

# **A Method for the Construction of Spheroidal Mass Distributions Consistent With the Harmonic Part of the Earth's Gravity Potential**

**C. C. Tscherning, H. Sünkel**

## Abstract

*The problem of determining a unique mass distribution ( $\rho$ ) consistent with the harmonic part of the corresponding gravity potential ( $V$ ) is solved for a mass distribution bounded by a spheroid by requiring  $\rho$  to fulfil a condition of the type  $\Delta(f \cdot \rho) = 0$ , where  $\Delta$  is the Laplace operator and  $f$  is a simple function. The solution is expressed as a system of equations relating the coefficients of  $V$  expressed as a series in solid (external) spheroidal harmonics and the coefficients of  $\rho$  expressed as a series in solid (internal) spheroidal harmonics (multiplied by a weight function).*

*Numerical results with  $f$  of polynomial type and  $V$  given by a set of low degree coefficients did not give geophysically satisfactory results. The method was then applied to small perturbations  $\rho_1 = \rho - \rho_0$ , where  $\rho_0$  is taken from a parametric earth model given by geophysical sources. Since  $\rho_1$  becomes very small, the technique may be useful in the construction of density reference models.*

## 1. Introduction

It is well known that the problem of determining the density distribution of the Earth (-) from the external potential ( $V$ ) has no unique solution. This does not mean that solutions cannot be found. On the contrary, we have many different methods available, and each of them is suited for different purposes. In prospecting, e.g., solutions are generally obtained by expressing an unknown density as a linear combination of a finite number of simple functions, each constant inside a certain rectangular box and zero outside. The density values are then obtained by solving a system of linear equations.

If the density distribution is supposed to fulfil some (possibly physically meaningful) minimum condition, like having minimum gravitational energy (Rubincam, 1980), then the density distribution will be the solution to a partial differential equation.

In many cases, this will result in the fact that simple relations can be found between the coefficients of the external potential expressed as a series in (external) solid spherical (or spheroidal) harmonics and the coefficients of the density distribution expressed in some system of base functions similar to the solid spherical harmonics (Tscherning, 1974). Thereby, we avoid the problem associated with the use of several other techniques, namely the problem of solving a large system of linear equations.

Unfortunately, the most simple relation between the two sets of coefficients does not result in geophysically realistic density distributions. In geodesy, however, we may be satisfied in many cases with a "model" for the density distribution, which is easily computable. This is the case, e.g., if we want to treat the deviations of such a model by some statistical technique (Tscherning, 1977 or Jordan, 1978).

When working with density anomalies (6), it is permissible to work in spherical approximations: the earth is regarded as a sphere with radius  $R = 6\,371.0$  km. The anomalous potential  $T$ , produced by these density anomalies,

$$T(P) = G \int_{\Omega} \delta(Q) / \|P - Q\| dQ \quad (1.1)$$

may in spherical approximation be expressed as a series in solid spherical harmonics outside the surface of the Earth. ( $\Omega$  is the set inside the surface of the Earth,  $P$  and  $Q$  are points and  $G$  is the gravitational constant).

Let the point  $P$  outside  $R$  have spherical coordinates  $\phi =$  latitude,  $\lambda =$  longitude, and  $r =$  distance from the origin. Then

$$T(P) = \frac{GM}{r} \sum_{i=2}^{\infty} \left(\frac{R}{r}\right)^i \sum_{j=0}^i \bar{P}_{ij}(\sin\phi) [\bar{C}_{ij}^* \cos j\lambda + \bar{S}_{ij} \sin j\lambda] , \quad (1.2)$$

where  $M$  is the mass of the earth,  $\bar{P}_{ij}$  are the fully normalized solid spherical harmonics and  $\bar{C}_{ij}$  and  $\bar{S}_{ij}$  are constants. (The symbol  $*$  on  $\bar{C}_{ij}$  signifies that these are the differences between the coefficients of  $W$  and the corresponding coefficients of  $U$ , which here is supposed to be of the Somigliana-Pizzetti type.)

When the density anomaly function  $\delta$  is required to fulfil the simple condition

$$\Delta(r^m \delta) = 0 , \quad (1.3)$$

where  $\Delta$  is the Laplace operator, then  $\delta$  can be expressed as a series in internal solid spherical harmonics multiplied by  $r^{-m}$ ,

$$\delta(P) = r^{-m} \sum_{i=0}^{\infty} \left(\frac{r}{R}\right)^i \sum_{j=0}^i \bar{P}_{ij}(\sin\phi) [c_{ij} \cos j\lambda + s_{ij} \sin j\lambda] . \quad (1.4)$$

In this case there is a simple relationship between the coefficients of the two series, cf. Tscherning (1974),

$$\begin{Bmatrix} c_{ij} \\ s_{ij} \end{Bmatrix} = \frac{(2i - m + 3)(2i + 1)}{R^{3-m} 4\pi G} \begin{Bmatrix} C_{ij}^* \\ S_{ij} \end{Bmatrix} \quad (1.5)$$

for  $i > -m$ , i.e.  $i$  generally  $\geq 0$ .

When working with the density distribution  $\rho$  itself, it is not feasible to work in spherical approximation, because of the magnitude of the second order zonal harmonic coefficient,  $C_{20}$ . We must use a spheroidal (or ellipsoidal) reference model. Let the Earth be approximated by a spheroid with semi-major axis  $a$ , semi-minor axis  $b$ , and hence half focal distance  $E = (a^2 - b^2)^{1/2}$ . We will also introduce spheroidal coordinates, i.e. the point  $P$  will have the coordinates  $(u, \beta, \lambda)$  where  $u$  is the semi-minor axis of the spheroid confocal with the reference spheroid and passing through  $P$ ,  $\beta$  is the reduced latitude and  $\lambda$  is the longitude, cf. (Heiskanen and Moritz, 1967, sec. 1-19). The gravity potential of the Earth,  $V$ , can then be approximated by an abbreviated series in spheroidal harmonics.

In the following, we will work with both internal and external spheroidal harmonics, which we will denote  $V_{nm}^i(P)$  and  $V_{nm}^e(P)$ , respectively. Then

$$V_{nm}^i(P) = P_n^m\left(i\frac{u}{E}\right) P_n^m(\sin\beta) \begin{cases} \cos m \lambda & 0 \leq m \leq n \\ \sin |m| \lambda & -n \leq m < 0 \end{cases} \quad (1.6)$$

$$V_{nm}^e(P) = Q_n^m\left(i\frac{u}{E}\right) P_n^m(\sin\beta) \begin{cases} \cos m \lambda & 0 \leq m \leq n \\ \sin |m| \lambda & -n \leq m < 0 \end{cases} \quad (1.7)$$

Here we put  $P_n^m = P_n^{-m}$ ,  $Q_n^m = Q_n^{-m}$ , where  $Q_n^m$  are the associated Legendre polynomials of the second kind. The "i" appearing within the arguments of  $P_n^m$  and  $Q_n^m$  is the imaginary unit.

We then have

$$V(P) \approx GM \sum_{n=0}^N \sum_{m=-n}^n A_{nm} V_{nm}^e(P) \quad (1.8)$$

where the coefficients  $A_{nm}$  can be determined from one of the standard earth models like the Smithsonian or Goddard Earth models. These models are expressed as series in solid spherical harmonics with coefficients  $C_{1j}$  and  $S_{ij}$  like in eq. (1.2). The coefficients  $A_{nm}$  may be computed from these coefficients using e.g. eq. (22.59) of Hotine (1969).

The summation limit  $N$  formally becomes infinite, but the coefficients of degree 4 to 6 larger than the maximal degree of the spherical harmonic coefficients are very small and can be neglected.

In order to obtain density distributions related to the external potential in a simple way similar to the spherical case, we now regard distributions  $\rho$  which fulfil

$$\Delta(f(u, \beta, \lambda) \cdot \rho) = 0, \quad (1.9)$$

where  $f(u, \beta, \lambda) \neq 0$ . Then we may express  $\rho$  as a modified series in solid spherical harmonics,

$$\rho(P) = \sum_{n=0}^{\infty} \frac{1}{f(u, \beta, \lambda)} \sum_{m=-n}^n a_{nm} V_{nm}^i(P). \quad (1.10)$$

Selecting  $f(u, \beta, \lambda)$  in a suitable way, we will be able then (as shown in the sequel) to obtain relationships between the coefficients  $A_{nm}$  and  $a_{nm}$  similar to these given in eq. (1.5).

In sections 2 - 3 we will determine these relationships, simply by computing the potential of a mass distribution  $F(u, \beta, \lambda) \cdot V_{nm}^i(P)$ ,  $F(u, \beta, \lambda) = 1/f(u, \beta, \lambda)$  being a simple function. In section 4 we will describe how these relations can be used for the construction of mass distributions, which are only slight perturbations of geophysically realistic distributions. Finally, in sec. 5, we will point out some future research problems.

## 2. The potential of $V_{nm}^i(u, \beta, \lambda)$

In this section we will treat the simplest case :  
 $F = \text{constant}$ . In this case we must compute the potential of a  
 harmonic mass distribution at a point  $Q$  with coordinates  
 $(u', \beta', \lambda')$  ,

$$Y_{nm}(u', \beta', \lambda') = \int_{u=0}^{u=b} \int_{\lambda=0}^{2\pi} \int_{\beta=-\pi/2}^{\pi/2} V_{nm}^i(u, \beta, \lambda) (u^2 + E^2 \sin^2 \beta) / L \cdot \cos \beta d\beta d\lambda du \quad (2.1)$$

where  $L = ||P - Q||$  and  $(u^2 + E^2 \sin^2 \beta) \cos \beta d\beta d\lambda du$  is the  
 volume element.

According to Hotine (1969, eq. (22.55)) we have

$$\begin{aligned} \frac{E}{L} = & i \sum_{n=0}^{\infty} (2n + 1) [Q_n(i\frac{u'}{E}) P_n(i\frac{u}{E}) P_n(\sin \beta') P_n(\sin \beta) + \\ & + 2 \sum_{m=1}^n (-1)^m \left\{ \frac{(n-m)!}{(n+m)!} \right\}^2 Q_n^m(i\frac{u'}{E}) P_n^m(i\frac{u}{E}) P_n^m(\sin \beta') P_n^m(\sin \beta) \cdot \\ & \cdot \cos m(\lambda' - \lambda)] . \end{aligned} \quad (2.2)$$

When evaluating eq. (2.1) we can then take advantage of  
 the orthogonality properties of the Legendre polynomials. We  
 also have

$$\frac{1}{2\pi} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} [P_n(\sin \beta)]^2 \cos \beta d\beta d\lambda = \frac{2}{2n+1} , \quad (2.3)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} [P_n^m(\sin \beta)]^2 \left\{ \begin{array}{l} \cos^2 m\lambda \\ \sin^2 m\lambda \end{array} \right\} \cos \beta d\beta d\lambda = \frac{1}{2n+1} \frac{(n+m)!}{(n-m)!} \quad (2.4)$$

where  $m \neq 0$  in the last equation.

Let us regard the part of the integrand equal to

$$\begin{aligned}
I &= -E^2 P_n^m \left( i \frac{u}{E} \right) P_n^m (\sin \beta) \left\{ \frac{\cos m\lambda}{\sin m\lambda} \right\} \left( \left( i \frac{u}{E} \right)^2 - \sin^2 \beta \right) \\
&= -E^2 P_n^m (q) P_n^m (t) \left\{ \frac{\cos m\lambda}{\sin m\lambda} \right\} (q^2 - t^2) , \quad (2.5)
\end{aligned}$$

where  $q = i \frac{u}{E}$  and  $t = \sin \beta$ . We will express  $I$  only using Legendre polynomials.

Using the well-known recursion formula

$$(n - m + 1) P_{n+1}^m (t) = (2n + 1) t P_n^m (t) - (n + m) P_{n-1}^m (t) , \quad (2.6)$$

we have

$$t P_n^m (t) = \frac{n - m + 1}{2n + 1} P_{n+1}^m (t) + \frac{n + m}{2n - 1} P_{n-1}^m (t) , \quad (2.7a)$$

$$t P_{n-1}^m (t) = \frac{n - m}{2n - 1} P_n^m (t) + \frac{n + m - 1}{2n - 1} P_{n-2}^m (t) , \quad (2.7b)$$

$$t P_{n+1}^m (t) = \frac{n - m + 2}{2n + 3} P_{n+2}^m (t) + \frac{n + m + 1}{2n + 3} P_n^m (t) , \quad (2.7c)$$

and

$$\begin{aligned}
t^2 P_n^m (t) &= \frac{(n - m + 1)(n - m + 2)}{(2n + 1)(2n + 3)} P_{n+2}^m (t) + \\
&+ \frac{(n - m + 1)(n + m + 1)}{(2n + 1)(2n + 3)} P_n^m (t) + \\
&+ \frac{(n + m)(n - m)}{(2n + 1)(2n - 1)} P_n^m (t) + \quad \dots
\end{aligned}$$

$$+ \frac{(n+m)(n+m-1)}{(2n+1)(2n-1)} P_{n-2}^m(t) , \quad (2.8)$$

which we will write

$$= A P_{n+2}^m(t) + (B+C)P_n^m(t) + D P_{n-2}^m(t) , \quad (2.9)$$

thereby defining the quantities  $A, B, C$  and  $D$ .

Hence

$$\begin{aligned} I &= -E^2 [ (A P_{n+2}^m(q) + (B+C)P_n^m(q) + D P_{n-2}^m(q)) P_n^m(t) \left\{ \begin{array}{l} \cos m\lambda \\ \sin m\lambda \end{array} \right\} \\ &\quad + (A P_{n+2}^m(t) + (B+C)P_n^m(t) + D P_{n-2}^m(t)) \left\{ \begin{array}{l} \cos m\lambda \\ \sin m\lambda \end{array} \right\} ] \\ &= E^2 [ A P_n^m(q) P_{n+2}^m(t) - (A P_{n+2}^m(q) + D P_{n-2}^m(q)) P_n^m(t) + \\ &\quad + D P_n^m(q) P_{n-2}^m(t) ] \cdot \left\{ \begin{array}{l} \cos m\lambda \\ \sin m\lambda \end{array} \right\} . \quad (2.10) \end{aligned}$$

Using the orthogonality property and eq. (2.3) and (2.4), we get

$$\begin{aligned} Y_{nm} &= -E^2 4\pi \left[ \int_{q=0}^{ib/E} (A P_{n+2}^m(q) + D P_{n-2}^m(q)) P_n^m(q) dq \cdot \right. \\ &\quad \cdot Q_n^m(q') P_n^m(t') \cdot B' - \int_{q=0}^{ib/E} A P_n^m(q) P_{n+2}^m(q) dq \cdot \\ &\quad \cdot Q_{n+2}^m(q') P_n^m(t') \cdot A' - \int_{q=0}^{ib/E} D P_n^m(q) P_{n-2}^m(q) dq \cdot \\ &\quad \left. Q_{n-2}^m(q') P_{n-2}^m(t') \cdot D' \right] \end{aligned}$$



with  $A' = B' = D' = 1$  for  $m = 0$ ,

$$B' = \frac{(n-m)!}{(n+m)!} (-1)^m \begin{Bmatrix} \cos m\lambda' \\ \sin m\lambda' \end{Bmatrix},$$

$$A' = \frac{(n-m+2)!}{(n+m+2)!} (-1)^m \begin{Bmatrix} \cos m\lambda' \\ \sin m\lambda' \end{Bmatrix}$$

and

$$D' = \frac{(n-m-2)!}{(n+m-2)!} (-1)^m \begin{Bmatrix} \cos m\lambda' \\ \sin m\lambda' \end{Bmatrix} \quad \text{for } m, 0.$$

Then

$$\begin{aligned} Y_{nm} = & -E^2 \cdot 4\pi \left[ \int_{q=0}^{ib/E} A P_{n+2}^m(q) P_n^m(q) dq (B' Q_n^m(q') P_n^m(t') - \right. \\ & \left. - A' Q_{n+2}^m(q') P_{n+2}^m(t')) + \int_{q=0}^{ib/E} D \cdot P_n^m(q) P_{n-2}^m(q) dq \cdot \right. \\ & \left. \cdot (B' \cdot Q_n^m(q') P_n^m(t') - D' \cdot Q_{n-2}^m(q') P_{n-2}^m(t')) \right]. \end{aligned}$$

Using eq. (A.1.3) from the appendix, we have

$$\begin{aligned}
Y_{nm} = E^2 \cdot 4\pi & \left[ \frac{b^2/E^2 + 1}{-2(2n+3)} \left( P_{n+2}^m \left( i\frac{b}{E} \right) \frac{d}{dq} P_n^m \left( i\frac{b}{E} \right) - \right. \right. \\
& - P_n^m \left( i\frac{b}{E} \right) \frac{d}{dq} P_{n+2}^m \left( i\frac{b}{E} \right) \left. \right) \cdot A \cdot (B' \cdot Q_n^m (i\frac{U'}{E})) \\
& \cdot P_n^m (\sin \beta') - A' \cdot Q_{n+2}^m \left( i\frac{U'}{E} \right) P_{n+2}^m (\sin \beta') \left. \right) + \\
& + \frac{b^2/E^2 + 1}{2(2n-1)} \left( P_{n-2}^m \left( i\frac{b}{E} \right) \frac{d}{dq} P_n^m \left( i\frac{b}{E} \right) - P_n^m \left( i\frac{b}{E} \right) \frac{d}{dq} P_{n-2}^m \left( i\frac{b}{E} \right) \right) \cdot \\
& \cdot D \cdot (B' \cdot Q_n^m (i\frac{U'}{E}) P_n^m (\sin \beta) - D \cdot Q_{n-2}^m (i\frac{U'}{E}) P_{n-2}^m (\sin \beta')) \left. \right] .
\end{aligned}
\tag{2.11}$$

which may be slightly simplified by factoring  $\frac{1}{2}(b^2/E^2 + 1) = a^2/(2E^2)$  .

Example 1 : m = 0

Here

$$A = \frac{(n+1)(n+2)}{(2n+1)(2n+3)} \quad , \quad D = \frac{n(n-1)}{(2n+1)(2n-1)} \quad ,$$

$$A' = B' = D' = 1 \quad ,$$

so

$$\begin{aligned}
Y_{n0} = \frac{-2\pi a^2}{2n+1} & \left[ \frac{(n+1)(n+2)}{(2n+3)^2} \left( P_{n+2} \left( i\frac{b}{E} \right) \frac{d}{dz} P_n \left( i\frac{b}{E} \right) - \right. \right. \\
& - P_n \left( i\frac{b}{E} \right) \frac{d}{dz} P_{n+2} \left( i\frac{b}{E} \right) \left. \right) \cdot (V_{n0}^e (u', \beta', \lambda')) - \\
& - V_{n+2,0}^e (u', \beta', \lambda') \left. \right) - \frac{n(n-1)}{(2n-1)^2} \left( P_{n-2} \left( i\frac{b}{E} \right) \frac{d}{dz} P_n \left( i\frac{b}{E} \right) - \right. \\
& - P_n \left( i\frac{b}{E} \right) \frac{d}{dz} P_{n-2} \left( i\frac{b}{E} \right) \left. \right) (V_{n0}^e (u', \beta', \lambda') - V_{n-2,0}^e (u', \beta', \lambda')) \left. \right]
\end{aligned}$$

Example 2 :  $n = 0$  .

Here

$$A = \frac{2}{3}, \quad D = 0, \quad A' = B' = D' = 1,$$

so

$$\begin{aligned} Y_{00} &= \frac{-2\pi a^2}{3} \left( -P_0 \left( i\frac{b}{E} \right)^3 \frac{d}{dq} P_2 \left( i\frac{b}{E} \right) \right) \frac{2}{3} \left( Q_0 \left( i\frac{u'}{E} \right) P_0(\sin\theta') - \right. \\ &\quad \left. - Q_2 \left( i\frac{u'}{E} \right) P_2(\sin\theta') \right) \\ &= \frac{4\pi a^2 b i}{3E} \left( V_{00}^e(u', \theta', \lambda') - V_{20}^e(u', \theta', \lambda') \right). \end{aligned}$$

This agrees with MacMillan (1958, eq. (39.2)).

Example 3 :  $m = 0, n = 1$  .

$$\begin{aligned} Y_{1,0} &= -\frac{2\pi a^2}{3} \left[ \frac{2 \cdot 3}{25} \left( \frac{5}{2} \left( i\frac{b}{E} \right)^3 - \frac{3i b}{2E} - \left( i\frac{b}{E} \right) \left( \frac{15}{2} \left( i\frac{b}{E} \right)^2 - \frac{3}{2} \right) \right) \right] \\ &\quad \cdot \left( V_{10}^e - V_{30}^e \right) \\ &= -\frac{2\pi a^2 b i}{25E} \left( -5 \frac{b^2}{E^2} - 3 + 15 \frac{b^2}{E^2} + 3 \right) \left( V_{1,0}^e - V_{3,0}^e \right) \\ &= -\frac{4\pi a^2 b^3 i}{5E^3} \left( V_{10}^e - V_{30}^e \right). \end{aligned}$$

### 3. The potential of some simple non-harmonic mass-distributions.

In section 2 we found that the (harmonic part of the) potential of an internal spheroidal harmonic  $V_{nm}^i$  was equal to a linear combination of generally three external spheroidal harmonics  $V_{n+2,m}^e$ ,  $V_{nm}^e$  and  $V_{n-2,m}^e$ . The reason for this was the occurrence of the factor  $t = \sin^2 \beta$  in eq. (2.5). This factor will disappear when we compute the potential of functions

$$F(u, \beta, \lambda) V_{nm}^i(u, \beta, \lambda) = \frac{F_1(q)}{(q^2 - t^2)} V_{nm}^i(q, t, \lambda), \quad (3.1)$$

where we again have put  $q = i \frac{u}{E}$ .

If we choose  $F_1(q)$  to be equal to a polynomial in  $q$ , then we may carry out the needed integrations without difficulty. We may even choose  $F_1(q)$  to be equal to different polynomials in different intervals of  $q$ , i.e.  $F_1(q)$  may be a discontinuous function.

In order to carry out the integrations we must first convert the quantities  $q^k P_n^m(q)$  into a linear combination of associated Legendre polynomials :

$$q^k P_n^m(q) = \sum_{j=-k}^k C_n^m(k, j) P_{n+j}^m(q). \quad (3.2)$$

From eq. (2.7a) we have  $C_n^m(0, 0) = 1$ ,

$$C_n^m(1, -1) = \frac{n + m}{2n - 1}, \quad C_n^m(1, 0) = 0 \quad \text{and} \quad C_n^m(1, 1) = \frac{n - m + 1}{2n + 1}$$

(3.3)

Then we get the following recursion formula for the determination of the coefficients

$$\begin{aligned}
 C_n^m(k, j) &= C_n^m(k-1, j-1) C_{n+j-1}^m(1, 1) + C_n^m(k-1, j+1) C_{n+j+1}^m(1, -1) \\
 &= C_n^m(k-1, j-1) \frac{n+j-m}{2(n+j)-1} + C_n^m(k-1, j+1) \cdot \\
 &\quad \cdot \frac{n+j+m+1}{2(n+j)+3} .
 \end{aligned} \tag{3.4}$$

(We must put  $C_n^m(k, j) = 0$  for  $|j| > k$ .)

For  $F_1(q) = \sum_{i=0}^I c_i q^i$  we then have

$$\begin{aligned}
 F_1(q) P_n^m(q) &= \sum_{i=0}^I c_i q^i P_n^m(q) \\
 &= \sum_{i=0}^I c_i \sum_{j=-i}^i C_n^m(i, j) P_{n+j}^m(q) ,
 \end{aligned} \tag{3.5}$$

which we will write as

$$= \sum_{j=-I}^I c_j^* P_{n+j}^m(q) ,$$

which also will define the constants  $c_j^*$

We then have

$$\begin{aligned}
Y_{nm}(u', \beta', \lambda') &= \int_{u=0}^b \int_{\lambda=0}^{2\pi} \int_{\beta=-\pi/2}^{\pi/2} F_1(q) V_{nm}^i(u, \beta, \lambda) / L \cos \beta d\beta d\lambda du \\
&= -E^2 4\pi \left[ \int_{q=0}^{q=ib/E} \sum_{j=-I}^I c_j^* P_{n+j}^m(q) P_n^m(q) dq \right] \cdot \\
&\quad \cdot Q_n^m(q') P_n^m(t') \cdot B' \\
&= -E^2 4\pi \left[ \sum_{j=-I}^I c_j^* \int_{q=0}^{iu/b} P_{n+j}^m(q) P_n^m(q) dq \right] \cdot \\
&\quad \cdot Q_n^m(q') P_n^m(t') B' \\
&= B_{nm} V_{nm}^e(u', \beta', \lambda') , \tag{3.6}
\end{aligned}$$

where

$$B_{nm} = -4\pi E^2 \frac{(n-m)!}{(n+m)!} (-1)^m \sum_{j=-I}^I c_j^* \int_{q=0}^{ib/E} P_{n+j}^m(q) P_n^m(q) dq . \tag{3.7}$$

In a similar way we can handle functions  $F(u, \beta, \lambda)$ , which are polynomials in  $q$  and  $t$  (or in  $r^2 = u^2 - E^2 \sin^2 \beta$ ).

The potential  $Y_{nm}$  of a function  $F(u, \beta, \lambda) V_{nm}^i(u, \beta, \lambda)$  will be a linear combination of external solid harmonics

$$Y_{nm}(u', \beta', \lambda') = \sum_{k \in J} B_{nm}^k V_{km}^e(u', \beta', \lambda') , \tag{3.8}$$

where  $J$  is a suitable index-set, e.g., equal to  $n+2$ ,  $n$  and  $n-2$  when  $F$  is equal to a constant as used in section 2.

4. Construction of mass distributions.

Let us suppose that the external potential is given by eq. (1.8) or equivalently by a set of coefficients  $GM_{nm}$ ,  $n \leq N$ .

If we require the density distribution to fulfil the condition (1.9) with  $F(u, \beta, h)$  equal to a polynomial in  $q = \frac{u}{E}$  and  $t = \sin\beta$  in each interval  $u_j < u < u_{j+1}$ ,  $j = 0, \dots, M$ ,  $u_1 \leq u_{j+1}$ ,  $u_0 = 0$  and  $u_M = b$ , then eq. (3.8) is valid. For the density distribution given by

$$\rho(u, \beta, \lambda) = \sum_{n=0}^N \sum_{m=-n}^n a_{nm} F(u, \beta, \lambda) V_{nm}^i(u, \beta, \lambda), \tag{4.1}$$

we must then have the relationship

$$\sum_{n \in J} B_{nm}^j a_{nm} = GM \cdot A_{jm}. \tag{4.2}$$

For a given set of coefficients it will only in a few cases be possible to solve the set of linear equations (4.2).

If the coefficients  $B_{nm}^j$ , which cannot be solved for, are very large, one may artificially adopt some very small potential coefficients  $A_{nm}$ , and then solve the equations. Or, using an elimination method, one may start with the high order and degree terms, ending up with a mass distribution  $\rho_1$  and some residuals  $A''_{nm}$  for the low degree terms. These residuals can then be used for the computation of a mass distribution  $\rho_0$ , using one of the functions  $F$ , which gives a one to one correspondence between  $A_{nm}^c$  and the

coefficients  $a_{nm}^0$  of a mass distribution  $\rho_0$ . We will then have  $\rho = \rho_0 + \rho_1$ .

We have performed calculations only with the simple case, described in section 3, where  $F(u, \beta, \lambda) = F_1(q) / (q^2 - t^2)$ . Then we simply have

$$a_{nm} = GM \cdot A_{nm} / B_{nm}^0. \quad (4.3)$$

In Figure 1 the radial variations of the mass distribution are shown, obtained by using the set of coefficients GEM10B and  $F_1(q) = 1, q, q^2$  and  $q^3$ , respectively. Note that for  $F_1(q) = q^2$ , we will get a mass distribution which is approximately harmonic. None of these results is geophysically meaningful.

A more meaningful result was obtained using  $F_1(q)$  equal to  $q^2$  times a polynomial expression  $\rho_0$  for the radial variation of the mass distribution, (with  $u/b$  substituted for  $r/R$  as a parameter), see Moritz (1968, eq. 88). The reason for this is obvious; the condition (1.9) in this case is fulfilled because  $\rho/\rho_0$  is approximately equal to the constant function 1, which certainly is harmonic.

(The use of a discontinuous function  $F_1(q) = \rho_0 \cdot q^2$ , will give a mass distribution which probably is geophysically more meaningful.)

Numerical experiments with an alternative method (inspired by methods discussed in Moritz (1968, 1973)), resulted in the following procedure, which will produce a geophysically meaningful mass distribution. Instead of starting by trying to solve the equations (4.2), the idea is to use a parametric earth (mass) model,  $\rho_0$ , and then determine a small perturbation to this model, using eq. (4.3).



We have carried out this procedure for one specific parametric earth model (PEM), given by Dziewonski et al., (1975, Table 1), and one set of potential coefficients, GEM10B (Lerch et al. 1978).

We chose the PEM describing an average structure (with respect to the continents and the oceans). This model is given in spherical approximation, by 8 polynomials in different continuity intervals with  $r/R$  as a parameter. We modified the polynomials, so that each of them was defined from 0 to  $R_i$ , where  $R_i$  is the radius of the  $i$ 'th surface of discontinuity.

Instead of using  $r/R$  as a parameter, we then used  $u_i/b_i$ , where  $b_i$  is the semi-minor axis of an ellipsoid having the same volume as the sphere with radius  $R_i$ .  $u_i$  is the ellipsoidal  $u$ -coordinate, so that  $u_i = b_i$  on the ellipsoid. We fixed  $b_8$ , so that it was equal to  $b$ , but varied all the others, so that the potential produced by the mass distribution agreed with  $GM$  and  $C_{2,0}$  of the set of potential coefficients used (GEM10B). We parameterized the  $b_i$  by putting

$$\frac{a_i^2 - b_i^2}{a_i^2} = \frac{a^2 - b^2}{a^2} \left( \frac{R_i}{R} \right)^c, \quad (4.4)$$

which through an iterative procedure gave the correct  $C_{2,0}$  value for  $c \approx 11.2$ . Let us denote this mass distribution by  $c_0$ , and the potential coefficients by  $h_{nm}^3$ . (Only the even zonal harmonics are different from zero because of the rotational and equatorial symmetry).

The set of potential coefficients GEM10B were slightly modified by subtracting the potential coefficients of the

isostatically compensated topography, computed as described in (Lachapelle, 1975). We then had the potential coefficients of an ellipsoidal Earth, without topography, which were converted into coefficients  $A_{nm}$  of an ellipsoidal harmonic series with  $N = 36 + 4 = 40$ .

The residual coefficients  $A_{nm}^1 = A_{nm} - A_{nm}^0$  were then converted using eq. (4.3), with  $F_1(u, \beta, \lambda) = u^2$ , resulting in a (residual) mass distribution  $\rho_1$ , which is not far from harmonic. (A geophysical interpretation of a harmonic residual mass distribution is given by (Schwiderski, 1967).) The total mass distribution  $\rho = \rho_0 + \rho_1$  will produce an external potential, with the ellipsoidal harmonic coefficients,  $A_{nm}$ . Figure 2 shows the variation of  $\rho_1$  at various depths. We see that the maximum absolute values numerically are below  $0.01 \text{ g/cm}^3$ , i.e.  $\rho$  is only a slight perturbation of  $\rho_0$ , and hence also a geophysically realistic PEM. A minor deficiency is the behavior near the Earth center, enforced by using ellipsoidal coordinates. The deficiency, however, disappears when  $\rho_1$  tends to zero near the Earth's center.

## 5. Conclusion.

In this paper we have given examples of possible methods of constructing mass density distributions, which will produce an approximation to the external gravity potential given by a set of low degree potential coefficients. We also have carried through a computation which gave a fairly realistic mass distribution as a result, or more correctly, the perturbations of the PEM used, were insignificant.

There are, however, some problems that need to be solved, primarily related to the use of PEM. Which excentricities should be adopted for the different discontinuity surfaces, which mean density should be used at the surface of the Earth and how do we get rid of the problems originating from the use of an ellipsoidal coordinate system? We have throughout used the condition eq. (1.9). This condition, however, is just one which makes the construction of density distributions possible in a unique way. The ideal situation would be to have a condition which corresponded to the minimalization of an integral formula like the one discussed in Moritz (1968, sec. 12), see also Rubincam (1979).

#### Acknowledgement.

The cooperation of the authors was initiated in June 1979, when the first author, supported by a NATO Grant, visited the Ohio State University, where the second author worked at that time. The support of NATO Grant No. 1378 is therefore gratefully acknowledged.

This paper is a revised version of a paper presented at the 7th International Symposium "Geodesy and Physics of the Earth", Karl-Marx-Stadt, GDR, May 12 - 17, 1980.

Appendix.

Integrals of products of associated Legendre polynomials of the same order.

The associated Legendre polynomials satisfy the well-known differential-equation

$$(1 - t^2) \frac{d^2}{dt^2} P_n^m(t) - 2t \frac{d}{dt} P_n^m(t) + [n(n+1) - \frac{m^2}{t^2}] \cdot P_n^m(t) = 0, \quad (\text{A } 1.1)$$

which we will use in order to evaluate the following integral for  $n \neq k$ ,

$$I = \int_0^1 P_n^m(z) P_k^m(z) dz. \quad (\text{A } 1.2)$$

We have

$$\begin{aligned} (n - k)(n + k + 1) &= n^2 + nk + n - nk - k^2 - k = \\ &= n^2 - n - k^2 - k = n(n + 1) - k(k + 1) \end{aligned}$$

and

$$\begin{aligned} n(n + 1)P_n^m(z) &= (z^2 - 1) \frac{d^2}{dz^2} P_n^m(z) + 2z \frac{d}{dz} P_n^m(z) + \\ &+ \frac{m^2}{1 - z^2} P_n^m(z), \end{aligned}$$

hence

$$\begin{aligned}
 & [n(n+1) - k(k+1)] P_n^m(z) P_k^m(z) = \\
 & = P_k^m(z) [n(n+1) P_n^m(z)] - P_n^m(z) [k(k+1) P_k^m(z)] \\
 & = P_k^m(z) \left[ (z^2 - 1) \frac{d^2}{dz^2} P_n^m(z) + 2z \frac{d}{dz} P_n^m(z) \right] - \\
 & \quad - P_n^m(z) \left[ (z^2 - 1) \frac{d^2}{dz^2} P_k^m(z) + 2z \frac{d}{dz} P_k^m(z) \right] \\
 & = (z^2 - 1) \left[ \frac{d^2}{dz^2} P_n^m(z) P_k^m(z) - \frac{d^2}{dz^2} P_k^m(z) \cdot P_n^m(z) \right] + \\
 & \quad + 2z \left[ P_k^m(z) \frac{d}{dz} P_n^m(z) - P_n^m(z) \frac{d}{dz} P_k^m(z) \right] \\
 & = \frac{d}{dz} \left[ (z^2 - 1) \left( P_k^m(z) \frac{d}{dz} P_n^m(z) - P_n^m(z) \frac{d}{dz} P_k^m(z) \right) \right] \\
 & = \frac{d}{dz} \left\{ P_k^m(z) \left[ \frac{n(n+m+1)}{2n+1} P_{n+1}^m(z) - \frac{(n+1)(n+m)}{2n+1} P_{n-1}^m(z) \right] \right. \\
 & \quad \left. - P_n^m(z) \left[ \frac{k(k-m+1)}{2k+1} P_{k+1}^m(z) - \frac{(k+1)(k+m)}{2k+1} P_{k-1}^m(z) \right] \right\}
 \end{aligned}$$

For  $z = 0$  we know that either  $P_n^m(z) = 0$  or  $\frac{d}{dz} P_n^m(z) = 0$ ,  
 so for  $k - n$  even and  $\neq 0$  we have

$$\begin{aligned}
 & \int_0^t P_n^m(z) P_k^m(z) dz \\
 & = \frac{t^2 - 1}{(n-k)(n+k+1)} \left[ P_k^m(t) \frac{d}{dz} P_n^m(t) - P_n^m(t) \frac{d}{dz} P_k^m(t) \right] \\
 & \hspace{25em} (A 1.3)
 \end{aligned}$$

(A 1.3)

$$\begin{aligned}
&= \frac{1}{(n-k)(n+k+1)} \left[ P_k^m(t) \left( \frac{n(n-m+1)}{2n+1} P_{n+1}^m(t) - \right. \right. \\
&\quad \left. \left. - \frac{(n+1)(n+m)}{2n+1} P_{n-1}^m(t) \right) - \right. \\
&\quad \left. - P_n^m(t) \left( \frac{k(k-m+1)}{2k+1} P_{k+1}^m(t) - \frac{(k+1)(k+m)}{2k+1} \right. \right. \\
&\quad \left. \left. \cdot P_{k-1}^m(t) \right) \right] .
\end{aligned}$$

For  $k - n$  odd, terms with  $P_n^m(0)$  or  $P_k^m(0)$  must be added.

#### REFERENCES

- DZIEWONSKI, A.M., A.L. HALES, and E.R. LAPWOOD : *Parametrically simple earth models consistent with geophysical data*. Physics of the Earth and Planetary Interiors, Vol. 10, pp. 12-48, 1975.
- HEISKANEN, W.A. and H. MORITZ : *Physical Geodesy*. W.H. Freeman, San Francisco, 1967.
- BOTINE, M. : *Mathematical Geodesy*. ESSA Monograph No.2, Washington, D.C., 1969.
- JORDAN, S.K. : *Statistical model for gravity, topography and density contrast in the earth*. J.Geophys.Res., Vol. 83, No. B4, pp. 1816 - 1827, 1978.
- LERCH, F.J., C.A. WAGNER, S.M. KLOSKO, R.P. BELOTT, F.E. LAUBSCHER, and W.A. TAYSOR : *Gravity model improvement using GEOS-3 altimetry (GEM10A and 10B)*. Presented at the AGU Spring Meeting, April 1978.
- LACHAPPELLE, G. : *Determination of the geoid using heterogeneous data*. Mitteilungen der geodätischen Institute der Technischen Universität Graz, Folge 19, 1975.
- MACMILLAN, W.D. : *The theory of the potential*. Dover reprint, New York, 1958.

- MORITZ, H. : *Density distributions for the equipotential ellipsoid*. Report No. 115, Department of Geodetic Science, The Ohio State University, Columbus, Ohio, 1968.
- MORITZ, H. : *Ellipsoidal mass distributions*. Report No. 206, Department of Geodetic Science, The Ohio State University, Columbus, Ohio, 1973.
- RUBINCAM, D.P. : *Gravitational potential energy of the earth: A spherical harmonic approach*. Journal of Geophysical Research, Vol. 64, No. B11, pp. 6219 - 6225, 1979.
- SCHWIDERSKI, E.W. : *The deep structure of the earth inferred from a satellite's orbit, Part I : The density anomaly*. U.S. Naval Weapons Laboratory, Technical report No. 2077, 1967.
- TSCHERNING, C.C. : *Some simple methods for the unique assignment of a density distribution to a harmonic function*. Report No. 213, Department of Geodetic Science, The Ohio State University, Columbus, Ohio, 1974.
- TSCHERNING, C.C. : *Models for the auto- and cross covariances between mass density anomalies and first and second order derivatives of the anomalous potential of the earth*. Proceedings 3rd International Symposium "Geodesy and Physics of the Earth", Part 2, pp. 261 - 268, 1977.

received: 1980 - 10 - 21

accepted: 1981 - 04 - 25

addresses:

Christian Tscherning  
 Geodetic Institute  
 Geodetic Department I  
 Gamlehavet Alle 22  
 DK - 2920 Charlottenlund  
 DEhMARK

Hans Sünkel  
 Institute of Physical Geodesy  
 Technical University Graz  
 Steyrergasse 17  
 A - 8010 Graz  
 AUSTRIA

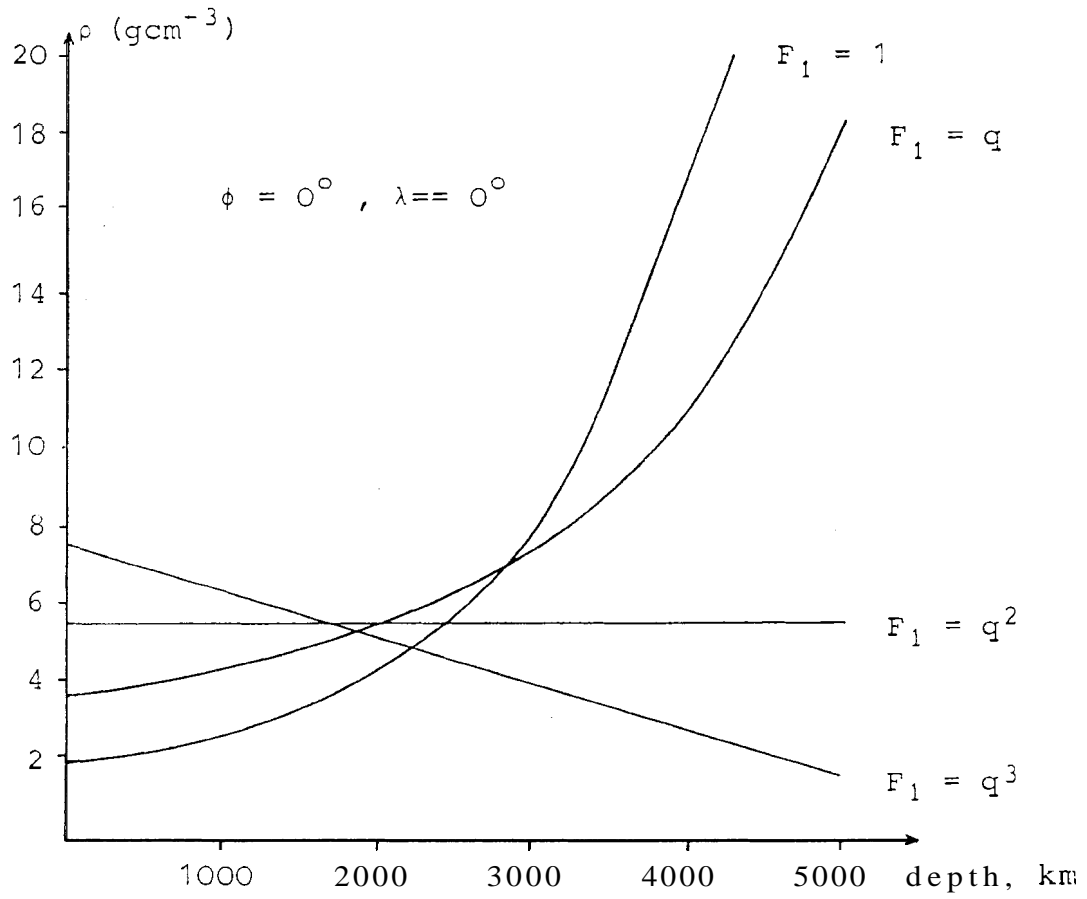


Fig. 1 Density variation for varying depths corresponding to different weight functions,  $F_1$ . GEM10B with  $n=12$  used

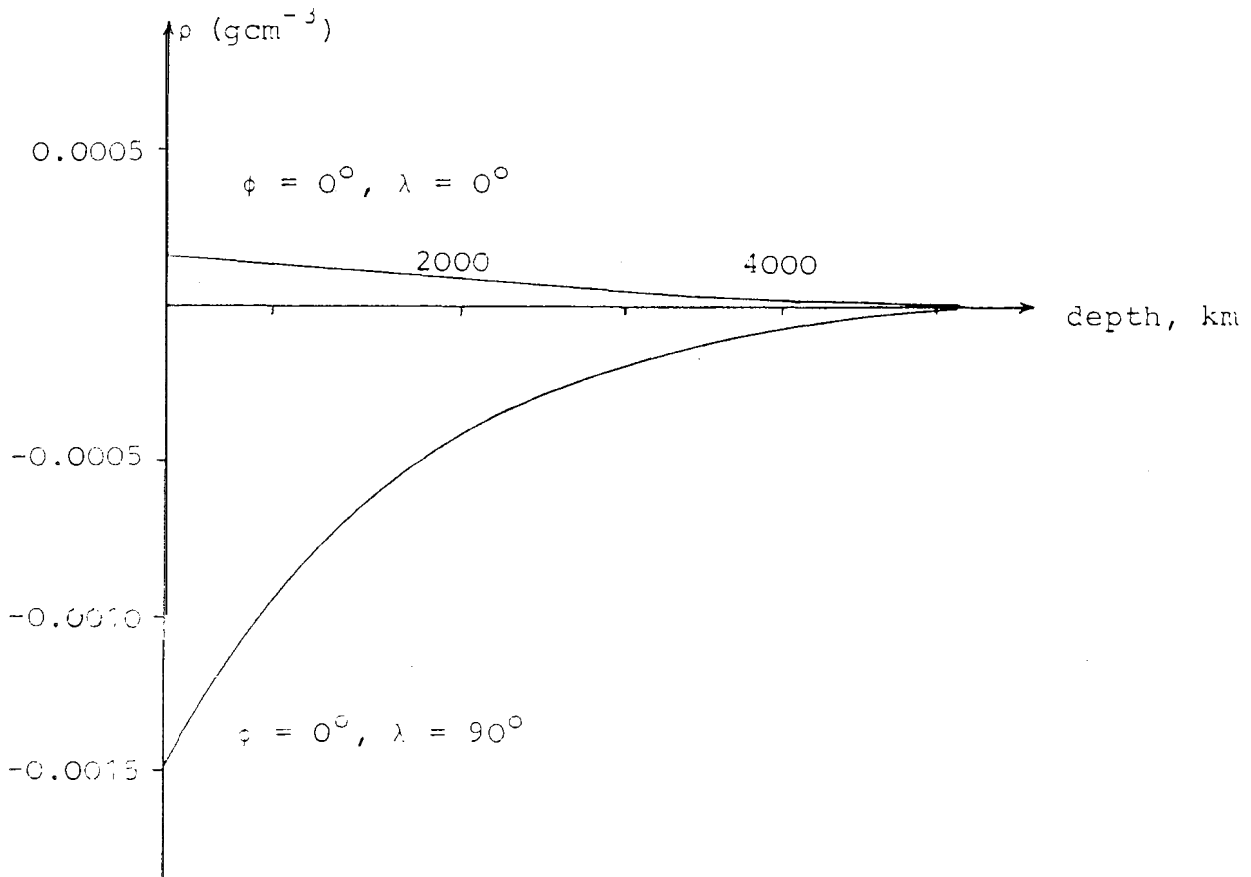


Fig. 2 Density anomaly variation with depth. GEM10B with  $n=12$  and  $F = q^2$  used







