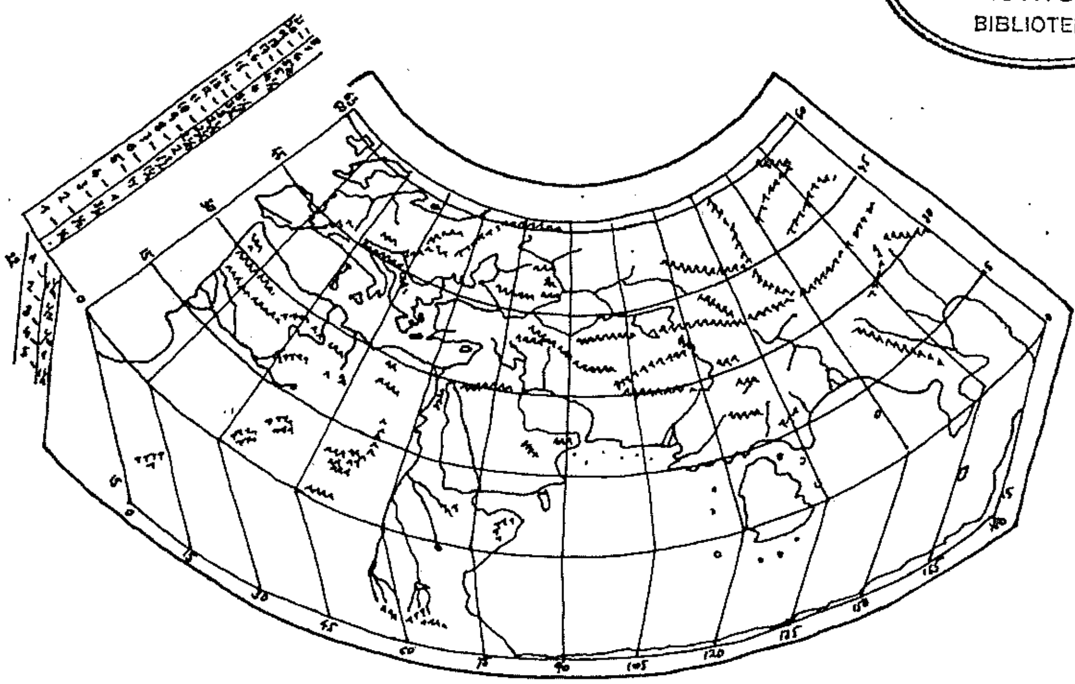
ISSN 0250-6351

ΓΕΩΔΑΙΤΙΚΑ ΤΕΤΡΑΔΙΑ QUATERNIONES GEODAESIAE



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Περιοδικη Εκδοση της Τοπογραφιας και Ανωτερας Γεωδαισιας και Χαρτογραφιας της Πολυτεχνικης Σχολης του Αριστοτελειου Πανεπιστημιου Θεσσαλονικης	*Reports from Topography and Higher Geodesy & Cartography Faculty of Engineering University of Thessaloniki Thessaloniki Greece*

Γεωδαιτικά Τετράδια / Quaterniones Geodaesiae, 3,1 / 1982, (1 - 15)

6351

F. Sansó and C.C. Tscherning

MIXED COLLOCATION : A PROPOSAL

(F. Sansó and C.C. Tscherning : Μικτή Σημειακή Προσαρμογή : Μία πρόταση)

Summary

The application of the collocation method, taking advantage of the knowledge of height data has been recently proposed and applied by using some mean value for the density of the topographic masses.

In the present work the method is reproposed on a new theoretical base, where the density anomalies can be treated as unknown parameters.

The simplest models are discussed, in view of possible numerical applications.

Prof. Fernando Sansó gave a lecture at the faculty of Engineering, University of Thessaloniki, after an invitation from Higher Geodesy and Cartography on May 22, 1981. This work is published on account of Prof. Sansó's visit.

Περίληψη

Ἔχει προταθεῖ πρόσφατα ἡ ἐφαρμογή τῆς μεθόδου τῆς σημειακῆς προσαρμογῆς λαμβάνοντας ὑπόψη γνωστά ὑψόμετρα, μέ τή χρήση κάποιων μέσων τιμῶν γιά τίς πυκνότητες τῶν τοπογραφικῶν μαζῶν.

Στήν ἐργασία αὐτή προτείνεται μιά μέθοδος βασισμένη σέ νέα θεωρητική βάση, ὅπου οἱ ἀνωμαλίες τῶν πυκνοτήτων ἀντιμετωπίζονται σάν ἄγνωστοι παράμετροι.

Ἐξετάζονται τά πιό ἁπλά ἀντίστοιχα μοντέλα, μέ προοπτική δυνατῶν ἀριθμητικῶν ἐφαρμογῶν.

Ὁ Καθηγητής Fernando Sansó τοῦ Ἰνστιτούτου Τοπογραφίας, Φωτογραμμετρίας καί Γεωφυσικῆς τοῦ Πολυτεχνείου τοῦ Μιλάνου ἔδωσε διάλεξη στήν Πολυτεχνική Σχολή τοῦ Α.Π.Θ. μετά ἀπό πρόσκληση τῆς Ἕδρας τῆς Ἀνωτέρας Γεωδαισίας καί Χαρτογραφίας στίς 22 Μαΐου 1981. Ἡ ἐργασία αὐτή δημοσιεύεται μέ τήν εὐκαιρία τῆς ἐπίσκεψης αὐτῆς.

ISSN 0250 - 6351

§1. Introduction

Periodically geodesists feel the need to base their computational formulas for the gravity field on a more safe ground, claiming the necessity of finding it in geophysical models.
It seems to the authors that this guess is misleading: it is in the nature of geodesy (and in its potential-theoretical basis) to be theoretically selfsufficient.
Indeed this doesn't mean that a good knowledge of the geophysics of the exterior crust, down to some Bjerhammar sphere, wouldn't be useful in approximating the anomalous potential: this knowledge as a matter of fact can be used to compute a good terrain correction, thus yielding a "smoothed field" on which every "prediction" becomes easier.
On the other hand the density of the outer layers of the crust is not so accurately known to serve as a firm datum.
It is however true that in applying the usual approximation methods (collocation, buried masses, etc.) the necessity was felt of introducing into the estimate of the anomalous field some information coming from the topography and the topographic masses. This in particular also for the sake of numerical stability of the solution. Think for instance of the usual collocation solution: it is perfectly known that if two measuring points P, P' are at a very small distance apart, as compared to their heigth on the Bjerhammar sphere, then the resulting normal equations are very poorly conditioned. Moreover a good deal of the fine structure of the gravity field depends heavily on the distribution of the topographic masses, so that "reducing" the observations one gets a much smoother field on which one can make more reliable predictions.
This problem has been recently treated in some papers: one for instance is by Tscherning and Forsberg |2|, who introduced topographic corrections before applying collocation; another is by Sansò |5| who proposed the application of the collocation principle to the internal densities (internal collocation), following and developping an idea of T. Krarup |4|.
Both ways however have some unpleasant features: the application of topographic corrections implies the knowledge of the density

of the topographic masses or, in lack of this, an arbitrary choice for its mean value.

On the other hand the straightforward application of the internal collocation principle results in the construction of an artificial density which is a harmonic function inside the earth: this density generates a field in agreement with measurements, however it is clearly far from any reasonable physical model of the interior distribution of mass.

This might be of no importance at all if a complete coverage of data is given on the surface of the earth (what is more far away from reality then a single layer solution, many times adopted by geodesists?), but since this is not the case the open violation of physical reality might be the source of larger errors in the estimate of the exterior field.

In what follows a new proposal is formulated, which arises from yearly discussions of the authors (Canberra 1979, Copenhagen 1980, Como 1981).

The main idea is to represent the anomalous gravity field by a combination of an anomalous potential T_s harmonic down to some Bjerhammar sphere S_o and an anomalous potential T_t, generated by a layer of topographic masses lying between the sphere S_o and the topographic surface S: the new point here is that the topographic anomalous density σ_t is considered as an unknown too, and determined on the basis of the data.

§2. The mixed collocation principle

Assume that the earth is a body B with known surface S (topography) and partition B into an internal ball B_o, with surface S_o, and a topographic layer C (crust) like in Fig. 2.1.

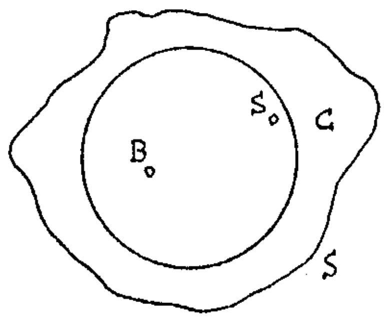

FIG. 2.1

Let us consider the anomalous potential $T(P)$ of B, defined as usual by $T = W - U$, where W is the actual potential of the gravity field and U the normal potential.
If ρ is the actual density of the masses inside B and ρ_o some suitable model of "normal" density in accordance with the normal potential U (see for instance Tscherning and Sunkel |7|), one can set $(\sigma(Q) = \rho(Q) - \rho_o(Q))$

$$(2,1) \qquad T(P) = k \int_B \frac{\sigma(Q)}{|P-Q|} dB_Q = k \int_C \frac{\sigma(Q) dC_Q}{|P-Q|} + k \cdot \int_{B_o} \frac{\sigma(Q)}{|P-Q|} dB_{oQ}$$

We shall refer to the first integral as to the topographic anomalous potential

$$(2,2) \qquad T_t(P) = k \int_C \frac{\sigma(Q)}{|P-Q|} d\, C_Q$$

as for the second integral, the spherical anomalous potential

$$T_s(P) = k \int_{B_o} \frac{\sigma(Q)}{|P-Q|} dB_{oQ} \quad ,$$

we can see that we are concerned with a function which is harmonic down to the sphere S_o and we shall suppose that T_s belongs to some Hilbert space H with reproducing Kernel $k(P,Q)$, such that

$$(2,3) \qquad T_s(P) = \langle k(P,Q),\ T_s(Q) \rangle_H$$

For the sake of simplicity we shall suppose that $k(P,Q)$ is invariant under rotations, i.e.

$$k(P,Q) = \sum_{2}^{+\infty}{}_n\ \sigma_n^2\ P_n(\cos\psi) \quad (\psi = \text{spherical distance between } P \text{ and } Q)$$

where σ_n^2 might or might not be modelled on the degree variances. The important point here is that T_s is rightly a member of H, which will in the sequel be approximated by another member $\hat{T}_s$ of H without invoking Runge-Krarup's theorem(.).

(.) Note: as a matter of fact it is the use of estimates $\hat{T}$ harmonic on a larger domain than T that induces an instability in the approximation process and causes f.i. the numerica problems recalled in the introduction.

Now let us suppose that at some points $\{P_i,\ i=1,2,\ldots n\}$, on or outside S, we have performed some measurements; if we disregard the observational errors, such measurements can always be represented, possibly after a linearization, as linear functionals of T, i.e.

(2,4) $$L_i(T) = y_i$$

for instance we have

(2,5) $$T(P)\big|_{P_i} = T(P_i)$$ evaluation of T

(2,6) $$-\frac{\partial T}{\partial r} - \frac{2}{r}\,T\Big|_{P_i} = \Delta g(P_i)$$ gravity anomaly

(2,7) $$\frac{1}{R}\,\frac{\partial T}{\partial \phi}\Big|_{P_i} = \xi(P_i)$$ N-deflection of the vertical

(2,8) $$\frac{1}{R\ \cos\phi}\,\frac{\partial T}{\partial \lambda}\Big|_{P_i} = \eta$$ E-deflection of the vertical

Using (2,1) the observation equation (2,4) can be splitted into

$$L_i(T) = L_i(T_t) + L_i(T_s) = y_i$$

Setting

$$\rho_P(Q) = \frac{k}{|P-Q|}\ ,\ L_i(\rho_P(Q)) = f_i(Q);\ L_i k(P,Q) = l_i(Q)$$

and recalling (2,2), (2,3) we have then

(2,9) $$L_i(T) = \int_C f_i(Q)\ \sigma(Q)\ dC + \langle l_i, T_s\rangle_H = y_i$$

Considering (2,9) we meet the first difficulty in defining the solution space, in which σ is supposed to range. In fact as we see (2,9) is very simply represented if we suppose $\sigma \in L^2(C)$; but this is acceptable only if $f_i(Q) \in L^2(C)$ too, so that one can write

$$\int_C f_i(Q)\ \sigma(Q)\ dC = \langle f_i, \sigma \rangle_{L^2}$$

This is true for instance for L_i as in (2,5) but not true for L_i as in (2,6), (2,7), (2,8).
We disregard for the moment this difficulty and we shall suppose that $f_i \in L^2(C)$. This can even be achieved in practice by some trick, smoothing L_i by considering for instance the average of the measurements on a block (e.g. mean gravity anomalies) or by slightly moving P_i inside the field of harmonicity: neither of these solutions is satisfactory, but we shall elaborate more convincing (hopefully) proposals in next paragraph.
Thus we consider observation equations of the form

$$(2,10) \qquad \langle f_i, \sigma \rangle_{L^2} + \langle l_i, T_s \rangle_H = y_i \qquad (i=1,2,\ldots n)$$

The unknown in (2,10) is the vector $(\sigma, T_s)^t$ belonging to the product space $L^2(C) \otimes H_k$. In such a space there are clearly infinite solutions of (2,10) and one can be selected by applying the minimum norm-collocation principle:

$$(2,11) \qquad \| \sigma \|^2_{L^2} + \lambda \| T_s \|^2_H = min.$$

The hybride norm constant λ plays the role of distributing the weight between the topographic potential and the spherical potential; it can be chosen on the basis of physical considerations or by numerical optimization.
The minimum (2,11) with side conditions (2,10) can be searched for, by introducing suitable Lagrange multipliers; this leads to minimize

$$\| \sigma \|^2_{L^2} + \lambda \| T_s \|^2_H - 2\Sigma\lambda_i \{ \langle f_i, \sigma \rangle_{L^2} + \langle l_i, T_s \rangle_H - y_i \}$$

i.e. to solve the normal system

$$(2,12) \qquad \begin{cases} \hat{\sigma} - \Sigma\lambda_i f_i = 0 \\ \lambda\ \hat{T}_s - \Sigma\lambda_i l_i = 0 \end{cases}$$

After setting

$$F_{ij} = \langle f_i, f_j \rangle_{L^2} \quad , \quad L_{ij} = \langle l_i, l_j \rangle_H$$

we get the estimates $\hat{\sigma}$, $\hat{T}_s$,

$$(2,13) \qquad \begin{aligned} \hat{\sigma} &= \Sigma_{i,j}\, y_j \,\{F_{ij} + \tfrac{1}{\lambda} L_{ij}\}^{-1} f_i \\ \hat{T}_s &= \tfrac{1}{\lambda}\, \Sigma_{i,j}\, y_j \,\{F_{ij} + \tfrac{1}{\lambda} L_{ij}\}^{-1} l_i \end{aligned}$$

As one can easily verify, for $\lambda \to 0$ or $\lambda \to \infty$ we get

$$(2,14) \qquad \lim_{\lambda \to 0} \hat{\sigma} = 0 \; ; \qquad \lim_{\lambda \to 0} \hat{T}_s = \Sigma_{i,j}\, y_j\, L_{ij}^{-1}\, l_i$$

$$(2,15) \qquad \lim_{\lambda \to \infty} \hat{\sigma} = \Sigma_{i,j}\, y_j\, F_{ij}^{-1}\, f_i \; ; \quad \lim_{\lambda \to \infty} \hat{T}_s = 0$$

i.e., respectively, the usual collocation solution with Bjerhammar sphere S_o, or the internal collocation solution with respect to the layer C: indeed, (2,13), combining both principles can be reasonably indicated as a mixed collocation solution.
It should also be noted that, in accordance with internal collocation, it happens that the estimated density $\hat{\sigma}$ is a harmonic function in its domain of definition, i.e. C. This is more easily understood, by realizing that the observations y_i are functions of σ only through

$$T_t(P) = \langle \rho_P, \sigma \rangle_{L^2}$$

so that, if σ is decomposed in two orthogonal components $\sigma = \sigma_H + \sigma_\perp$ such that σ_H is a $L^2(C)$ harmonic function and $\sigma_\perp$ is orthogonal to $L^2(C)$ harmonic functions, we get

$$T_t(P) = \langle \rho_P, \sigma \rangle_{L^2} = \langle \rho_P, \sigma_H \rangle_{L^2} \quad :$$

whence from external observations (P_i on or outside S) only σ_H can be estimated.
Once $\hat{\sigma}$, $\hat{T}_s$ are computed as in (2,13), a global estimate of the anomalous potential $\hat{T}$ is obtained, recalling (2,1), in the form

(2,16) $$\hat{T}(P) = \hat{T}_t(P) + \hat{T}_s(P) = \langle \rho_P, \hat{\sigma} \rangle_{L^2} + \hat{T}_s(P) =$$

$$= \Sigma_{i,j}\, y_j\, \{F_{ij} + \frac{1}{\lambda} L_{ij}\}^{-1} \{\langle \rho_P, f_i \rangle_{L^2} + \frac{1}{\lambda} l_i(P)\}.$$

Some remarks on the meaning and possible applications of (2,16) will be done in the next paragraph.

§3. Applications of mixed collocation

We shall first of all make some general remarks on the solution of the mixed collocation method.

Remark 1: let us introduce a Hilbert space H_C of functions harmonic outside S, endowed with a reproducing Kernel defined as

(3,1) $$C(P,Q) = \langle \rho_P, \rho_Q \rangle_{L^2} + \frac{1}{\lambda} K(P,Q) \ , \ (C(P,Q) = C(Q,P)).$$

We have then

$$L_i C(P,P_i) = \langle \rho_P, L_i \rho_{P_i} \rangle_{L^2} + \frac{1}{\lambda} L_i K(P,P_i) =$$

$$= \langle \rho_P, f_i \rangle_{L^2} + \frac{1}{\lambda} l_i(P)$$

$$L_j L_i C(P_i,P_j) = \langle f_j, f_i \rangle_{L^2} + \frac{1}{\lambda} L_j l_i(P) =$$

$$= F_{ij} + \frac{1}{\lambda} L_{ij}$$

so that (2,16) appears as

(3,2) $$\hat{T} = \Sigma_{i,j}\, y_j\, \{L_j L_i C(P_i,P_j)\}^{-1}\, L_i C(P_i,P)$$

i.e. the usual solution of the minimum norm principle, with the norm induced by the reproducing Kernel $C(P,Q)$. Indeed this is true, but with some particular features depending on the choice (3,1). First of all we have a more "realistic" Kernel C, non invariant under rotations

because of the introduction of the "topographic" kernel $\langle \rho_P, \rho_Q \rangle_{L^2}$ which doesn't enjoy this invariance property. On the other hand the kernel K for the spherical potential T_s, which should account for most of the information on the lower frequencies of T, can be modelled on realistic values of the low degree variances, as the least squares collocation principle would require.
What is most important however is that, contrary to the l.s. collocation method, if the asymptotic behaviour of σ_n^2, in $K(P,Q)$, is properly chosen, we can claim that $T = T_t + T_s$ belongs to H_C as $\hat{T}$. This allows the interpretation of $\hat{T}$ as the projection of T on $Span\ \{L_i C(P_i, P)\}$ and yields the error estimate

$$(3,3) \qquad \|T - \hat{T}\|^2_{H_C} = \|T\|^2_{H_C} - \Sigma\, y_i \{L_i L_j C(P_i, P_j)\}^{-1}\, y_i$$

The convergence of $\hat{T}$ to T is hence guaranteed.

Remark 2: if we choose in (3,1), $\lambda = 1$ and

$$K(P,Q) = \int_{B_o} \rho_P(M)\ \rho_Q(M)\ dB_M$$

we get

$$C(P,Q) = \int_B \rho_P(M)\ \rho_Q(M)\ dB_M$$

i.e. the typical kernel of internal collocation (see for instance Sansò |5|).
If we make another choice for K, λ should be determined by some optimization criterion, e.g. by minimizing (3,3). Alternatively, if one has at least a physical guess for the true $\|\sigma\|^2_{L^2}$ and $\|T_s\|^2_H$, one could choose in analogy with the least squares method

$$\lambda = \frac{\|\sigma\|^2_{L^2}}{\|T_s\|^2_H} \qquad \text{(a priori values)}$$

Remark 3: when we are globally estimating the gravity field, we must always take into account two further conditions that the estimating potential $\hat{T}$ has to satisfy, i.e.

to be generated by a distribution of total mass zero and to have the "barycenter" at the origin of the coordinates.
These, recalling the two relations

$$\frac{1}{kR_o} \frac{1}{4\pi} \int_{S_o} T_s \, dS = M_S \qquad (M_S = \text{spherical mass})$$

$$\frac{R_o}{k} \frac{1}{4\pi} \int_{S_o} T_s \, \vec{r} \, dS = M_S \vec{r}_S \quad (\vec{r}_S = \text{spherical barycenter})$$

valid for the spherical potential T_s, can be written very simply as

$$\text{(3,4)} \qquad \int_C \hat{\sigma} \, dC + \frac{1}{kR_o} \frac{1}{4\pi} \int_{S_o} T_s \, dS = 0$$

$$\text{(3,5)} \qquad \int_C \hat{\sigma} \, \vec{r} \, dC + \frac{3R_o}{k} \frac{1}{4\pi} \int_{S_o} T_s \, \vec{r} \, dS = 0$$

We have also to note that for a global estimate the largest radius R_o of the internal sphere, suitable for mixed collocation is b, the minor semiaxis of the earth's ellipsoid.
If, on the contrary, we are estimating T only on a local basis we might impose the recalled conditions on T_s only(.) and leave $\hat{T}_t$ more free to assume any value suitable to interpolate the data: this because $\hat{T}_t$ will take care of the global behaviour of T and $\hat{T}_t$ will account for the local variations only. In this case R_o can be chosen even larger than b, with the sole condition that all the portion of the earth's surface considered is lying outside the ball B_o.

(.) Note: i.e. we consider $\hat{T}_s$ as defined by a series of spherical harmonics of order higher than one.

Now we can turn our attention to the problems related to a practical computation of the mixed collocation solution.
The main problem is indeed to suitably define a solution space H_σ for $\hat{\sigma}$, such that at least functionals of the type (2,6), (2,7), (2,8) (i.e. first order derivatives of T) are admissible.
Since, as we have already observed, $L_i \rho_P = f_i$ do not belong to $L^2(C)$, we have to restrict the solution space $H_\sigma \subset L^2$ in such a way that $f_i \in H_\sigma^*$, the dual of H_σ.
This is most simply done if we take as H_σ a finite dimensional variety in $L^2(C)$.
Suppose for instance to partition C into smaller blocks C_i such that on each C_i, σ can be considered as constant, by geophysical reason or simply by hypothesis: then, calling $\chi_i(P)$ the characteristic function of C_i ($\chi_i = 1$ on C_i, $\chi_i = 0$ elsewhere), we could represent σ as (see Fig. 3.1)

$$\sigma(P) = \Sigma_i \; \sigma_i \; \chi_i(P) \quad H_\sigma$$

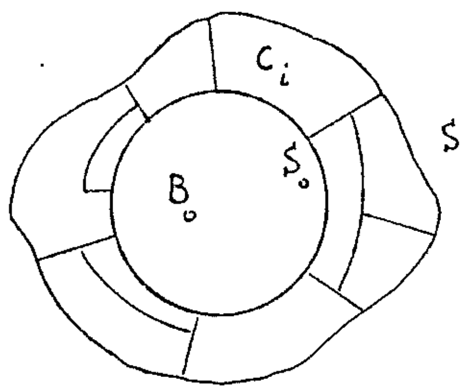

FIG. 3.1

H_σ can be normed by the L^2 norm, i.e.

$$(3,6) \qquad \|\sigma\|^2_{L^2} = \Sigma_{ij} \; \sigma_i \; \sigma_j \int_C \chi_i \; \chi_j \; dC = \Sigma_i \; \sigma_i^2 \; C_i$$

with this definition H_σ becomes a hilbert space with reproducing kernel

$$K_\sigma(P,Q) = \Sigma_i \; \frac{1}{C_i} \; \chi_i(P) \; \chi_i(Q)$$

as it is easily verified.

Now we can note that $\sigma \in H_\sigma$ yields

(3,7) $$\sup_C |\sigma| < +\infty$$

on the other hand, since the first derivatives of $\rho_P(Q)$ are of the order $\rho_P^{-2}(Q)$ as P tends to Q, we have that $L_i\rho_P = f_i(P)$ are integrable functions on C so that, setting $\Phi_{ij} = \int_{C_j} f_i dC$,

(3,8) $$L_i \quad \langle\rho_{P_i},\hat{\sigma}\rangle_{L^2} = \Sigma_j \hat{\sigma}_j \Phi_{ij}$$

are well defined and bonded quantities.
Whence the mixed collocation principle can be rewritten

(3,9) $$\Sigma\hat{\sigma}_i^2 \, C_i + \lambda\|T_s\|_H^2 = min$$

(3,10) $$\Sigma_j \hat{\sigma}_j \Phi_{ij} + \langle l_i, \hat{T}_s\rangle_H = y_i$$

the corresponding normal system is

(3,11) $$\begin{cases} \sigma_j C_j - \Sigma\ \lambda_i \Phi_{ij} = 0 \\ \lambda T_s - \Sigma\ \lambda_i l_i = 0 \end{cases}$$

which, taking (3,10) into account, is reduced to

(3,12) $$\Sigma_i (\Sigma_j \Phi_{hj} C_i^{-1} \Phi_{ij})\ \lambda_i + \frac{1}{\lambda}\ \Sigma_i L_{hi} \lambda_i = y_h$$

After solving (3,12) and substituting in (3,11), we find the estimate

(3,13) $$\hat{T} = \Sigma\hat{\sigma}_i \ \langle\rho_P, \chi_i\rangle_{L^2} + \hat{T}_s$$

§4. Discussion

Two points are still open to discussion in this preliminary proposal: the difficulty of computing practically the estimate (3,13)

and the possibility of including some geophysical information into the estimated potential. Possible answers to these questions will be given in remarks 4 and 5.
From the computational point of view, the greatest difficulty in the solution procedure lies in calculating functions of the type

$$\int_{C_i} \frac{k}{|P-Q|}\, dC$$

which are essential to recover $\hat{T}_t$, as well as the related quantities

$$\Phi_{ij} = L_i \int_{C_j} \frac{k}{|P-Q|}\, dC_Q$$

however one should consider that, specially in a local framework, when C_i are chosen of a simple geometrical form like prisms, cylinders etc., these functions are not too different from the ones we are accustomed to compute in applying the old reduction theories (cfr. Heiskanen-Moritz |3| and Forsberg-Tscherning |2|).

Remark 4: a further step in the direction of a simpler computational scheme, can be done by representing the topographic potential T_t as generated by point masses. In this case in fact we have a very simple representation of both the potential and the functionals L_i, i.e.

(4,1) $$\hat{T}_t(P) = \Sigma_j \hat{m}_j \rho_{P_j}(P)$$

(4,2) $$L_i T_t(P) = \Sigma_j \hat{m}_j L_i \rho_{P_j}(P_i) = \Sigma_j \hat{m}_j R_{ij}$$

Now the mixed collocation principle could be taken as

(4,3) $$\Sigma_i \frac{\hat{m}_i^2}{C_i} + \lambda \|\hat{T}_s\|_H^2 = min$$

(4,4) $$\Sigma_j \hat{m}_j R_{ij} + \langle l_i, \hat{T}_s \rangle_H = y_i$$

Let us underline the weighting factor C_i which is introduced in order to obtain a variational principle homogeneous with (3,9): here C_i could be roughly taken as the same blocks as in Fig. 3.1, where however the masses have been shrinked to single points $P_i \in C_i$.

Remark 5: the mixed collocation principle has the interesting feature to be suitable to the introduction of some relevant information coming from the geophysical side, specially if used on a local scale.
The first information may come already in deciding the blocks C_i on which the density can be considered as constant: in fact these blocks can be drown for instance following the boundaries of density discontinuities.
Moreover, we could have some information on the magnitude of the density anomalies. This could be supplied for instance in the form of the inequalities

$$(4,5) \qquad ||\sigma_i|| < k_i$$

where the constants k_i might depend on the block C_i or be a unique constant for all the blocks, according to the vagueness of our prior information.
Algorithms for the solution of least squares with inequalities can be used to solve the corresponding problem.
More advantageous might be a global information on σ_i of the type

$$(4,6) \qquad \Sigma(\sigma_i - \bar{\sigma})^2 < k \qquad \text{(given constant)}$$

where $\bar{\sigma} \doteq \frac{1}{n} \Sigma\sigma_i$: (4,6) could be interpreted as a constraint on the sample variance of the local densities.
A constraint like (4,6) can be easily dealt with by using a Lagrange multiplier, if the "free" solution doesn't fulfil it spontaneously. The effect of (4,6) should be similar to (4,5), with a unique constant k_i, although computationally simpler.
We would expect that the solution of the mixed collocation principle with geophysical constraints, like

(4,5) or (4,6), would display better approximation properties than the usual solutions from both points of view of the magnitude of the estimation error and of the numerical stability of the solution.

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