REVIEWS OF GEOPHYSICS AND SPACE PHYSICS, VOL. 19, NO. 1, PAGES 213–221, FEBRUARY 1981

73

Comparison of Some Methods for the Detailed Representation of the Earth's Gravity Field

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Currently, two types of methods are used for the detailed representation of the earth's gravity potential, namely, integral formulae and approximation techniques. The theoretical advantages and disadvantages of the methods (or of a combination of the methods) are discussed, and the results obtained are compared. If the methods are judged according to their general applicability, flexibility, and possibilities for obtaining error estimates, then approximation techniques like least squares or minimum norm collocation must be preferred. Integral formulae techniques are, on the other hand, well suited in areas with a homogeneous data coverage.

1. Introduction

Let W denote the gravitational potential of the earth and U a suitable approximation to W, for example, defined as a linear combination of low-order solid spherical harmonic plus the rotational potential. We will not presuppose that the anomalous potential $T = W - U$ is harmonic outside the surface of the earth. Neither will we presuppose that the usual regularity conditions at infinity are fulfilled. This means that we do not presuppose the precise knowledge of the rotational axis, the gravity center, or the total mass of the earth.

We will keep in mind the fact that W is produced by a volume distribution of mass and the rotation of the earth. This means that T not only is smooth in the open set Ω outside the surface of the earth but also behaves well at the boundary ω [*Sansò*, 1980].

A representation of the earth's gravity field will in this paper be any numerical procedure capable of computing an approximation $\tilde{T}$ to the anomalous potential or to the value of some functional F (linear or nonlinear) applied on T, $\tilde{FT}$. (Sometimes an approximation $\tilde{T}$ will be constructed, in which case we may be able to compute $F(\tilde{T})$.) The main point is that we do not make a distinction between T and the values of functionals applied on T.

The functionals, which we will actually regard, will all be linearized and will be denoted by L, contingently with a subscript, for example, L_i. The functionals will be of the kind which, when applied on T, give linear combinations of some of the following quantities:

Height anomaly

$$\zeta = T/\gamma \tag{1}$$

Gravity anomaly

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r}T \tag{2}$$

Meridian and prime vertical components of the deflection of the vertical

$$\xi = -\frac{1}{\gamma \cdot r}\frac{\partial T}{\partial \varphi} \tag{3}$$

$$\eta = -\frac{1}{\gamma \cdot r \cdot \cos\varphi}\frac{\partial T}{\partial \lambda} \tag{4}$$

Gravity gradients

$$\partial^2 T/\partial x_i\, \partial x_j$$

where the quantities are evaluated at a point P, with latitude φ, longitude λ, radial distance r, and the normal gravity $\gamma = |\nabla U|$, x_i being the coordinates of a suitable local or global Cartesian coordinate system.

Gravity field representations are used for many different purposes, and some types of representations are suitable for one purpose but not necessarily for the others. A representation of $\tilde{T}$ as a series in solid spherical harmonics is suitable for the analysis of certain satellite orbit perturbations but is certainly not suitable for the computation of a precise gravity value needed at a leveling point.

Let us mention different areas where a precise, detailed representation of the gravity field is needed: (1) three-dimensional adjustment of geodetic networks or 'reduction' of observations for a two-dimensional adjustment (standard deviations σ needed: $\sigma(\tilde{\zeta}) \approx \pm 0.5$ m, $\sigma(\tilde{\xi}) = \sigma(\tilde{\eta}) \approx \pm 1''$), (2) computation of orthometric heights H from ellipsoidal heights h ($\sigma(\tilde{H} - h) = \sigma(\tilde{\zeta}) < 1$ m for mapping purposes and <0.05 m for geodynamic purposes), (3) computation of potential differences in a leveling network from observed height differences ($\sigma(\tilde{\Delta g}) \approx \pm 1$ mGal), (4) determination of a reference surface for oceanography ($\sigma(\tilde{\zeta}) \approx \pm 0.1 - \pm 0.01$ m), (5) computation of gravity vector components for inertial navigation and surveying and support of advanced military weapon systems ($\sigma(\tilde{\xi}) = \sigma(\tilde{\eta}) \approx \pm 0.5''$, $\sigma(\tilde{\Delta g}) \approx \pm 1$ mGal), and (6) computation of the density distribution inside the earth (prospecting): ($\sigma(\tilde{\Delta g}) \approx \pm 10 - \pm 0.1$ mGal, depending on actual purpose).

One method of obtaining a quantity $L(T)$ is, naturally, to observe the quantity. If this is not feasible or possible, a representation $\tilde{T}$ (or $L(\tilde{T})$) must be constructed by using our knowledge of the gravity field, that is, the many different types of observations. These do not consist only of the observed values of ζ, Δg, ξ, η, etc., but also of the coordinates of the points in which they have been observed. Furthermore, angles between points, observed by using a theodolite, contain gravity field information, because the scale readings are carried out with the instrument axis near to the plumb line [see *Eeg and Krarup*, 1975].

The purpose of this paper is to compare different representation techniques. This will be done not only on the basis of actual results obtained by using the different techniques but also on the basis of whether the methods can handle the type of information we actually have: we do not know station positions precisely; the position of the coordinate systems with respect to the earth's center of mass and rotational axis is not yet

Paper number 80R1457.
0034-6853/81/080R-1457$06.00

known exactly, and the total mass is given approximately.

In section 2 we have tried to review the different methods in a systematic manner, and in section 3 we confront the methods with three types of questions concerning (1) theoretical properties, (2) flexibility and reliability, and (3) computational properties. Finally, in section 4 some of the results actually obtained are reviewed.

2. METHODS FOR GRAVITY FIELD REPRESENTATION

A General Overview of the Methods

A wide range of techniques are used, or have been proposed, for the representation of the gravity field. Following *Moritz* [1978*a*], we may regard the methods as belonging to one of two approaches to physical geodesy: the model approach or the operational approach.

In the model approach a mathematical model is set up, which from a given set of data will determine the required quantity. The problem is then to obtain the data.

The operational approach starts with the available data and asks how these data can be combined in the best way in order to obtain the required quantity.

Now these approaches are in a certain sense dual: the justification for large-scale observation campaigns of gravity was that the data could be used for the solution of the geodetic boundary value problem. On the other hand, data have often been collected for some specific reason (like rock density values for geological investigations), and the problem has then been how to use this kind of data for gravity field representation.

Therefore the model approach will sometimes fit the real-world situation, at least to a certain degree. Mostly, it will not fit, but in reality we will see that the two approaches go hand in hand. In fact, a model approach, which has no restrictions whatsoever in the use of data, is equivalent to an operational approach, and an operational approach may (partially) consist of finding such 'corrections' to or 'reductions' of the observations that they will suit the model.

The Model Approach (Not an Exhaustive Review)

In the model approach, known relations between T and different types of observations are used. Presupposing harmonicity of T in Ω and regularity at infinity, $\tilde{T}$ will be the solution to different boundary value problems. The solutions manifest themselves as integral formulae: Stokes' integral, Poisson's integral (with data given on a sphere), or the proposed integral formula solution to Molodensky's problem. Application of the respective linear functionals on these formulae gives corresponding integral formulae for these functionals, known, for example, as Vening Meinesz integral formulae.

The general expression for $\tilde{T}$ is

$$\tilde{T}(P) = \int_{\omega} K(P, Q) \left[D(Q) - \sum_{i=1}^{n} C_i(Q) \right] d\omega_Q \qquad (5)$$

where P is a point in $\tilde{\Omega}$, Q is a point on the boundary ω, $K(P, Q)$ is a kernel function, $D(Q)$ is the data, and the $C_i(Q)$ are certain correction terms (compare, for example, *Heiskanen and Moritz* [1967, equation (8-50)]). Note that the data necessary for the numerical evaluation of (5) must be available in a regular grid around P (or around P's projection on ω, if P is a point in free space). The grid values are generally determined from irregularly spaced data by methods developed in the operational approach.

The technique of astrogeodetic geoid determination also belongs to the model approach. The model is very simple, and harmonicity or regularity of T is not presupposed. The anomalous potential restricted to the reference ellipsoid is regarded as a function of latitude and longitude only. The deflections of the vertical are then associated in a simple manner with T through the horizontal derivatives (compare (3) and (4)). $\tilde{T}$ (or $\zeta = \tilde{T}/\gamma$) is then determined through numerical integration. This model requires, however, that the deflections of the vertical in some way be reduced to the ellipsoid (or geoid). The method has the drawback that the two-dimensional numerical integration may break down if only one component of the deflection of the vertical is available. In this situation, only geoid profiles can be determined, or the one-component stations are left out! (See, for example, *Wenzel* [1978, Figure 1]).

The 'worst' example of the model approach is probably the harmonic sampling functions of *Giacaglia and Lundquist* [1972]. This method requires that the values of ζ or Δg be available in a specific, predefined set of points in space. The approximation $\tilde{T}$ is then given as a linear combination of harmonic functions with the observed values as coefficients. (Because of the global character of the sampling functions they are not well suited for the detailed representation of the gravity field.)

The Operational Approach

While the model approach, until quite recently, has been used exclusively when dealing with the gravity field of the earth, the operational approach (namely, least squares adjustment) has always dominated geometrical geodesy. This is obviously due to the simple relation between model and data. First, through recent developments in mathematics and statistics it has been possible to generalize least squares techniques from finite dimensional spaces to spaces of infinite dimension.

We will now presuppose that T can be expressed as the sum of a harmonic and a regular function (which we will also denote by T) and some correction functions $A \cdot X$, where X is a vector of constants describing corrections to the reference system orientation and position with respect to the earth's rotation axis and gravity center, for example. (A modeling of time-dependent changes can also be made here.) We may also introduce other parameters, such as station position corrections. The important point here is that we only deal with a finite number of parameters.

Here T will be an element of a reproducing kernel Hilbert space of harmonic functions (because it is the potential of a difference between volume density distributions), and X will be an element of a finite dimensional real vector space, which we may furnish with a norm, for example, inherited from the statistical distribution of the observation errors. This means that our task is the determination of a point in an infinite dimensional Hilbert space spanned by the harmonic functions and the parameters X. We will suppose that there are a total of k parameters, X_j, $j = 1, \cdots, k$.

The observations D_i will be related to T and X through linearized functionals

$$D_i = L_i(T) + L_i(A) \cdot X + \text{observational error} \qquad i = 1, \cdots, n \qquad (6)$$

We are hence faced with an improperly posed problem, namely, the determination of the T and X in all infinitely many coordinates.

Different techniques can be used in order to find a solution; however, only the following two seem to have been used in geodesy: (1) a change of the space and/or topology and (2) the use of a regularization operator.

Moritz [1978*a*] also lists the use of probabilistic methods. However, the methods, which are used for gravity field representation and are called probabilistic, do not seem to me to be truly probabilistic. (The proposed methods are under all circumstances equivalent to regularization methods, because the covariance functions used are reproducing kernels belonging to some Hilbert space of harmonic functions.)

A large number of methods have been based on a change in the solution space. A finite set of functions h_i, $i = 1, \cdots, m$ are selected ($m + k \leq n$), and the approximation $\tilde{T}$ is obtained as a linear combination

$$\tilde{T}(P) = \sum_{i=1}^{m} a_i h_i \tag{7}$$

The observation equations

$$D_i = L_i(\tilde{T}) + L_i(A) \cdot X = \sum_{j=1}^{m} a_j L_i h_j + L_i(A) \cdot X \tag{8}$$

form, for $m + k < n$, an overdetermined system, which can be solved by using traditional least squares techniques, using the variance-covariance matrix of the observation errors $\{P_{ij}\}^{-1}$. In case the resulting system of normal equations is not of full rank, or in case the solution oscillates 'too much,' a regularization procedure may be used. Let $\| \quad \|$ be a norm on the $m + k$ dimensional space spanned by the functions h_i and the parameters, with

$$\| (\tilde{T}, X) \|^2 = \| \tilde{T} \|_0^2 + X^T Q X \tag{9}$$

where $\| \quad \|_0$ is the norm in the Hilbert space and Q is a positive definite $k \times k$ matrix. A solution is then obtained by minimizing

$$\lambda \cdot \| (\tilde{T}, X) \|^2 + \{D_j - L_j(\tilde{T}) - L_j(A) \cdot X\}^T \{P_{ij}\} \{D_i - L_i(T) - L_i(A) \cdot X\} \tag{10}$$

where $\lambda > 0$. The result is a new set of normal equations. (This is the procedure used, for example, for the construction of the satellite solutions GEM 9 and 10 [see *Lerch et al.*, 1979].

In the case $k = 0$ and $m = n$ the coefficients $\{a_i\}$ are determined as solutions to the system of equations

$$L_j(\tilde{T}) = D_j = \sum_{i=1}^{n} a_i L_j h_i = \{a_i\}^T \{L_j h_i\} \tag{11}$$

This set of equations may be solved when the functions h_i and the linear functionals L_i (regarded as elements of the space dual to the space spanned by $\{h_j\}$) are linearly independent. This technique is called 'pure collocation,' because an exact agreement between $\tilde{T}$ and the data is enforced.

If $m + k < n$, we are again faced with an improperly posed problem, which is generally solved by using a regularization technique (see discussion below).

If the main goal is not to determine $\tilde{T}$ but some operator F applied on $\tilde{T}$, the same procedure can be used. F can, for example, be the restriction operator which restricts T to a set R, for example, the surface of the earth, an ellipsoid, or a sphere. In these cases, T_R is a function of two variables, which are generally taken to be the latitude and the longitude. This requires that the data refer to points in R. This fact is sometimes overlooked and results in a misleading terminology (such as 'inverse Stokes operator,' the simple, correct version of which is (2); see also *Nash and Jordan* [1978], who never take into account the topography of the earth).

The different methods vary mostly according to the kind of base functions h_i which are used. Preferably, harmonic functions are used, such as the potentials of point masses or surface densities or harmonic kernel functions [*Lelgemann,* 1978*a*]. These functions can be chosen so that they are well suited for local, detailed gravity field approximation. A suitable choice of base functions can make $\tilde{T}$ identical to a solution obtained by using regularization.

Also, nonharmonic functions such as spline functions and traditional polynomials are used. So-called finite elements (for example, a set of polynomials, each valid for disjoint sets in $\mathbf{R}^3$) have also been proposed [*Junkins and Engels,* 1979], and *Batdorf* [1970] has considered some nearly harmonic 'B functions.'

Such methods require that the harmonicity of T also be represented as an observation, a fact which sometimes seems to have been forgotten. In the same way, additional conditions have to be introduced, when using potentials of point masses, in order not to violate the condition that T be regular at infinity (see, for example, *Blaha* [1977]).

The 'method of regularization' consists of minimizing linear combinations of norms (e.g., those given by (10)), where $\tilde{T}$ is now regarded as an element of an infinite dimensional Hilbert space in which T is an element. In this case the minimization of the quantity in (10) will furnish us with an n-dimensional subspace spanned by harmonic functions. These functions are equal to the linear functional L_i applied on the reproducing kernel $K(P, Q)$, that is,

$$\tilde{T}(Q) = \sum_{i=1}^{n} a_i L_i K(\cdot, Q) \tag{12}$$

(where the dot indicates that L_i has been applied on $K(P, Q)$ with respect to the variable P). The coefficients can be determined by pure collocation, (11) for $k = 0$. This technique is known as minimum norm collocation [cf. *Tscherning,* 1978*b*], and the normal equations become

$$\{L_i L_j K(\cdot, \cdot) + P_{ij}'\} \{a_j\} = \{D_i\} \tag{13}$$

which are symmetric ($\{P_{ij}'\} = \{P_{ij}\}^{-1}$). For $k > 0$, see, for example, *Tscherning* [1978*a*].

Using regularization, we are faced with the problem of selecting a suitable Hilbert space, that is, a set of functions and a quadratic norm. This choice will determine the reproducing kernel and vice versa.

Until recently, the only reproducing kernels which were explicitly known and computable were for a Hilbert space consisting of functions harmonic outside a sphere. So in order to use regularization one had to change solution space first.

These kernels could in many cases be represented as closed expressions, which greatly facilitated the computations [see *Tscherning and Rapp,* 1974; *Tscherning,* 1976*a*; *Heller and Jordan,* 1979]. The kernel proposed by *Krarup* [1978] does not require a change in solution space, but integration along the true earth's surface is necessary.

Hence numerical reasons have until now made it necessary to change solution space. (This is quite a serious operation [e.g., *Sjöberg,* 1978, Figure 6].) However, the change still permits us to associate the linear functionals with their true position in space. On the other hand, it is customary to work in spherical approximation, and the error introduced thereby is small but not negligible.

We lose, however, the important property that $\tilde{T}$ is an element of the same Hilbert space as T. If they are elements of the same space, then maximal error estimates can be computed [see *Dermanis*, 1976; *Krarup*, 1978; *Tscherning*, 1977].

Solutions optimal in a least squares sense may be obtained by choosing a norm which is derived from T itself. This is the well-known method of least squares collocation. The only implemented version of the method requires that we immediately change solution space by turning to spherical approximation. Here the earth is bounded by a sphere with radius R, and T can be expressed as a series in fully normalized solid spherical harmonics with coefficients $\bar{C}_{ij}$, $\bar{S}_{ij}$, $i = 0, \cdots, \infty$, $0 \leq j \leq i$. Then the empirical covariance function is $K_E(P, Q)$, where

$$K_E(P, Q) = \sum_{n=0}^{\infty} \sigma_n \left(\frac{R^2}{r_P r_Q} \right)^{n+1} P_n(\cos \psi) \tag{14}$$

$$\sigma_n = \sum_{j=0}^{n} (\bar{C}_{ij}^2 + \bar{S}_{ij}^2) \tag{15}$$

Here ψ is the spherical distance between P and Q; r_P and r_Q are the radial distances of P and Q, respectively; and P_n is the nth Legendre polynomial. The σ_n are the (potential) degree variances. We can use $K_E(P, Q)$ as our reproducing kernel, thereby forcing $\tilde{T}$ to be an element of the corresponding Hilbert space. The use of the technique requires the estimation of the degree variances σ_n. This is a difficult process [see *Tscherning and Rapp*, 1974; *Moritz*, 1977; *Jekeli*, 1978; *Rapp*, 1979*b*].

The least squares principle, the use of which results in (14) and (15), is of a global character [see *Heiskanen and Moritz*, 1967, chapter 7]. For this reason some authors have rejected the use of least squares collocation for the local, detailed approximation of T [e.g., *Kearsley*, 1976]. However, one must keep in mind that the use of a least squares principle, resulting in the isotropic covariance function (14), does not imply that T must fulfill any isotropy conditions.

If a high-order reference field U is used, the 'global' least squares principle may be adapted for local use. A sufficiently high order reference field has the significance that the gravity field outside the regarded area is of minor importance when constructing $\tilde{T}$ for the area. Hence the gravity field outside the area can be regarded as having the same variations globally as locally. The global sampling process used when estimating $K_E(P, Q)$ may hence be restricted to the local area, as was done by *Tscherning* [1974, section 2.3]. In this manner the least squares collocation method may be regarded as being optimal in a least squares sense for a certain local area.

The method of least squares collocation will in theory enable the estimation of mean square errors. The theoretical interpretation of this type of error is somewhat doubtful, as there is no explicit probability distribution given for the random or stochastic variables.

The techniques described above may, as was explained above for the model approach, be used for the restriction of T to a two-dimensional surface in $\mathbf{R}^3$ or for the approximation of another operator applied on T. The restriction T_R may be regarded as a function of (φ, λ) or of some plane set of coordinates (X, Y). The two techniques (change in solution space and regularization) are then adapted to this situation. The change in solution space will result in the choice of a finite set of functions of two variables, for example, surface harmonics, polynomials in two variables, multiquadratic functions, or the binary sample functions used by *Brown* [1975]. Regularization techniques will appear through the use of reproducing kernels and covariance functions depending only on the spherical distance ψ and not on r_P and r_Q as in (14) (see, for example, *Nash and Jordan* [1978] and *Hein and Lenze* [1979]).

As was mentioned above, this type of procedure has the drawback that data must refer to the selected reference surface. This problem is considerably reduced if the surface of the earth or a sphere outside the earth is used and not a sphere inside the earth.

Combined Methods

Techniques which combine the model approach and the operational approach are in frequent use. As was mentioned above, the integral formula (5) requires that data be available in a regular grid. Here two-dimensional operational approaches are followed in order to obtain the gridded data (see, for example, *Sünkel* [1977] and *C. R. Schwarz* [1978]).

The integral equation (5) must in principle be evaluated over the whole surface ω. In order to save computational time the integration is frequently only carried out to a certain distance (depending on the reference potential U used). The error arising will be approximately constant in a small area. A comparison with contingently observed values of the geoid undulations or deflections of the vertical may then (for example, through a least squares process) determine these 'systematic errors.' An example of this type of technique is the method of astrogravimetric geoid determination (see, for example, *Heiskanen and Moritz* [1967, p. 203] and *Merry and Vaniček* [1974]).

Least squares or collocation techniques require the solution of systems of linear equations. Here the number of equations can be reduced by first computing an approximation $\tilde{T}_1$ exclusively from data given in a local area ω_1. As a second step, $T_2 = T - \tilde{T}_1$ may be determined by using an integral formulae method, where the integration will only have to be performed in the area outside ω_1, because $T - \tilde{T}_1$ is zero in ω_1. This technique is developed by *Moritz* [1976], *Lachapelle* [1977], and *Lachapelle and Tscherning* [1978].

The prediction methods developed by Bjerhammar [e.g., *Bjerhammar*, 1978] can to a certain extent also be regarded as combined methods. A discretized integral equation is used for the determination of quantities (e.g., the so-called reduced gravity anomalies Δg^*) associated with points or surface elements on a sphere inside the earth. A harmonic function is then determined outside the sphere by using an integral formula. By a suitable choice of 'carrier points' and values we may arrive at techniques identical to the collocation methods, which use reproducing kernel Hilbert spaces of functions harmonic down to an internal sphere [see, *Sjöberg*, 1978].

3. COMPARISON OF THE METHODS

Realizing which kind of treatment our poor T is given by the different methods, one may be a little suspicious about the performance of some of the methods. Do they really work?

All the methods mentioned have been tried in practice with some success.

However, it is one thing to test a method in a scientific environment, where users really know what they are doing. It is quite another in a production environment, where the methods really are applied. They must be as foolproof as possible. This again means that different user groups will favor different techniques. The theoretically minded will favor methods which do not use reductions of data and which have been proved mathematically correct. The practician may prefer methods which, for a small cost, deliver a precise result within prespecified limits and may not worry too much about the method used.

In Table 1 an overview of the methods and the types of observations they can use is given. In Table 2 we have summarized the answers to 22 questions which have been 'asked' of every method. The questions have been asked so that a 'yes' indicates something positive about a method. In a few cases some of the questions are not relevant for judging a method. In this case no answer is given.

We have asked three types of questions: (1) about theoretical properties, (2) about the flexibility and reliability, and (3) about computational properties.

The most important theoretical question to be raised is whether or not a method converges. If it does not converge, there is a limit to the precision which can be achieved by using the method. Convergence has been proved for most of the methods which are based on the operational approach [see *Moritz,* 1976; *Tscherning,* 1978*a*; *Sanso' and Tscherning,* 1980; *Krarup,* 1980; *Keller,* 1978; *Sjöberg,* 1979; *Bjerhammar and Svensson,* 1979]. The reasons for a nonconvergence appear clearly in Table 2. The methods which require spherical approximation or the reduction of data to a sphere or ellipsoid will not converge. It should also be noted that a set of harmonicity which is bigger than Ω (or even nonharmonicity) does not exclude convergence. Also, the condition $\Delta\tilde{T} = 0$ may be better and better approximated when the data density increases.

Naturally, it is a condition for convergence that station positions are improved simultaneously with an improvement in $\tilde{T}$. This relationship is also clearly seen in the table.

For large-scale and/or routine computations the flexibility and reliability of a method is very important. There is here again a remarkable difference between the 'model approach' methods and the 'operational approach' methods, a difference which nearly makes it justifiable to call the model approach methods inoperable. There are, however, situations where all data are nicely homogeneously distributed, have insignificant errors, and are available to such an extent that a required precision can be achieved. We should also keep in mind that error estimates may be computed (by comparison with an external standard) in cases where they cannot be computed as a part of the computational process (see, for example, *Strange and Fury* [1979]). However, no method can really give 'true'

TABLE 1. Methods for Gravity Field Representation

Method	Short Name	Permitted Data Types
Model Approach		
Integral formulae		(all data gridded)
Poisson's integral	Pois. I	geoid heights on sphere
Stokes' integral	Stok. I	Δg
Molodensky series	Mol. Se.	Δg on ω
Numerical integration		
Astrogeodetic leveling	As. Ge. Le.	(ξ, η) pairs of ξ or η in profiles on ellipsoid
Harmonic sampling functions	Harm. Sa. Fct.	T or Δg in fixed points on sphere
Operational Approach		
Change in solution space		
Least squares approximation in three dimensions ($n \geq m$)		
Harmonic base functions	LSAH3D	all data types
Nonharmonic base functions	LSA3D	all data types
Least squares approximation in two dimensions ($n \geq m$)	LSA2D	data which can be 'reduced' to the same two-dimensional surface
Regularization methods		
Least squares collocation	LSCOL	all data types
Minimum norm collocation	MNCOL	all data types
Combined Methods		
Regularization, then integration		
Least squares collocation, then Stokes integral	LSCOL + Stok. I	all data types in the inner area Δg on sphere, outside
Change in solution space, then integration		
Least squares approximation in two dimensions, then Poissons integral	LSA2D + Pois. I	data which can be reduced to two-dimensional surface
Integration, then change in solution space		
Astrogravimetric leveling	As. Gra. le	Δg on part of Ω, ξ, η, in area of interest, plus one ζ value

TABLE 2. A Comparison of the Different Methods

	Pois. I	Stok. I.	Mol. Se.	As. Ge. Le.	Harm. Sa. Fct.	LSAH3D	LSA3D	LSA2D	LSCOL	MNCOL	LSCOL ⊥ Stok. I	LSA2D + Pois. I	As. Gra. Le.
Theoretical Properties													
T harmonic, regular at infinity	(+)	(+)	(+)		(+)	(+)	–	–	(+)	(+)	(+)	(+)	(+)
Set of harmonicity correct	–	–	+		–	(+)(1)	(–)		–	(+)(1)	–	–	(–)
Parameter estimation possible	–	–	–	–	–	+	+	(–)	+	+	(+)	–	(+)
Regularity at infinity not required	+	–	–	(+)	+	+	+		+	(+)(4)	–	+	+
Spherical approximation not needed	–	–	(+)	(+)	–	+	+	(+)(2)	–	+	–	+	(+)
Ultimate precision possible (convergence)	–	–	?	–	–	+	+	(–)(2)	(–)	+	(–)	+	(+)
'Reduced' data not needed	–	–	+	–	(–)	+	+	–	+	+	(+)	+	–
Flexibility and Reliability													
All functionals can be evaluated	+	+	+	only ζ	+	+	+	(–)	+	+	+	+	–
All data types can be used	–	–	–	–	–	+	+	(–)	+	+	+	(–)	–
Data quality (σ) taken into account	–	–	–	(–)	–	+	+	+	+	+	+	+	(–)
Error estimates computable	–	–	–	(–)	–	(+)	(+)	(–)	(+)	(+)	(+)	–	–
Global data coverage not required	(–)	(–)	(–)	+	(–)	+	+	+	+	+	(+)	–	+
Stable in cluster/downward continuation	(+)	(+)	(+)	(+)	(+)	(+)(3)	(+)(3)	(+)(3)	(+)(4)	(+)(4)	(+)(4)	(+)(3)	(+)
New data/correction easily introduced	–	–	–	(–)	–	(+)	(+)	(+)	+	+	+	–	–
Behavior in data holes known	(+)	(+)	(+)	+	(+)	+	+	+	(+)	+	+	+	+
Computational Properties													
Method easily implemented	(+)	(+)	(+)	+	+	+	+	+	(–)	+	(–)	+	(+)
Method can be fully automated	(+)	(+)	(+)	+	(–)	(+)	(+)	(+)	–	+	–	+	(–)
Data management uncomplicated	(+)	(+)	(+)	–	(+)	+	+	(+)	(+)	+	(–)	(+)	(–)
No solution of linear equations	+	+	+	+	–	–	–	–	–	–	–	–	(–)
Gridded data not needed	–	–	–		–	+	+	+	+	+	(+)	+	(–)
Fast method	(+)	(+)	–	+	(+)	(+)	+	+	(+)	(–)	+	(+)	+
Automatic selection of base function	+	+	+	(+)	+	(–)	(–)	(–)	(+)	+	(+)	+	(+)

The explanation of the symbols is as follows: absolutely yes, +; yes with modifications, (+); unknown, ?; no with modifications, (–); absolutely no, –; depends on choice of base functions, but correct set of harmonicity can be achieved, (1); depends on choice of two-dimensional surface, (2); depends on selection of base functions, (3); and depends on norm and/or set of harmonicity, (4). For explanation of method name abbreviations, see Table 1.

error estimates: In least squares collocation the empirical covariance function must be known, and in minimum norm collocation it is necessary to know the norm of T, $\| T \|$.

Solutions obtained by least squares or minimum norm collocation can be easily updated or corrected. If the reduced normal equations are stored, an updated value of an observation only means that the right-hand side must be reduced again and a back substitution must be carried out. A new observation just means that a new column is added, which must be reduced, etc. Least squares approximation requires a change in the unreduced normal equations when a new observation is added [see *Tscherning,* 1978*a*; *Wolf,* 1979].

The stability of the methods in situations where data are lacking, occur in clusters, or must be continued downward from satellite altitude depends very much on the selected base functions and the degree of regularization used. This is a field where more investigations must be carried out [see *Tscherning,* 1975; *Lelgemann,* 1978*b*; *Rummel,* 1979].

The computational complexities of the different methods are of the same magnitude. Methods which do not require the solution of linear equations will need gridded data, and the process of producing these data is comparable to the effort spend for solving systems of equations. The main reason for this is that a different grid must be used for each computational point.

We should also keep in mind that a solution $\tilde{T}$ computed by using collocation or an approximation technique will be valid for a certain (smaller or larger) area. A data grid usable for precise computations by an integral formula will be only partly valid for a set of different points. This is because of the strong singularities of the kernels used in the integral formulae, which require special precautions during the numerical evaluation. The methods may be compared with respect to computational speed. This is, however, somewhat dangerous, because it depends on the smartness of the programer writing the computer algorithms and other computer dependent factors. But it is obvious that the most complex methods will need longer computational time. However, the computational time used must be balanced against the precision needed for the result. A very precise $\tilde{T}$ must necessarily cost more than a less precise $\tilde{T}$.

The considerations made when selecting a specific method should ideally be of the cost-benefit type: How much computer time will be spent in order to obtain a certain precision? Whether a 'better' result could have been obtained for the same cost seems, however, never to have been taken into consideration. This is because the 'capital' investments for implementing one method are very great in terms of manpower. We must realize that there are very few geodetic organizations which are capable of designing, implementing, and maintaining a system of computer programs for gravity field representation.

Let us conclude this section by again reviewing Table 2. Here the operational approach methods like least squares approximation and collocation stand out with many pluses. This is also why many authors favor these methods [*K.-P. Schwarz and H. Sünkel,* 1978; *Moritz,* 1978*a*; *Lachapelle and Tscherning,* 1978; *Reinhart et al.,* 1978].

However, in many cases a homogeneous gravity coverage exists in areas with moderately varying topography. Altimeter-derived geoid undulation at sea also forms a nice homogeneous set. Hence integral formulae methods are feasible in many situations [*Strange and Fury,* 1979; *C. R. Schwarz,* 1978].

4. RESULTS FROM METHODS USED FOR THE REPRESENTATION OF THE DETAILED GRAVITY FIELD

In this section we will review some of the results obtained with a gravity field representation method for the computation or prediction of 'point values' of gravity anomalies and deflections of the vertical. Such results are presented by different investigators in the form of comparisons between (1) computed and observed values, (2) computed and observed values where all values have been artificially generated, and (3) values computed with different methods.

First, it is well known that the computational results will depend strongly on the local gravity field variations (see, for example, *Tscherning and Forsberg* [1978, Table 1]). So different methods must be compared in terms of

$$\rho = \frac{\text{root-mean-square variation (observed} - \text{computed)}}{\text{root-mean-square variation of the observations}}$$

Second, the result (ρ) will depend on the kind of data used and the spacing of the data. Through theoretical investigations we know that some of the methods are able, in principle, to give an unlimited precision if the necessary data are available. However, methods which involve the use of reduced data or spherical approximation may also, in practice, give fine results. This is primarily because the errors in the 'model' will in many cases be small in comparison to the errors in the observations. Furthermore, different 'corrections' can be applied as a weapon against the 'model errors.'

The dependence of ρ on the data spacing is shown in the numerical studies of *Tscherning* [1975, Figures 5–7] and *K.-P. Schwarz* [1976]. When the spacing s of the data is smaller than the first zero point of the used covariance function, ρ will decrease almost linearly toward zero for $s \to 0$. The decrease will depend not only on the spacing of the data in the vicinity of the point of computation but also on the extent of the data distribution.

Figures 1 and 2 show results obtained by comparing computed and observed deflections of the vertical and gravity anomalies. We clearly see the decrease of ρ for decreasing s and the significant improvement obtained by adding a small

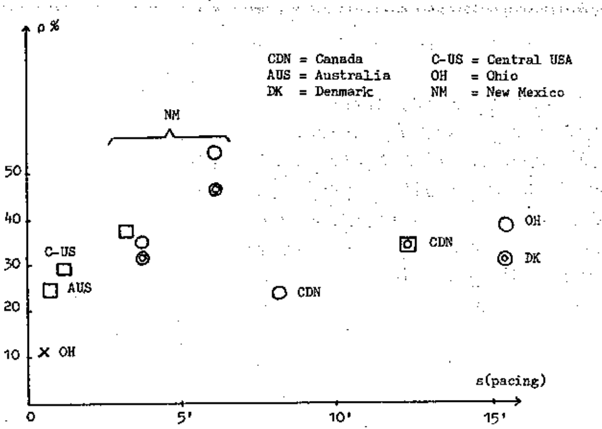

Fig. 1. Results obtained when computing deflections of the vertical: circles, using collocation with Δg only; double circles, using collocation with Δg and (ξ, η) spaced $\frac{1}{2}°$–$1°$ apart; squares, using Vening Meinesz formula; circles within squares, using collocation and Vening Meinesz formula combined; and crosses, using numerical integration of horizontal gravity gradients and (ξ, η) spaced $\frac{1}{2}°$ apart. The results are from different authors, collected by *Tscherning and Forsberg* [1978, Table 1], *Kearsley* [1976] and *C. R. Schwarz* [1978], and computations by the author in the New Mexico area.

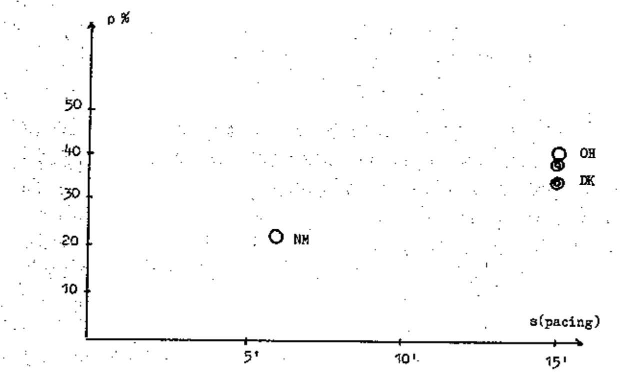

Fig. 2. Results obtained when computing gravity anomalies. For explanation of symbols, see Figure 1. These results were collected by *Lachapelle and Tscherning* [1978, Table 1].

number of deflections of the vertical (Figure 1). The results obtained in the same area with different methods (collocation and Vening Meinesz integral) are also of the same quality. But we also see that the decrease of ρ for decreasing s is different in New Mexico and in Canada. This is without doubt caused by the circumstance that the topography in the New Mexico area varies much more than the topography in the Canadian plains. Remember that we have taken into account the different variations of the gravity through the definition of ρ.

Height anomalies (or geoid undulations) have recently become available as observations. One type is obtained as the difference between the ellipsoidal height determined by Doppler satellite observations and the orthometric height. The other is obtained by setting the geoid height equal to the sea surface topography as determined by radar altimetry. These 'observational' values have standard deviations between 1 and 0.3 m.

The values have been compared with values obtained by different computational procedures (see, for example, *Rapp and Rummel* [1976], *Lachapelle* [1979], *Torge* [1980], *Monka et al.* [1978], *Balmino et al.* [1979], and *Chapman and Talwani* [1979]). Unfortunately, the authors have not given enough information that the results can be expressed in terms of ρ and s. The general conclusion is, however, that quite different methods give nearly the same results. Furthermore, the values have been used for mean gravity anomaly prediction and for the prediction of point values [see *Rapp,* 1978, 1979*a*].

Over the years, several investigations have been carried out by varying a specific method [*Tengström,* 1971; *Tscherning,* 1975; *Kearsley,* 1976; *Sjöberg,* 1978] or by comparing different methods [*Rapp and Agajelu,* 1975] while data spacing or type was not varied. Artificially generated data have also been used, mainly to test a specific method [see *K.-P. Schwarz,* 1978; *Sjöberg,* 1978]. These investigations confirm the impression that different methods will give results of approximately the same quality when the same set of data is used.

Some of the results have a special interest, because they show how much or how little the results depend on the 'changes in solution space' or the regularization used, that is, on the covariance function (see, for example, *Sjöberg* [1978] and *Rapp* [1979*a*]).

Finally, let us notice that very few investigations have been carried out in which a set of parameters and a local (detailed) gravity field approximation have been determined simultaneously [*Tscherning,* 1976*b*].

5. CONCLUSION

Our investigations of the different methods for gravity field representation show that the least squares approximation and collocation methods are conceptually superior to the integral formulae methods. These methods will, however, give results comparable to the operational methods in situations where only one kind of data is available and these data are sufficiently homogeneously distributed.

Today it is impossible to single out one method as the best, because the different methods have not been compared in a systematic way. We know only very little about how the different changes of the solution space and the different regularizations influence the precision of $\tilde{T}$, how data should be distributed, and which kinds of data should be used in order to obtain a given precision at the lowest cost. Answers to these important questions can only be found through international cooperation.

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(Received August 6, 1980;
accepted October 14, 1980.)