^2_{u,n}$ goes to zero, as seen from the empirical analysis of the gravity field expressed for instance in the form of Kaula's rule(*) [6]

$$\sigma^2_{u,n} \cong \frac{c}{n^4} \quad . \tag{4.14}$$

In fact if we assume $\rho \in L^2(B)$ we must have

$$||\rho||^2_{L^2} = 4\pi \sum_n \int_0^1 \sum_m \rho^2_{nm}(r)\, r^2\, dr \quad , \quad < +\infty$$

what implies that

$$\int_0^1 \sigma_{\rho n}(r,r) r^2\, dr = \frac{1}{2n+1} \int_0^1 \sum_m \rho^2_{nm}(r)\, dr = 0(\frac{1}{n}) \quad . \tag{4.15}$$

Even more if we believe that

$$\int_0^1 \frac{1}{2n+1} \sum \rho^2_{nm}(r)\, r^2 dr = \frac{c}{n^\alpha} \quad , \tag{4.16}$$

we must have $\alpha > 2$.

On the other hand we have, from Cauchy-Schwarz inequality,

$$\sigma_{\rho,n}(r_1,r_2) \le \sqrt{\sigma_{\rho,n}(r_1,r_1)} \quad \sqrt{\sigma_{\rho,n}(r_2,r_2)} \quad ,$$

and from Schwarz inequality applied to (4.7) we derive

$$\sigma^2_{u,n} \le \frac{1}{(2n+1)^2} \quad (\int_0^1 \sqrt{\sigma_{\rho,n}(r,r)} \quad r^{n+2}\, dr)^2 \le$$

$$\le \frac{1}{(2n+1)^2} \int_0^1 \sigma_{\rho,n}(r,r)\, r^2\, dr \;.\; \frac{1}{(2n+3)} \quad .$$

Whence, if (4.15) has to be true, we find

$$\sigma^2_{u,n} \le o(\frac{1}{n^4}) \quad , \tag{4.17}$$

(*) Attention should be paid that the degree variances here defined are equivalent to the variances of the individual coefficients in Kaula's book (cfr. formula (5.15)).

which is already in contrast with (4.14): even more if (4.16) holds, we should find $\sigma^2_{u,n} = O(1/n^5)$. This contradiction was already perceived by Kaula as well as by several authors (e.g. [8]).

We notice that a purely random noise density in the body B

$$E\{\rho(P)\ \rho(Q)\} = C\ \delta(P-Q) \quad ,$$

would generate a potential with degree spectrum

$$\sigma^2_{u,n} = O(1/n^3) \quad ,$$

so that we can say that judging from the actual gravity field we find a behaviour compatible with a density ρ a little milder than a body white noise, but rougher than a square integrable function.

It is for that reason that some models of "layered white noise" have been proposed in the literature, with more or less artificially chosen radius in order to satisfy (4.14), [8].

What we want to show here is which kind of constraint (4.14) puts on a layered model.

We call a layered model of density, one where the density is divided in N layers

$$0 = R_o < R_1 < \dots < R_N = 1 \tag{4.18}$$

in each of which the density is radially constant. If we assume that the density is independently distributed from layer to layer(*), we arrive, according to <u>Example</u> 4.3, at a gravity spectrum of the form

$$\sigma^2_{u,n} = \sum_{1}^{N}{}_i \frac{\sigma^2_{\rho,n}(i)}{(2n+1)^2(n+3)^2} [R_i^{n+3} - R_{i-1}^{n+3}]^2 \quad , \tag{4.19}$$

where $\sigma^2_{\rho,n}(i)$ is the density degree variance of the i-th layer, delimited by spheres of radius R_{i-1}, R_i.

(*) This hypothesis is an extreme one, and hardly acceptable from a physical point of view; however, it is possible to see that the same argument holds even if we admit some correlation between layers.

Since

$$R_i < 1 \quad , \quad i = 1,\dots,N-1 \quad ,$$

we see that the contribution of the inner layers decays necessarily exponentially, so that the only opportunity to have

$$\sigma^2_{u,n} \sim c/n^4$$

is that

$$\sigma^2_{\rho,n}(N) \sim \text{const} \quad (n \to \infty) \tag{4.20}$$

because this degree variance is the only one multiplied by a factor

$$[1 - R_{N-1}^{n+3}]^2 = O(1) \quad .$$

Whence (4.20) is the only real mathematical constraint that Kaula's rule puts on a layered distribution. From a statistical point of view (4.20) means that there is a "lateral" white noise distribution in the last, N-th layer, since $\sigma^2_{\rho,n}(N)$ has the meaning of the variance of each single coefficient $\rho_{nm}(N)$.
From the physical point of view, for a planet like the earth, this is not such bad a conclusion since for wavelengths at least of ~100 Km the roughness of the topography and of its isostatic compensation is likely to be seen as a kind of white noise in the lateral density variations.
Obviously, in order to be confirmed, this reasoning would require a deeper check on the powers implied.

5. INTERNAL COLLOCATION AND OTHER PROPOSALS

What has been presented up to now is only a review on the theoretical properties of the i.g.p. and a few examples; the question is whether all this can be transformed into a viable algorithm to be applied to real data.

The first characteristic of real data is that they are always in a finite number and refer to a discrete set of points.

If we disregard for the moment the presence of observational noise or even model noise, we could define here our problem as: to determine an internal density ρ in such a way that the external gravity potential u satisfies a number of observation equations, which, in an already linearized form, we write

$$\langle F_i, u\rangle = Q_i \qquad \left(u = \int_B \frac{\rho(Q)}{r_{PQ}}\, dB\right) \tag{5.1}$$

Here the observational functionals F_i depend on the type of measurement performed and are subject to the only condition that the expression (5.1) be finite (i.e. F_i should be bounded functionals on the space of potentials u considered).

Since we know that in the most favourable hypothesis we can determine the harmonic component ρ of ρ (cfr. Theorem 2.5), characterized by having a minimum L^2 norm, it is natural that we agree to set up a procedure aiming at identifying ρ only: the most natural choice seems to look for a ρ such that

$$\begin{aligned} &\|\tilde{\rho}\|^2_{L^2} = \min \\ &\langle F_i, \tilde{u}\rangle = Q_i \\ &\tilde{u} = \int_B \frac{\tilde{\rho}}{r_{PQ}}\, dB \end{aligned} \tag{5.2}$$

It is elementary to see that such a ρ is given by the projection of ρ on the span $\{F_i,\ i=1,2,..,N\}$ in $L^2(B)$ and that the corresponding potential u has the form

$$u(P) = \Sigma \langle F_i, K(P,P_i)\rangle \{\langle F_i, \langle F_j, K(P_i,P_j)\rangle\rangle\}^{-1} Q_j \quad : \tag{5.3}$$

this is nothing but the usual collocation formula applied with the reproducing kernel K (cfr. (2.5))

$$K(P,Q) = \int \frac{1}{r_{PM} r_{MQ}}\, dB_M \quad .$$

We give this formula not just because it is of practical use, but rather to criticize it and realize what attempts have been done to overcome its main drawbacks.

Remark 5.1. The first important drawback is that the use of a weak norm like L^2 for ρ forbids the use of many important observational functionals. Not even $(\partial/\partial r_P)u$, which corresponds to a gravity anomaly observation, has a bounded norm since $(\partial/\partial r_P)(1/r_{PQ})$ is not in $L^2(B)$ when $P \in S$. Obviously what one has to do is to suitably strengthen the norm. As it is shown in [2], one can use a Sobolef norm on the spherical lateral variables (i.e. σ) with coefficients chosen in such a way that ρ is still harmonic.
Even better, considering that between gravity degree variances and harmonic density degree variances there is a one to one correspondence, namely (4.8), one can use this relation to go from the outer harmonic covariance of u to the inner harmonic covariance of ρ. In fact the covariance of u is estimable empirically and $\sigma^2_{\rho,n}$ can be derived accordingly. This is described in [18] and [5] where also a practical example is worked out.
Yet this idea has proved to be insufficient for practical purposes since together with the auto-covariance $C_{\rho\rho}$, the harmonic model (4.8) implies a fixed cross-covariance between u and ρ (considered at the boundary surface) namely(*)

$$C_{u,\rho}(P,Q) = \Sigma\ (2n+1)(2n+3)\ \sigma^2_{u,n} \cdot (2n+1)P_n(\cos \psi_{PQ}) \quad .$$

A comparison with empirical data in an area where density data were available (cfr. [5]) has shown that this model was to be rejected. In fact, although the two covariances $C_{uu}, C_{\rho\rho}$ derived one from the other according to (4.8) were reasonably consistent with the empirical values, the cross-covariance $C_{u,\rho}$ derived with the same rule, was not matching the empirical behaviour.

(*) As a matter of fact in [5] the covariances $C_{\delta g \delta g}$ and $C_{\delta g \rho}$ have been considered, which however can be derived from $C_{u,u}$, $C_{u,\rho}$.

Remark 5.2. Any model of density derived in the way described in Remark 5.1, displays the typical behaviour of harmonic functions, namely it gets its extreme values at the boundary; on the contrary one could be willing to model a density with large variations at considerable depth.

This can still be achieved with the so-called quasi-harmonic densities ρ, satisfying, instead of Laplace equation, an equation of the type

$$\Delta[w(r)\rho] = 0 \quad , \tag{5.4}$$

with $w(r)$ a suitable weighting function.

This approach is described in [17],[18].

Remark 5.3. If we go back to the motivation for studying the i.g.p. in the framework of gravity field approximation, we see that a useful norm for ρ should take into account the real form of the earth's surface; but in this case the computation of the relevant reproducing kernel can be very difficult. We shall dwell on this problem in the next paragraph, where the problem of estimating ρ in the outermost layer, which is the more interesting for geodesy, will be separated from the rest.

6. MIXED COLLOCATION AND ITS DEVELOPMENT

When the accent of the i.g.p. is put on the most external layer of the body either because we are mainly interested in knowing the matter density only there, or because this is particularly useful for deriving a gravity field approximation, we can split the contribution of the density into two parts.

In terms of a minimum norm principle, if we define $B\backslash B_o$ to be the outer layer and B_o to be an internal body, this can be written

$$\int_{B\backslash B_o} (A\rho)^2 \, dB + \int_{B_o} (A_o\rho_o)^2 \, dB = \min \quad , \tag{6.1}$$

to which the observation equations should be added

$$\langle F_i,u\rangle + \langle F_i,u_o\rangle = Q_i \tag{6.2}$$

$$u = \int_{B\setminus B_o} \frac{\rho}{r_{PQ}}\, dB \qquad , \qquad u_o = \int_{B_o} \frac{\rho_o}{r_{PQ}}\, dB \quad :$$

in (6.1) A, A_o represent in general different operators specifying the type of norm chosen for the various parts of B.
As for the outer part $B\setminus B_o$, if it has really to account for the actual shape of the boundary it seems difficult to avoid a simple L^2 norm, because this is the simplest from the computational point of view.
As we have observed in Remark 5.1, this topology seems to be too weak to permit the use of many functionals, however this point can be overcome by suitably restricting ρ (in $B\setminus B_o$) to a subspace of $L^2(B\setminus B_o)$, still preserving the same easily computable norm.
As for the inner part B_o (second term in (6.1)) we observe that if we are not directly interested in finding ρ_o, we can substitute it with a suitable norm of the outer potential u_o, thus arriving at the mixed collocation principle [11]

$$\int_{B\setminus B_o} \rho^2\, dB + ||u_o||^2_{H_{K_o}} = \min \tag{6.3}$$

$$\langle F_i,u\rangle + \langle F_i,u_o\rangle = Q_i$$

$$u = \int_{B\setminus B_o} \frac{\rho}{r_{PQ}}\, dB$$

$\rho \in$ suitable subspace of $L^2(B\setminus B_o)$.
It has to be noted that now u_o is generated by a really internal body so that the reproducing kernel of u_o can be assumed in full right to have the typical spherical form

$$K_o(P,Q) = \Sigma\, K_{on}\left(\frac{r_B}{r_P r_Q}\right)^{n+1} (2n+1)\, P_n(\cos\ _{PQ}) \tag{6.4}$$

which is harmonic down to the sphere of radius r_B (Bjerhammar sphere).
The reproducing kernel K_o can be estimated from surface data as in [4].

The problem remains to be solved of choosing a suitable subspace of $L^2(B\backslash B_0)$ to which we restrict ρ.
First of all we notice that if we require ρ to depend on lateral variables only(*), $\rho=\rho(\sigma)$, then we already have a regularization of the radial derivative, which represents a gravity observation. To be stronger we can choose to regularize even more ρ deciding to keep it constant blockwise.
The situation is illustrated in Fig. 6.1: the blocks have bases all at the same depth and variable heights. The densities in this case attains only a discrete number of values ρ_i and we have

$$\int_{B\backslash B_0} \rho^2 \, dB = \Sigma \rho_i^2 V_i \qquad (V_i = S_i H_i) \quad . \tag{6.5}$$

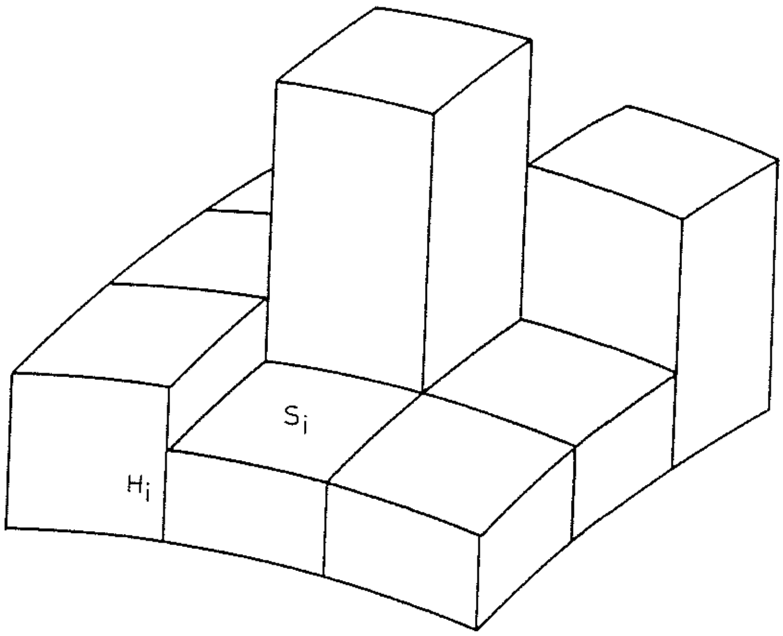

Fig. 6.1 - Block partitioning of the layer $B\backslash B_0$.

(*) This hypothesis of a radially constant ρ can be justified only for layers of limited depth.

Once the blocks are fixed, this subspace $H(\{B_i\})$ of $L^2(B\backslash B_0)$ becomes a reproducing kernel Hilbert space, with

$$K(P,Q) = \Sigma \frac{I_i(P)\ I_i(Q)}{V_i} \tag{6.6}$$

$$I_i(P) = \begin{cases} 1 & P \in B_i \\ 0 & \text{otherwise} \end{cases} .$$

To recognize that (6.6) is the reproducing kernel of $H(\{B_i\})$, apart from a direct check, we can see that $\{I_i(P)\}$ is complete in $H(\{B_i\})$; it is also orthogonal in this space with respect to the L^2 topology, so that the system

$$\bar{I}_i(P) = \frac{I_i(P)}{\sqrt{V_i}}$$

is a complete orthonormal system in this space.
Whence we really have

$$K(P,Q) = \Sigma\ \bar{I}_i(P)\ \bar{I}_i(Q)$$

as well as

$$||\rho||^2 = \Sigma(\langle\rho,\bar{I}_i\rangle)^2$$

to be compared with (6.6),(6.5) respectively.

Remark 6.1. If we interpret K(P,Q) as a covariance function, we find for an arrey of blocks like in Fig. 6.2 a covariance of the form shown in Fig. 6.3.a). In Fig. 6.3.b) we show the first line of K(P,Q). As we can see it is by far too simple a function to make us able to model the empirical covariance of possibly available ρ values,

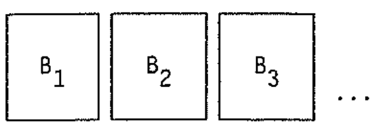

Fig. 6.2

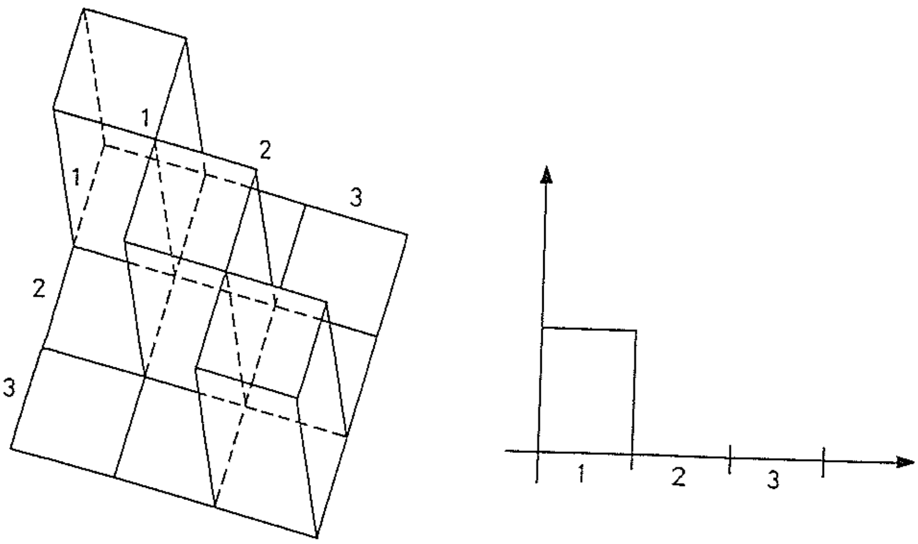

Fig. 6.3 a) b)

like in the example worked out in [5]. In the effort of constructing more flexible models, the authors in [5] have proposed the use of overlapping blocks, like in the array shown in Fig. 6.4. Obviously in this case $\{I_i(P)\}$ is still complete but neither normalized nor orthogonal. However an orthonormal basis of $H(\{B_i\})$ can be computed by applying a Gram-Smith orthonormalization process. In the example in Fig. 6.4 we have, assuming all the volumes V_i to be unitary

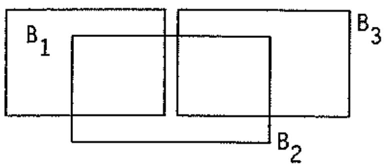

Fig. 6.4

$$\bar{I}_1 = I_1$$

$$\bar{I}_2 = \frac{2}{\sqrt{3}} \left(I_2 - \frac{1}{2} I_1\right)$$

$$\bar{I}_3 = \frac{3}{\sqrt{2}} \left(I_3 - \frac{2}{3} I_2 + \frac{1}{3} I_1\right) \quad .$$

The reproducing kernel now has the form

$$K(P,Q) = \sum_{1}^{3}{}_i \, \bar{I}_i(P) \, \bar{I}_i(Q)$$

the first line of which is shown in Fig. 6.5. So, by varying the block size and the overlapping, one can think to give different shapes to the reproducing kernel, following the suggestion of the empirical covariance of ρ, when available.

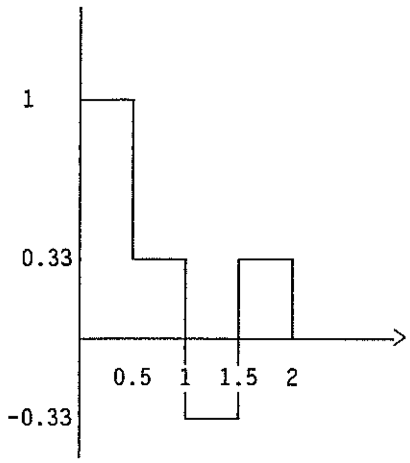

Fig. 6.5

Naturally the little example presented here can be expanded to any number of blocks by observing that if $\{I_i(P),\ i=1...N\}$ is a (non orthonormal) basis of $H(\{B_i\})$ with Gram-matrix

$$\langle I_i, I_j\rangle = \{C_{ij}\} \quad ,$$

then it is enough to decompose C by the Cholesky's formula

$$C = U^+U$$

$$(C_{ij} = \Sigma_K\ u_{Kj}\ u_{Ki})$$

to derive a new orthonormal basis

$$\bar{I}_i = \Sigma_j (U^{-1})_{ji}\ I_j \quad . \tag{6.7}$$

The corresponding reproducing kernel is then

$$K_\rho(P,Q) = \Sigma_{1}^{N}{}_i\ \bar{I}_i(P)\ \bar{I}_i(Q) \tag{6.8}$$

Remark 6.2. Since the potential $u_i(P)$ corresponding to each block B_i is known in closed form (see e.g. [2]), we define that to any reproducing kernel $K_\rho(P,Q)$ as in (6.8) we can make to correspond a known reproducing kernel for potentials in the form