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Notes on Convergence in Collocation Theory (*)

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Summary. — The method of collocation may be used for the construction of approximations ($\tilde{T}$) to the anomalous potential of the Earth (T), assuming T to be an element of a reproducing kernel Hilbert Space, $H_K(\Omega)$, (K : reproducing kernel, Ω area of harmonicity). The problem whether a $\tilde{T}$ may be found which approximates T arbitrarily well is equivalent to the problem of convergence of $\tilde{T}$ towards T when the number of observations increase.

This problem is discussed under the hypothesis that Ω is the set outside a sphere representing the surface of the Earth. Convergence is then assured when T is an element of $H_K(\Omega)$ and the observations are associated with a set of linear functionals, which form a complete set in the space dual to $H_K(\Omega)$.

The so-called empirical covariance function is a reproducing kernel in a Hilbert space, $H_C(\Omega)$, which does *not* include T as an element. In this case convergence is not generally assured. However, it is proved that a kind of convergence is assured when $\tilde{T}$ is constructed using smoothed data. The convergence will not be towards T but towards a T' smoothed in the same manner as the observations.

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0. — INTRODUCTION

The method of (least squares) collocation was introduced in Geodesy by Krarup (1969) as a tool for determining an approximation ($\tilde{T}$) to the anomalous potential of the Earth (T) using heterogeneous data types. and with only a finite number of values being available. In order to apply the method several choices will have to be made :

(1) choice of a linear vector space (of which $\tilde{T}$ will be an element),

(2) choice of an inner product,

preferably so that the vector space becomes a so-called reproducing kernel Hilbert space.

Krarup proposed to use a linear vector space of functions harmonic in a set (Ω), which include the actual set of harmonicity (Ω should be the set in $\mathbb{R}^3$ outside a sphere totally enclosed in the Earth). For the inner products he proposed that some which corresponded to simple reproducing kernels should be used.

Having established this model it is naturally of interest to know whether an approximation $\tilde{T}$ may be constructed which will be arbitrarily «close» to T. The existence of such good approximants in between the functions harmonic in Ω is assured by the Runge Theorem, cf. Krarup (1969). Another question is then whether the method of collocation will furnish us with good approximants. This question is discussed in Krarup (1978).

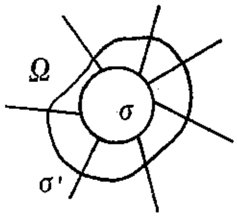
A related convergence problem has been treated in Moritz (1976) and Tscherning (1978). Here the set of harmonicity is the set outside the surface of the Earth. In this case arbitrarily good approximations can be found provided that the used Hilbert space includes T as an element and that the linear functionals associated with the observations form a complete set in the space dual to the Hilbert space.

If we work in the spherical approximation (where the Earth is approximated by a sphere) the so-called empirical covariance function can be defined. This function is a reproducing kernel in a Hilbert space, which does not include T as an element. In the following sections 1 — 3 we will analyze this situation in detail. Unfortunately, it has not been possible to arrive to a definite conclusion about the convergence in this case. However, we have in section 4 proved a kind of convergence for a class of smoothed functions. We find this result quite satisfactory, because it indicates how actual data should be handled in practice, in case a huge number of observations are available.

1. — THE PRECISE FORMULATION OF THE CONVERGENCE PROBLEM

a) — According to Krarup (1969) the method of collocation solves the following problem :

– we have a true anomalous potential T which is a function harmonic outside the earth's surface σ' and we have performed some measurements on T at points P_i on σ' or outside it. Let us call $m_i (i = 1,2 ..., N)$ the results of these measurements ;



– we want to find an approximation $\tilde{T}$ to T in such a way that

I - $\tilde{T}$ is harmonic outside the sphere σ, that is in Ω ;

II - $\tilde{T}$ is a member of a Hilbert space $H_K(\Omega)$ of harmonic functions (in Ω), with reproducing kernel $K(P, Q)$

$$K(P, Q) = \sum_{n=2}^{+\infty} \sum_{m=-n}^{n} v_{nm}(P)\, v_{nm}(Q) \tag{1.1}$$

$$\begin{cases} v_{nm}(P) = \rho_n^2\, u_{nm}(P) \\ u_{nm}(P) = \left(\dfrac{R}{r_P}\right)^{n+1} Y_{nm}\,(\Theta_P, \lambda_P) \end{cases} \tag{1.2}$$

where ρ_n^2 are simple numerical coefficients chosen in such a way that $K(P, Q)$ is an appropriately regular function ;

III - $\tilde{T}$ represents the same measurements m_i as T : This point has also an analytic representation if we notice that the measurements at points P_i may be considered as linear functionals L_i on T as well as on $\tilde{T}$ and that, if we suppose L_i to be bounded in H_K, L_i can also be represented by means of the formula

$$L_i \varphi = \langle L_i K(P_i, Q), \varphi(Q) \rangle = \\ = \langle K_i(Q), \varphi(Q) \rangle \\ \forall \varphi \in H_K \tag{1.3}$$

The point III then can be written as

$$L_i \tilde{T}(P_i) = \langle K_i (Q), \tilde{T}(Q) \rangle = m_i \tag{1.4}$$

(*Note* : we cannot say $m_i = \langle K_i(Q), T(Q) \rangle$ unless we add the hypotesis that $T \in H_K$!)

IV - $\tilde{T}$ must satisfy the variational principle

$$\| \tilde{T} \| = \min$$

The solution of the problem from I to IV is

$$\tilde{T}(P) = \sum_{1}^{N}{}_{i,j} K_i(P) \{\langle K_i, K_j \rangle\}^{-1} m_j \tag{1.5}$$

(By $\{\langle K_i, K_j \rangle\}^{-1}$ we mean the inverse of the matrix with elements $\{\langle K_i, K_j \rangle\}$).
Formula (1.5) has a rather simple interpretation if we make the hypothesis that

$$T \in H_K \tag{1.6}$$

Naturally (1.6) is a strong hypothesis but not stronger than the one we make by defining the covariance function of T! (we shall return to this point). If (1.6) is verified we can write

$$m_i = L_i T = \langle K_i, T \rangle$$

so that (1.4) becomes

$$\langle K_i, \tilde{T} \rangle = \langle K_i, T \rangle. \tag{1.7}$$

Let us call

$$S_N = \text{span}\{K_i(P), \quad i = 1,2 \dots N\}, \tag{1.8}$$

$$P_N = \text{orthogonal projector on } S_N : \tag{1.9}$$

it is evident that (1.7) implies

$$P_N \widetilde{T} = P_N T \tag{1.10}$$

so that the element T of minimum norm satisfying (1.10) is necessarily

$$\widetilde{T} = P_N T \tag{1.11}$$

(1.11) can also directly be verified by observing that under hypothesis (1.7), (1.5) becomes

$$\widetilde{T}(P) = \sum_{1}^{N}{}_{i,j} K_i(P) \{\langle K_i, K_j \rangle\}^{-1} \langle K_j(Q), T(Q) \rangle \tag{1.12}$$

It is easy to verify that the operator

$$\widetilde{S}_N(\cdot) = \sum_{1}^{N}{}_{i,j} K_i \{\langle K_i, K_j \rangle\}^{-1} \langle K_i, \cdot \rangle$$

satisfies the two relations

$$\begin{cases} \widetilde{S}_N^2 = \widetilde{S}_N \\ (\widetilde{S}_N)^t = \widetilde{S}_N \end{cases} \tag{1.13}$$

so that $\widetilde{S}_N(\cdot)$ is seen to be an orthogonal projection, namely the projection P_N on S_N. From (1.11) we derive the convergence theorem (1.I) (cf. Meschkowski (1962, p. 118), Moritz (1976, section 2)).

Theorem (1.I). – If $T \in H_K$, then T converges strongly in H_K to T if and only if the sequence $\{K_i\}$ representing the measurement functionals $\{L_i\}$, is a complete sequence in H_K:

In fact if $\{K_i\}$ is complete, that is if

$$\left\{\begin{array}{l} \langle K_i, T \rangle = L_i T = 0 \\ \\ i = 1,2 \ldots \infty \end{array}\right. \Rightarrow T = 0, \tag{1.14}$$

we have that the limit

$$\lim_{N \to \infty} P_N T = T^* ,$$

which exists in any case, must also coincide with T, since

$$\langle K_i, T^* - T \rangle = 0, \quad \forall_i .$$

We have then that the approximation determined using the method proposed by Krarup, under the simplifying hypothesis (1.6), is a convergent solution. The problem of what happens when $T \notin H_K$ is discussed in Krarup (1978), however in the context of a hybrid norm.

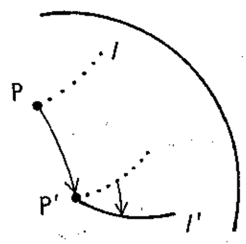
b) — We have also a different formulation of the collocation method, namely the Moritz' one, which is widely applied in practice.

In this formulation we look for a linear estimate of T,

$$\hat{T}(P) = \sum_{1}^{N}{}_i \lambda_i(P)\, m_i = \sum_{1}^{N}{}_i \lambda_i(P)\, L_i\, T(P_i) . \tag{1.15}$$

We must now from the very beginning work in the so-called spherical approximation. The Earth is here approximated by a sphere and Ω will now be the set outside this sphere. T will then be harmonic in Ω, so the following definition of a covariance function is meaningful:

– let us call by u_α any rotation of the space around the origin (α is here a 3-D parameter defining the displacement of the pole $P \to P'$ and the rotation of one direction $l \to l'$).



– the covariance function of T is then

$$C(P, Q) = \frac{1}{4\pi} \int T(u_\alpha P) \, T(u_\alpha Q) \, d\alpha, \tag{1.16}$$

the integration taking place over all the space of the parameters α. Following H. Moritz it is possible to show that if T admits the development (see (1.2))

$$T = \sum_{2}^{+\infty}{}_n \sum_{-n}^{n}{}_m \, \bar{a}_{nm} \, u_{nm} , \tag{1.17}$$

then we have

$$\begin{cases} C(P, Q) = \sum\limits_{n,m} \dfrac{\bar{\sigma}_n^2}{2n+1} \, u_{nm}(P) \, u_{nm}(Q) \quad (1) \\ \bar{\sigma}_n^2 = \sum\limits_{-n}^{n}{}_k \, \bar{a}_{nk}^2 \end{cases} \tag{1.18}$$

Let us notice immediately that if T belongs to a more restricted space, like H_K, then we have

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(1) *Note*: we shall suppose that $\bar{\sigma}^2{}_n \neq 0, \forall n$.

$$T = \sum_{2}^{+\infty}{}_n \sum_{-n}^{n}{}_m a_{nm} v_{nm} =$$

$$= \sum_{2}^{+\infty}{}_n \sum_{-n}^{n}{}_m a_{nm} \rho_n u_{nm} ,$$

$$\bar{a}_{nm} = a_{nm} \rho_n ,$$

$$\bar{\sigma}_n^2 = \sum_{-n}^{n}{}_m \bar{a}_{nm}^2 = \rho_n^2 \sum_{-n}^{n}{}_m a_{nm} = \rho_n^2 \sigma_n^2 ,$$

$$C(P, Q) = \sum_{n,m} \frac{\bar{\sigma}_n^2}{2n+1} u_{nm}(P) u_{nm}(Q) = \tag{1.19}$$

$$= \sum_{n,m} \frac{\sigma_n^2}{2n+1} \rho_n^2 u_{nm}(P) u_{nm}(Q) =$$

$$= \sum_{n,m} \frac{\sigma_n^2}{2n+1} v_{nm}(P) v_{nm}(Q)$$

$$(\sigma_n^2 = \sum_{-n}^{n}{}_m a_{nm}^2) ,$$

so that $C(P, Q)$ maintains its formal development in H_K on the condition that we replace $\bar{\sigma}_n^2$ with σ_n^2 and u_{nm} with v_{nm}. Moreover we observe that $C(P, Q)$ is invariant under rotations, that is for any u_α,

$$C(u_\alpha P; u_\alpha Q) = C(P, Q).$$

The Moritz' principle is based on an invariance principle of the estimate $\hat{T}$ and on a variational principle for the estimation error.

I - We require the coefficients $\lambda_i(P)$ to be rotationally invariant.

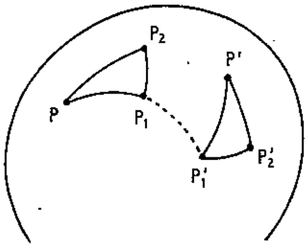
This will be better clarified by an example: suppose we have measured T at two points P_1, P_2 and that we estimate

$$\hat{T}(P) = \lambda_1(P) T(P_1) + \lambda_2(P) T(P_2) ;$$

suppose now that we measure T at the points

$$P_1' = u_\alpha P_1, \qquad P_2' = u_\alpha P_2$$

and we want to express T at the point $P' = u_\alpha P$



generally we'll have

$$\hat{T}(P') = \lambda_1'(P')\,T(P_1') + \lambda_2'(P')\,T(P_2')\,.$$

The invariance principle requires that

$$\lambda_1'(P') = \lambda_1(P), \qquad \lambda_2'(P') = \lambda_2(P). \tag{1.20}$$

II - We define the punctual estimation error as

$$e(P) = T(P) - \hat{T}(P)$$

and we notice that if we submit all the space to a rotation u_α, because of the invariance principle (1.20) we have a rotated error

$$U_\alpha\, e(P) = T(u_\alpha P) - \sum_1^N{}_i\, \lambda_i(P)\, L_i\, T(u_\alpha P_i). \tag{1.21}$$

We define the mean square estimation error

$$E^2 = \frac{1}{4\pi}\int \{\, U_\alpha\, e(P)\,\}^2\, d_\alpha =$$

$$= C(P, P) - 2 \sum_1^N{}_i\, \lambda_i(P)\, L_i\, C(P_i, P) +$$

$$+ \sum_1^N{}_{i,j}\, \lambda_i(P)\, \{\, L_i L_j\, C(P_i, P_j)\,\}\, \lambda_j(P). \tag{1.22}$$

The Moritz' principle is exactly

$$E^2 = \min : \tag{1.23}$$

(1.23) yields

$$\lambda_i(P) = \sum_1^N{}_j \{ L_i L_j C(P_i, P_j) \}^{-1} L_j C(P_j, P)$$

so that the estimated potential $\hat{T}$ is given by

$$\hat{T} = \sum_1^N{}_{i,j} L_j C(P_j, P) \{ L_i L_j C(P_i, P_j) \}^{-1} m_i =$$

$$= \sum_1^N{}_{i,j} L_j C(P_j, P) \{ L_i L_j C(P_i, P_j) \}^{-1} L_i T(P_i). \tag{1.24}$$

The solution (1.24) is perfectly equivalent to Krarup's solution if we choose $C(P, Q)$ as the reproducing kernel of the Hilbert space in which $\hat{T}$ is embedded.

In fact let us call H_C such a space and let us notice that

$$l_i(P) = L_i C(P_i, P) \tag{1.25}$$

are the representers of L_i in H_C.

Furthermore

$$L_i L_j C(P_i, P_j) = L_j l_i(P_j) = \langle l_j(Q), l_i(Q) \rangle_C \quad (^2) \tag{1.26}$$

so that (1.24) can be written

$$\hat{T} = \sum_1^N{}_{i,j} l_j(P) \{\langle l_j, l_i \rangle_C \}^{-1} m_i , \tag{1.27}$$

which is formally coincident with (1.5).

(2) we shall indicate by $\langle , \rangle_C$, $\| \ \|_C$ the scalar product and the norm in H_C in order to distinguish them from the analogous quantities in H_K.

— However in this case we cannot put any more the hypothesis $T \in H_C$ as shown in Tscherning (1977).

In fact, since from (1.19) we know that $\left\{ \frac{\sigma_n}{\sqrt{2n+1}} v_{nm} \right\}$ is an orthogonal basis of H_C, we have

$$\| T \|_C^2 = \| \sum_{n,m} a_{nm} v_{nm} \|_C^2 = \| \sum_{n,m} \frac{a_{nm} \sqrt{2n+1}}{\sigma_n} \frac{\sigma_n}{\sqrt{2n+1}} v_{nm} \|_C^2 =$$

$$= \sum_{2}^{+\infty}{}_n \frac{2n+1}{\sigma_n^2} \sum_{-n}^{n}{}_m a_{nm}^2 = \sum_{2}^{+\infty} (2n+1) = +\infty \quad (1.28)$$

Consequently *it is not allowed to put*

$$m_i = L_i T(P_i) = \langle l_i(Q), T(Q) \rangle_C ,$$

and the result

$$\hat{T} = P_N T$$

doesn't hold any more. Moreover, it is clear from (1.28) that we cannot put the problem whether in H_C

$$\lim_{N \to \infty} \hat{T} = T$$

or not, because this is simply impossible in the strong as well as in the weak sense.

However, we can always suppose that $T \in H_K$, for an opportune reproducing kernel $K(P, Q)$; we are therefore in the following situation

$$\begin{cases} T \in H_K & \text{rep.Ker. } K(P, Q) = \sum_{n,m} v_{nm}(P)\, v_{nm}(Q) \\ \hat{T} \in H_C & \text{'' \; ''} \quad C(P, Q) = \sum_{n,m} \frac{\sigma_n^2}{2n+1} v_{nm}(P)\, v_{nm}(Q) \end{cases} \quad (1.29)$$

But since H_C is densely embedded in H_K [3] we have also

$$\hat{T} \in H_K,$$

so that we can put the meaningful problem whether

$$\lim_{N \to \infty} \hat{T} = T$$

strongly or weakly in H_K.

This is exactly the convergence problem that we are going to analyze in the next paragraphs, firstly from the abstract point of view, determining a necessary and sufficient condition for the unconditional convergence of $\hat{T}$ to T; since however this condition is not very simple to handle, we give in the last paragraph a theorem of convergence for smoothed potentials which, we think, is « almost » the solution of the problem for practical purposes.

2. — DIFFERENT FORMULAE FOR THE COLLOCATION SOLUTION

Under hypothesis (1.29) we supposed that all the functionals L_i were bounded in H_K and that they there were represented by

$$\begin{aligned} & L_i\, T(P_i) = \langle K_i(Q), T(Q) \rangle \\ & (K_i(Q) = L_i\, K(P_i, Q)) \end{aligned} \tag{2.1}$$

It is worth noticing that the same L_i are represented in H_C by

$$\begin{aligned} & L_i\, \hat{T}(P_i) = \langle l_i(Q), \hat{T}(Q) \rangle_C \\ & (l_i(Q) = L_i\, C(P_i, Q)) \end{aligned} \tag{2.2}$$

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(3) in fact the set of linear combinations $(\sum_{n,m\,1}^{N} \mu_{nm}\, v_{nm})$, is dense in both H_K, H_C. Moreover if $\varphi \in H_C$,

$$\varphi = \sum \bar{a}_{nm} \frac{\sigma_n}{\sqrt{2n+1}} v_{nm} \qquad (\sum \bar{a}^2_{nm} < +\infty),$$

so that

$$\| \varphi \|^2 = \sum \bar{a}^2_{nm} \frac{\sigma^2_n}{2n+1} \leq \max \left(\frac{\sigma^2_n}{2n+1} \right) \sum \bar{a}^2_{nm} = \text{const.}\, \| \varphi \|^2_C$$

Now we would like to know the relation between these two sequences. We can observe that $C(P, Q)$ for any fixed P is an element of H_K : in fact

$$\| C(P, Q) \|^2 = \| \sum_{n,m} \frac{\sigma_n^2}{2n+1} v_{nm}(P)\, v_{nm}(Q) \|^2 =$$

$$= \sum_{n,m} \frac{\sigma_n^4}{(2n+1)^2} [v_{nm}(P)]^2 =$$

$$= \sum_{2}^{+\infty}{}_n \frac{\sigma_n^4}{(2n+1)^2} \rho_n^2 \left(\frac{R}{r_P}\right)^{n+1} \sum_{-n}^{n}{}_m Y_{nm}^2(P) =$$

$$= \sum_{2}^{+\infty}{}_n \frac{\sigma_n^4}{2n+1} \rho_n^2 \left(\frac{R}{r_P}\right)^{n+1} P_n(1) \leq$$

$$\leq \left(\sum_{2}^{+\infty}{}_n \sigma_n^2\right) \mathrm{Max} \left(\frac{\sigma_n^2 \rho_n^2}{2n+1}\right).$$

Therefore we can write

$$L_i\, C(P, P_i) = \langle K_i(Q), C(Q, P) \rangle . \tag{2.3}$$

Let us introduce the operator C defined in H_K as

$$\begin{cases} C\,\varphi = \langle C(P, Q), \varphi(Q) \rangle \\ C \sum_{n,m} \varphi_{nm} v_{nm} = \sum_{n,m} \frac{\sigma_n^2}{2n+1} \varphi_{nm} v_{nm} \end{cases} \quad \forall\, \varphi \in H_K : \tag{2.4}$$

and we see that C has the eigenfunctions $\{v_{nm}\}$ with eigenvalues $\left\{\frac{\sigma_n^2}{2n+1}\right\}$, so that C is a nuclear operator, i.e.

$$\mathrm{Tr}\, C = \sum_{n,m} \frac{\sigma_n^2}{2n+1} = \sum_n \sigma_n^2 < +\infty . \tag{2.5}$$

(2.5) also implies that $C^{1/2}$ (the positive square root of C) is a completely continuous operator. Furthermore, since we suppose $\sigma_n^2 \neq 0\ \forall\, n$, C is an invertible operator, that is

$$C\,\varphi = 0 \Rightarrow \varphi = 0 .$$

From (2.3) and (2.4) we then derive

$$l_i(P) = L_i\, C(P_i, P) = C\, K_i, \tag{2.6}$$

a formula which answers our question. Moreover we have

$$L_i\, L_j\, C(P_i, P_j) = \langle l_i, l_j \rangle_C = L_i\, l_j(P_i) =$$

$$\langle K_i, l_j \rangle = \langle K_i, C\, K_j \rangle\,. \tag{2.7}$$

From (1.27), (2.6), (2.7) and taking into account that

$$m_i = \langle K_i, T \rangle \qquad (\text{since } T \in H_K)$$

we get the first form of the collocation solution :

$$\hat{T} = \sum_{i,j\,1}^{N} C\, K_i \{\langle K_i, C\, K_j \rangle\}^{-1} \langle K_j, T \rangle\,. \tag{2.8}$$

Now it isn't difficult to verify that $\hat{T}$ is invariant, for any fixed N, under any non-singular linear transformation of the sequence $\{K_j\}$. In fact let A be any non-singular matrix with elements $\{A_{sr}\}$: we have the identity (summation over repeated indexes is understood)

$$\hat{T} = \Sigma\, C\, K_i\, A_{ir}\, A^{-1}_{rs} \{\langle K_s, C\, K_m \rangle\}^{-1} A^{-1}_{nm}\, A_{jn} \langle K_j, T \rangle =$$

$$= \Sigma\, C\, (K_i\, A_{ir}) \{\langle (K_s\, A_{sr}), C(K_m\, A_{mn}) \rangle\}^{-1} \langle (K_j\, A_{jn}), T \rangle\,;$$

and setting

$$\sum_{i\,1}^{N} K_i\, A_{ir} = h_r \tag{2.9}$$

we get

$$\hat{T} = \sum_{r,n} C\, h_r \{\langle h_r, C\, h_n \rangle\}^{-1} \langle h_n, T \rangle\,, \tag{2.10}$$

which is formally identical with (2.8). We can use this invariance property to construct a new sequence $\{h_r\}$ satisfying a C-orthogonality relation, i.e.

$$\langle h_r, C\, h_n \rangle = \delta_{rn}\,. \tag{2.11}$$

This can be accomplished by applying the orthonormalization process of Gram-Schmidt : in this way we obtain a sequence

$$h_r = \sum_{1}^{r} {}_i A_{ir} K_r \tag{2.12}$$

so that (2.11) is satisfied.

We can notice that (2.12) yields also

$$q_r = C h_r = \sum_{1}^{r} {}_i A_{ir} C K_i = \sum_{1}^{r} {}_i A_{ir} l_i , \tag{2.13}$$

so that, recalling (2.7)

$$\langle q_r, q_n \rangle_C = \langle h_r, q_n \rangle = \langle h_r, C h_n \rangle = \delta_{rn} : \tag{2.14}$$

in other words the sequence $\{q_r\}$ which corresponds in H_C to $\{h_r\}$ of H_K, is an orthonormal sequence in H_C. From (2.10), (2.11), (2.13) we derive *the second equivalent form of* $\hat{T}$

$$\hat{T}(P) = \sum_{1}^{N} {}_r q_r(P) \langle h_r(Q), T(Q) \rangle. \tag{2.15}$$

Formula (2.15) is particularly suggestive : in fact let us observe that by (2.15) we have in particular

$$\langle h_r, q_n \rangle = \delta_{rn} , \tag{2.16}$$

which means the two sequences $\{h_r\}$, $\{q_r\}$ realize a biorthogonal system in H_K.

From (2.15) then we see that $\hat{T}$ is nothing but the partial sum (of order N) of the formal development associated with T

$$T \sim \sum_{1}^{+\infty} {}_r q_r \langle h_r, T \rangle. \tag{2.17}$$

The problem of the convergence of $\hat{T}$ to T in H_K is then equivalent to the problem of the convergence of the development of T with respect to the biorthogonal system $\{q_r, h_r\}$. This point of view will be examined in the next paragraph.

In any way, following the terminology of biorthogonal developments, we shall call the operator transforming T into $\hat{T}$, a partial sum operator

$$\hat{T} = S_N(T)$$

$$S_N(\cdot) = \sum_1^N {}_r\, q_r \langle h_r, \cdot \rangle. \tag{2.18}$$

Before going to the third formula for $\hat{T}$, we have an important point to underline.

From (2.12) we see [4] that if $\{K_i\}$ is a complete sequence in H_K, that is the linear manifold

$$\mathscr{H}_K = \{\varphi \mid \varphi = \sum_1^N {}_i\, \mu_i K_i, \quad N = 1,2, \dots, \forall \mu_i\}$$

is dense in H_K, then also $\{h_r\}$ is a complete sequence in H_K.

Moreover if $\{K_i\}$ is complete in H_K, so is $\{l_i\}$: in fact, on the contrary, we could find a $\varphi \neq 0$, $\varphi \in H_K$ orthogonal to Span $(l_i, i = 1,2 \dots)$ so that

$$\langle \varphi, l_i \rangle = \langle C\varphi, K_i \rangle = 0, \qquad \forall i$$

but in this case we have also

$$C\varphi = 0 \Rightarrow \varphi = 0$$

Consequently from (2.13) we see that also $\{q_r\}$ is a complete sequence.

After these remarks we go to the third form of $\hat{T}$ by observing that, because of (2.11), the sequence

$$v_i = C^{1/2} h_i \tag{2.19}$$

is an orthonormal sequence in H_K, in fact

$$\langle v_i, v_j \rangle = \langle h_i, C h_j \rangle = \delta_{ij}. \tag{2.20}$$

We can also notice that, repeating the proof of the above statement on the com-

(4) recall that the matrix $\{A_{ir}\}$ of the orthonormalization process of Gram-Schmidt is always invertible!

pleteness of $\{l_i\}$, we deduce that also $\{v_i\}$ is a complete sequence if $\{K_i\}$ is complete, i.e.

$$\text{Span } \{v_i, i = 1,2 \dots\} = H_K . \tag{2.21}$$

Using the obvious relations

$$\begin{cases} h_i = C^{-1/2} v_i \\ q_i = C^{1/2} v_i \end{cases} \tag{2.22}$$

we can write (2.15) in the form

$$\hat{T} = \sum_1^N {}_r C^{1/2} v_r \langle C^{-1/2} v_r, T \rangle . \tag{2.23}$$

If we put

$$V_N = \text{Span } \{v_i, i = 1,2 \dots N\}$$

and

$$Q_N = \sum_1^N {}_i v_i \langle v_i, \cdot \rangle \tag{2.24}$$

which is the orthogonal projection on V_N, we also have from (2.23)

$$\hat{T} = C^{1/2} Q_N C^{-1/2} T , \quad (5) \tag{2.25}$$

which is *the third form of* $\hat{T}$.

(2.25) shows that the operator

$$S_N = C^{1/2} Q_N C^{-1/2}$$

is a projection operator since

$$S_N^2 = C^{1/2} Q_N C^{-1/2} C^{1/2} Q_N C^{-1/2} = S_N , \tag{2.26}$$

---

(5) to be more *precise* one should say that

$$\hat{T} = \overline{(C^{1/2} Q_N C^{-1/2})} T$$

where with the over-bar we have indicated the closure of the operator $C^{1/2} Q_N C^{-1/2}$.

although in general not an orthogonal projector (in H_K !) since

$$S_N^t = C^{-1/2} Q_N C^{1/2} \neq S_N \tag{2.27}$$

unless Q_N commutes with C.

In *this last case* we have

$$S_N = C^{1/2} Q_N C^{-1/2} = C^{1/2} C^{-1/2} Q_N = Q_N$$

and the convergence of $\hat{T}$ to T is guaranteed from (2.21).

We have then :

lemma 1 — if (K_i) is a complete sequence and if

$$C\, Q_N = Q_N\, C \tag{2.28}$$

then $\hat{T}$ converges strongly to T in H_K.

This is the case, for example, when

$$\{K_i\} = \{v_{nm}\}$$

so that

$$\{l_i\} = \{C\, K_i\} = \left\{ \frac{\sigma_n^2}{2n+1} v_{nm} \right\}$$

$$\{q_i\} = \left\{ \frac{\sigma_n}{\sqrt{2n+1}} v_{nm} \right\}$$

$$\{h_i\} = \left\{ \frac{\sqrt{2n+1}}{\sigma_n} v_{nm} \right\}$$

In this case in fact

$$V_N = \text{Span}\{v_{nm}, \mid m \mid \leqq n\,, \quad n = 2.3 \dots N\}\,,$$

which is an invariant subspace with respect to C, so that (2.28) is verified and

$$\hat{T} = \sum_{2}^{N}{}_n \sum_{-n}^{n}{}_m \frac{\sigma_n}{\sqrt{2n+1}} v_{nm} \left\langle \frac{\sqrt{2n+1}}{\sigma_n} v_{nm}, T \right\rangle =$$

$$= \sum_{2}^{N}{}_n \sum_{-n}^{n}{}_m v_{nm} \langle v_{nm}, T \rangle \underset{H_K}{\rightarrow} T.$$

Finally we shall observe that for any $\varphi \in H_K$ of the form

$$\varphi = C^{1/2} \psi \quad (6), \qquad \psi \in H_K$$

we have

$$\varphi - S_N(\varphi) = C^{1/2} \psi - C^{1/2} Q_N C^{-1/2} C^{1/2} \psi =$$

$$= (C^{1/2} - C^{1/2} Q_N) \psi :$$

on the other hand, since $C^{1/2}$ is completely continuous

$$\| C^{1/2} - C^{1/2} Q_N \| \rightarrow 0$$

so that

$$\| \varphi - S_N(\varphi) \| \rightarrow 0 .$$

Indeed this is not useful for us *since we can never have*

$$T = C^{1/2}\psi, \qquad \psi = C^{-1/2} T, \qquad \psi \in H_K , \tag{2.29}$$

in fact (2.29) would be equivalent to $T \in H_C$, which is impossible !
However we have proved

lemma 2 — for $\forall \varphi = C^{1/2}\psi, \quad \psi \in H_K$, i.e. $\quad \forall \varphi \in H_C$,

(6) it is easy to prove that

$$\{ \varphi \mid \varphi = C^{1/2} \psi, \psi \in H_K \} \equiv H_C \subset H_K .$$

we have

$$S_N(\varphi) \underset{H_K}{\rightarrow} \varphi$$

in the strong sense.

3. — AN ABSTRACT CHARACTERIZATION OF THE CONVERGENCE PROBLEM

As we have seen at the end of the above paragraph (lemma 2), for $\forall\ \varphi \in H_C$ we have the strong convergence in H_K

$$S_N(\varphi) \underset{H_K}{\rightarrow} \varphi\,. \tag{3.1}$$

Indeed we had to expect this result : in fact if $\varphi \in H_C$, both relations

$$\hat{\varphi} = S_N(\varphi) \in H_C$$

$$\varphi \in H_C$$

hold so that we are within the hypothesis of theorem (1.I).

Consequently we have even

$$S_N(\varphi) \underset{H_C}{\rightarrow} \varphi \tag{3.2}$$

which clearly implies (3.2).

The question that we put now is : can we state that there is always in H_K some $\varphi \notin H_C$ such that we have at least a weak convergence

$$S_N(\varphi) \overset{*}{\underset{H_K}{\rightarrow}} \varphi \quad ? \qquad (^7) \tag{3.3}$$

(7) Let's recall that weak convergence implies pointwise convergence, since

$$\varphi_N(P) = \langle\, K(P,\cdot),\, \varphi_N \,\rangle \rightarrow \langle\, K(P,\cdot),\, \varphi \,\rangle = \varphi(P).$$

Moreover if P is any point on the boundary we have simultaneously

$$\forall P \in \sigma\,,\ \varphi_N(P) \rightarrow \varphi(P)\,;\ |\,\varphi_N(P)\,| \leqq \|\,K(P,\cdot)\,\|\ \|\,\varphi\,\| \lneqq \text{const.}$$

which is enough to guarantee that

$$\lim_{N \rightarrow \infty} \int_\sigma [\varphi_N(P) - \varphi(P)]^2\, d\sigma = 0\,:$$

but then we have also uniform convergence of φ_N to φ in any closed subset of Ω (open).

We shall divide this question into two parts:

a) we shall prove that if we have an unconditional weak convergence, i.e.

$$S_N(\varphi) \underset{H_K}{\overset{*}{\rightarrow}} \varphi \quad , \quad \forall\, \varphi \in H_K$$

then we have also an unconditional strong convergence

$$S_N(\varphi) \underset{H_K}{\rightarrow} \varphi \quad , \quad \forall\, \varphi \in H_K :$$

furthermore we shall give necessary and sufficient condition for this to happen.

b) we shall prove that even if we haven't an unconditional convergence, we have certainly in H_K some $\varphi \notin H_C$ such that

$$S_N(\varphi) \underset{H_K}{\overset{*}{\rightarrow}} \varphi \,;$$

this means that we have always at least the possibility that for T, which certainly doesn't belong to H_C, we have

$$\hat{T} = S_N(T) \underset{H_K}{\overset{*}{\rightarrow}} T \,.$$

This means that the convergence problem is at least a non trivial one. The proofs of the statements *a*) and *b*), that will follow, could be avoided since they are included in more general theorems on biorthogonal bases in Banach spaces (see (Singer. 1970)), however we think that they may be useful for further research on the subject, so that we find it worthwihle reproducing them [8].

A) — Let us first of all define some linear subspaces of H_K:

$$E_0 = \{\varphi \in H_K \mid \lim_{N \to \infty} S_N(\varphi) = \varphi\} \tag{2.4}$$

$$E_1 = \{\varphi \in H_K \mid \exists \lim_{N \to \infty} S_N(\varphi) \in H_K\} \tag{3.5}$$

$$E_2 = \{\varphi \in H_K \mid \operatorname{Sup}_N \; \| S_N(\varphi) \| < +\infty\} \tag{3.6}$$

(8) As a matter of fact the proofs of (Singer, 1970) are modified and simplified in accordance with the present case.

$$E^* = \{\varphi \in H_K \mid \lim_{N\to\infty}{}^* S_N(\varphi) = \varphi\} \tag{3.7}$$

where, in (3.7), by lim* we mean the weak limit.

The following inclusions are evident

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq H_K ; \tag{3.8}$$

since furthermore, due to lemma 2, have also

$$H_C \subseteq E_0 \tag{3.9}$$

and since H_C is dense in H_K, we may conclude that also E_0, E_1, E_2 are dense in H_K.

Moreover we have the following

lemma 3 — the sets E_2, E^* are coincident

$$E_2 = E^* . \tag{3.10}$$

In fact let us remind that, from the form (2.8) of the $S_N(\cdot)$ operator, we have

$$\langle K_n, S_N(\varphi) \rangle = \langle K_n, \varphi \rangle \qquad n = 1.2 \dots N$$

so that setting

$$S_N = \text{Span}\{K_n, \quad n = 1,2 \dots N\}$$

$$P_N = \text{orthogonal projection on } S_N$$

we have

$$P_N\, S_N(\varphi) = P_N\varphi . \tag{3.11}$$

Now let $\varphi \in E_2$, so that

$$\| S_N(\varphi) \| \leqq \text{const.} \qquad \forall N :$$

we have for $\forall \psi \in H_K$ and using (3.11)

$$\begin{aligned} \langle \psi, S_N(\varphi) \rangle &= \langle P_N \psi, S_N(\varphi) \rangle + \langle (I - P_N)\psi, S_N(\varphi) \rangle = \\ &= \langle \psi, P_N\varphi \rangle + \langle (I - P_N)\psi, S_N(\varphi) \rangle . \end{aligned} \tag{3.12}$$

Consequently, with the help of (3.12) and recalling that by hypothesis $P_N \to I$, $\| S_N(\varphi) \|$ is bounded

$$
\begin{aligned}
&| \langle \psi, \varphi - S_N(\varphi) \rangle | \leqq | \langle \psi, (I - P_N)\varphi \rangle | + \\
&+ | \langle (I - P_N)\psi, S_N(\varphi) \rangle | \leqq \| \psi \| \ \| (I - P_N)\varphi \| + \qquad (3.13) \\
&+ \| (I - P_N)\psi \| \ \| S_N(\varphi) \| \underset{(N\to\infty)}{\to} 0
\end{aligned}
$$

But (3.13) means that

$$\lim_{N\to\infty}{}^{*} \ S_N(\varphi) = \varphi$$

so that we have

$$\varphi \in E_2 \Rightarrow \varphi \in E^*$$

that is

$$E_2 \subseteq E^* . \qquad (3.14)$$

The converse

$$E^* \subseteq E_2 \qquad (3.15)$$

is also true since every weakly convergent sequence is also bounded, i.e.

$$\varphi \in E^* \Rightarrow S_N(\varphi) \overset{*}{\to} \varphi \Rightarrow \| S_N(\varphi) \| \leqq \text{const} \Rightarrow \varphi \in E_2 .$$

The relations (3.14), (3.15) prove (3.10).

Note : from lemma 3 it also follows that

$$E_0 = E_1 . \qquad (3.16)$$

In fact if $\varphi \in E_1$

$$S_N(\varphi) \underset{H_K}{\to} \psi \qquad (3.17)$$

for some $\psi \in H_K$; but since $E_1 \subseteq E_2 = E^*$ we have also

$$S_N(\varphi) \underset{H_K}{\overset{*}{\rightarrow}} \varphi$$

which, combined with (3.17), gives

$$\varphi \in E_1, \qquad S_N(\varphi) \underset{H_K}{\rightarrow} \varphi \Rightarrow \varphi \in E_0 .$$

But then

$$E_1 \subseteq E_0$$

which together with the first of (3.8) proves (3.16).

We are now ready to demonstrate the

Theorem (3.I) : If $S_N(\varphi)$ is unconditionally weakly convergent to φ, then it is also unconditionally strongly convergent to φ, i.e.

$$E_2 \equiv H_K \Rightarrow E_0 \equiv H_K \tag{3.18}$$

the converse being evidently true.

The necessary and sufficient condition for (3.18) to be verified is that

$$\| S_N \| \leqq \text{const} \qquad \forall N . \tag{3.19}$$

Let us start with (3.19) : it is clear that, if (3.19) is true

$$\forall \varphi \in H_K, \qquad \| S_N(\varphi) \| \leqq \| S_N \| \; \| \varphi \| \Rightarrow \varphi \in E_2 :$$

hence (3.19) is a sufficient condition for $E_2 \equiv H_K$.

Furthermore if $E_2 \equiv H_K$, then

$$\forall \varphi \text{ fixed in } H_K, \qquad \| S_N(\varphi) \| \leqq \text{const.}$$

hence, for the theorem of uniform boundedness we have also

$$\| S_N \| \leqq \text{const.}$$

i.e. (3.19) is also a necessary condition for $E_2 \equiv H_K$.

Let us come now to (3.18) : since by hypothesis $E_2 \equiv H_K$, we have

$$\forall \varphi \in H_K, \qquad S_N(\varphi) \underset{H_K}{\overset{*}{\rightarrow}} \varphi ;$$

consequently

$$\| \varphi \|^2 = \langle \varphi, \varphi \rangle =$$

$$= \lim_{N \to \infty} \langle \varphi, S_N(\varphi) \rangle \leqq$$

$$\leqq \lim \operatorname{Inf} \| \varphi \| \; \| S_N(\varphi) \| ,$$

so that

$$\| \varphi \| \leqq \lim \operatorname{Inf} \| S_N(\varphi) \| . \tag{3.20}$$

On the other hand for any $\varphi \in H_K$ we can fix $\varphi^* \in H_C$ so that

$$\| \varphi - \varphi^* \| \leqq \varepsilon \tag{3.21}$$

$$\| \varphi^* \| \leqq \| \varphi \| + \varepsilon \tag{3.22}$$

hence we can write

$$\| S_N(\varphi) \| = \| S_N(\varphi - \varphi^*) + S_N(\varphi^*) \|$$

$$\leqq \| S_N \| \; \| \varphi - \varphi^* \| + \| S_N(\varphi^*) \| .$$

Since, by (3.19), $\| S_N \| \leqq M$ (suitable constant) and recalling that

$$\varphi^* \in H_C \Rightarrow \varphi^* \in E_0 \Rightarrow \lim_{N \to \infty} \| S_N(\varphi^*) \| = \| \varphi^* \| ,$$

using also (3.22), we get

$$\operatorname{Lim} \operatorname{Sup} \| S_N(\varphi) \| \leqq M \varepsilon + \| \varphi \| + \varepsilon :$$

from the arbitrariness of ε we have also

$$\operatorname{Lim} \operatorname{Sup} \| S_N(\varphi) \| \leqq \| \varphi \| . \tag{3.23}$$

But (3.20) and (3.23) are compatible only if

$$\lim_{N \to \infty} \| S_N(\varphi) \| = \| \varphi \|$$

which, together with (3.20) [9] proves also that

$$\lim_{N\to\infty} S_N(\varphi) = \varphi .$$

Concluding, we have proved that, if $E_2 \equiv H_K$, then

$$\forall\, \varphi \in H_K,\ S_N(\varphi) \underset{H_K}{\to} \varphi \Rightarrow \varphi \in E_0 \Rightarrow E_0 \equiv H_K ,$$

q.e.d.

REMARK 1 — The Theorem (3.I) can also be expressed as: the necessary and sufficient condition for the weak as well as for the strong unconditional convergence of $S_N(\varphi)$ to φ is that, for any fixed $\varphi \in H_K$ we have

$$\| S_N(\varphi) \| \leqq \text{const.} \tag{3.24}$$

This condition has an equivalent formulation, more customary in the theory of biorthogonal systems:

— a constant exists, C_0, such that for any real sequence $\{\alpha_1, \alpha_2 \dots \alpha_{n+m}\}$ and any integers n, m

$$\| \sum_1^n{}_i \alpha_i q_i \| \leqq C_0 \| \sum_1^{n+m}{}_i \alpha_i q_i \| . \tag{3.25}$$

It is possible to show that, for instance, (3.25) holds if $h_i - v_{n_i m_i} \to 0$ $(i \to \infty)$ for an opportune sequence of (n_i, m_i), $(n_i \to \infty)$.

REMARK 2 — Before going on to point *b*) we shall give without any proof a useful theorem:

Theorem (3.II): A necessary and sufficient condition for $S_N(\varphi)$ to be strongly unconditionally convergent in H_K to φ, is that

$$E_1 \equiv E_2 \tag{3.26}$$

B) — We are now ready to prove the

Theorem (3.III): If $S_N(\varphi)$ is not unconditionally convergent in H_K to φ, then there is always some $T \in H_K$, $T \notin H_C$ such that

$$\hat{T} = S_N(T) \underset{H_K}{\overset{*}{\to}} T \tag{3.27}$$

---

(9) in fact

$$\| \varphi - S_N(\varphi) \|^2 = \| \varphi \|^2 - 2 \langle \varphi, S_N(\varphi) \rangle + \| S_N(\varphi) \|^2 \to \| \varphi \|^2 - 2 \| \varphi \|^2 + \| \varphi \|^2 = 0$$

Let us observe that the theorem (3.III) states essentially that whenever we don't have unconditional convergence we have

$$H_C \subset E_2 \equiv E^*$$

(inclusion in the strict sense).

But this is evident since, on the contrary, we should have

$$H_C \equiv E_2 ,$$

which, considering (3.8) inplies also

$$E_1 \equiv E_2 .$$

But, by theorem (3.II) this is in contradiction with the hypothesis that $S_N(\varphi)$ is not unconditionally convergent in H_K.

One should observe that in the hypothesis of theorem (3.III) we have for some $T \in H_K$, $T \notin H_C$

$$S_N(T) \underset{H_K}{\overset{*}{\rightarrow}} T ,$$

but we must have also some $T \in H_K$ such that

$$S_N(T) \underset{H_K}{\overset{*}{\nrightarrow}} T$$

in fact, on the contrary, E_2 would coincide with H_K, which is absurd by theorem (3.I).

Concluding this paragraph it is important to underline that we have only studied *some abstract properties* of the convergence problem ; we have even found necessary and sufficient conditions for the convergence, but these are not very practical to handle, at least in our opinion. The only thing we can assert is that if the sequence $\{h_i\}$ tends asymptotically to the sequence $\{v_{nm}\}$ of the eigenvectors of the operator C, then we have the unconditional convergence of $\hat{T}$ to T, cf. remark 1 above.

This isn't so much indeed! Particularly if we want to address the problem of the convergence of $\hat{T}$ when the measurements m_i are the values of T at the nodes P_i of an ε-grid on the boundary. But we shall be concerned with this problem in the next paragraph.

4. — A CONVERGENCE THEOREM FOR A CLASS OF SMOOTHED POTENTIALS

In this paragraph we shall derive an interesting theorem, by going back to the original principle which has caused the choice of $\hat{T}$ as approximation of T, that is to (1.23).

We shall start from formula (1.22) which gives the mean square estimation error for any rotationally invariant linear estimate and we observe that, choosing

$$\begin{aligned} \lambda_i(P) &= \sum_1^N{}_j \{L_i L_j C(P_i, P_j)\}^{-1} L_i C(P_j, P) = \\ &= \sum_1^N{}_j \{\langle l_i, l_j \rangle_C\}^{-1} l_j(P) \end{aligned} \tag{4.1}$$

we obtain for E^2

$$\begin{aligned} E^2 &= C(P, P) - \sum_1^N{}_{i,j} \lambda_i(P) \langle l_i, l_j \rangle_C \lambda_j(P) = \\ &= C(P, P) - \sum_1^N{}_{i,j} \lambda_i(P) \langle K_i, C K_j \rangle \lambda_j(P) . \end{aligned} \tag{4.2}$$

On the other hand we notice that

$$\hat{T} = S_N(T) = \langle \sum_1^N{}_i \lambda_i(P) h_i(Q), T \rangle = \sum_1^N{}_i \lambda_i(P) m_i ,$$

so that (4.2) can also be written

$$E^2 = \langle K(P, \cdot), [C - S_N C S_N^t] K(P, \cdot) \rangle : \tag{4.3}$$

(4.3) can be easily verified by direct computation. Using (2.25), we get

$$\begin{aligned} S_N C S_N^t &= C^{1/2} Q_N C^{-1/2} C C^{-1/2} Q_N C^{1/2} = \\ &= C^{1/2} Q_N C^{1/2} \end{aligned}$$

which tends uniformly to C, cf. Riesz and Sz. Nagy (1956), page 204.

Consequently, since we may presuppose

$$\| K(P, \cdot) \|^2 = K(P, P) \leq \text{const.} \qquad \forall P \in \bar{\Omega}$$

we derive the relation

$$\begin{cases} E^2 \leqq \| K(P, \cdot) \| \; \| C - C^{1/2} Q_N C^{1/2} \| \\ \text{uniformly in } P \in \bar{\Omega} \end{cases} \tag{4.4}$$

But, from its definition (1.22)

$$E^2 = \frac{1}{4\pi} \int \{U_\alpha e(P)\} \, d\alpha :$$

recalling now the invariance property (1.21) we can write

$$U_\alpha e(P) = U_\alpha T - \sum_1^N \lambda_i \langle h_i, U_\alpha T \rangle =$$

$$= U_\alpha T - S_N (U_\alpha T).$$

Therefore (4.4) yields

$$\begin{cases} \lim\limits_{N \to \infty} \frac{1}{4\pi} \int \{U_\alpha T - S_N(U_\alpha T)\}^2 \, d\alpha = 0 \\ \text{uniformly in } P \in \bar{\Omega}. \end{cases} \tag{4.5}$$

Formula (4.5) has also the following interpretation : for any $P \in \bar{\Omega}$, $U_\alpha T$ is a function of $L^2(\alpha)$ (space of square integrable functions of the parameter α), the same is true for $S_N(U_\alpha T)$ and we have

$$\begin{cases} \lim\limits_{N \to \infty} \| U_\alpha T - S_N(U_\alpha T) \|^2_{L^2\alpha} = 0 \\ \text{uniformly in } P \in \bar{\Omega} \end{cases} \tag{4.6}$$

As a consequence of (4.6) we can also assert that for any $\chi(\alpha) \in L^2(\alpha)$ the relation

$$\begin{cases} \lim\limits_{N \to \infty} \int \chi(\alpha) \, S_N(U_\alpha T) \, d\alpha = \int \chi(\alpha) \, U_\alpha T \, d\alpha \\ \text{uniformly in } P \in \bar{\Omega} \end{cases} \tag{4.7}$$

holds. Since $S_N(\cdot)$ is a linear continuous operator we have

$$\int \chi(\alpha) \, S_N(U_\alpha T) \, d\alpha = S_N\Big(\int \chi(\alpha) \, U_\alpha T \, d\alpha\Big)$$

so that (4.7) becomes

$$\left\{\begin{array}{l} \lim_{N\to\infty} S_N(\int \chi(\alpha)\, U_\alpha\, T\, d\alpha) = \int \chi(\alpha)\, U_\alpha T\, d\alpha \\ \text{uniformly in } P \in \bar{\Omega} \end{array}\right. \tag{4.8}$$

Now let $\chi_\omega(\alpha)$ be the characteristic function of an ensamble ω in the parameter space, containing the parameter vector α_0 corresponding to the identity:

$$u_{\alpha_0} = I$$

$$\omega = \text{open},\ \alpha_0 \in \omega,\ \text{mes}(\omega) = \omega > 0$$

$$\chi_\omega(\alpha) = \left\{\begin{array}{ll} \frac{1}{\omega} & \alpha \in \omega \\ 0 & \alpha \notin \omega \end{array}\right. \tag{4.9}$$

For instance, using the approximated form of a rotation matrix

$$u_\alpha = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix} \tag{4.10}$$

we can put

$$\alpha = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \qquad \alpha_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\omega = \{ \alpha \mid |a| < \delta, \quad |b| < \delta, \quad |c| < \delta \}, \tag{4.11}$$

for small values of the parameter δ.

Let us notice that, being a measurable and bounded function, $\chi_\omega(\alpha) \in L^2(\alpha)$, then we can always define an ω-smoothing operator

$$\chi_\omega(T) = \int \chi_\omega(\alpha)\, U_\alpha\, T\, d\alpha =$$

$$= \frac{1}{\omega} \int_\omega T(u_\alpha P)\, d\alpha\,. \tag{4.12}$$

We can observe that $\chi_\omega(T)$ is a harmonic function in Ω and that $\chi_\omega(T) \in H_K$; in fact

$$\| \chi_\omega(T) \| \leqq \frac{1}{\omega} \int_\omega \| T(u_\alpha P) \| \, d\alpha ,$$

and since the norm $\| \quad \|$ of H_K is rotationally invariant

$$\| T(u_\alpha P) \| = \| T(P) \|$$

so that

$$\| \chi_\omega(T) \| \leqq \| T \| \frac{1}{\omega} \int_\omega d\alpha = \| T \| : \tag{4.13}$$

i.e. $\chi_\omega(\cdot)$ is a contraction in H_K.

In order to better understand what this $\chi_\omega(\cdot)$ operator is, let us identify $\chi_\omega(T)$ by means of its boundary values and let us choose ω as in (4.11).

For any $P \in \sigma$ we have

$$\begin{aligned} x_P &= R \sin\varphi \cos\lambda \\ y_P &= R \sin\varphi \sin\lambda \\ z_P &= R \cos\varphi \end{aligned}$$

and also

$$u_\alpha P = \begin{bmatrix} x_P + a y_P + b z_P \\ - a x_P + y_P + c z_P \\ - b x_P - c y_P + z_P \end{bmatrix}$$

Consequently

$$\chi_\omega(T) = \frac{1}{8\delta^3} \iiint_{-\delta}^{\delta} da\, db\, dc\; T(x_P + a y_P + b z_P, - a x_P + y_P + c z_P, - b x_P - c y_P + z_P) \tag{4.14}$$

for α ranging in ω, $u_\alpha P$ will move in a neighbourhood of P on the sphere σ, so that (4.14) *is a kind of mean of $T(P)$ on a small block around P,* therefore $\chi_\omega(T)$ is the smoothing operator which substitutes the boundary values of $T(P)$ with opportune means of blocks around the points P.

It is worth noticing that this smoothing of the boundary values may be applied when we deal with actual data. Recalling (4.8) we arrive then to enunciate the

Theorem (4.I): for any $T \in H_K$ and any fixed ω let us consider the ω-smoothed potential $\chi_\omega(T)$: the collocation approximation for $\chi_\omega(T)$, $S_N[\chi_\omega(T)]$ is convergent to $\chi_\omega(T)$,

$$\lim_{N \to \infty} S_N[\chi_\omega(T)] = \chi_\omega(T)$$

the convergence being uniform in $P \in \bar{\Omega}$

This theorem is, in our opinion, quite satisfactory because it reflects a possible use of practical measurements, i.e. the block smoothing.

Concluding it is interesting to underline that the result obtained in (4.I) makes use of a different topology than the natural H_K-topology, so that the problem of convergence in H_K still requires further research.

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