67

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Gravity prediction using collocation and taking known mass density anomalies into account

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Summary. The anomalous (gravitational) potential of the Earth, T, is split in two parts, $T = T^c + T_M$. Here T_M is a harmonic function generated by known mass density anomalies and $T^c = T - T_M$. This function will also be a harmonic function, which therefore may be approximated using the method of collocation, based on known gravity anomalies or altimeter derived geoid undulations, for example. Gravity anomalies can then be predicted using the known linearized relationship between T and Δg. This procedure may give a 40–50 per cent increase in the precision of the prediction results as compared to a procedure where mass density anomalies are not taken into account.

1 Introduction

The anomalous potential of the Earth, T, is equal to the difference between the gravitational potential of the Earth, W, and a normal or reference potential, U, i.e. $T = W - U$.

We will in the following suppose that T is a harmonic function outside the surface of the Earth, and that it fulfils certain regularity conditions at infinity. This means that in U we have included the rotational part of W, the potential of the atmosphere, the Moon etc. and that there are no zero-order and first-order harmonic coefficients in the expansion of T in a series of solid spherical harmonics.

We may consider U as being generated by a density reference model (see, e.g. Moritz 1973). Then T can be considered as being generated by an anomalous density distribution, ρ, i.e.

$$T(P) = k \int_{\text{Earth}} \rho(Q)/\|P - Q\| \, dR^3, \tag{1}$$

where P and Q are points in R^3, $\|P - Q\|$ is the distance between the points, and k is the gravitational constant. The total mass of ρ is zero, i.e. it takes positive as well as negative values. Equation (1) is the basis for using gravity field information in geophysical investigations, but it is naturally also the basis for using density information when computing approximations to T or to quantities dependent on T such as gravity anomalies or deflections of the vertical.

In this paper we will consider only one method for the determination of approximations to T, namely the method of least squares or least norm collocation. The use of density information in connection with this method, however, illustrates well how density information can be used in connection with other techniques. In Section 2 we will give a brief introduction to collocation and describe two possible methods for the use of density information. In Section 3 we describe the results of a test computation, where gravity anomalies were predicted in a mountainous area of New Mexico, USA. Finally, in Section 4 we discuss the requirements on density information seen in relation to the requirements on the observed gravity field-dependent quantities.

2 The method of collocation and the use of density information

Let us regard a certain function (in our case T) for which we have obtained observations x_i, $i = 1, \ldots, n$. We will suppose that these observations are error free* and that they can be related to the function through a set of linear functionals, L_i, $i = 1, \ldots, n$, i.e. $L_i(T) = x_i$. The free air gravity anomalies, the geoid undulations and the deflections of the vertical, for example, are related to T through such linear(ized) functionals.

Collocation is then a method for the determination of an approximation to the function, so that it agrees with the values of the observations. Denoting the approximation to the anomalous potential by $\tilde{T}$ we must have

$$L_i(\tilde{T}) = L_i(T) = x_i, \qquad i = 1, \ldots, n. \tag{2}$$

Let us require the approximation to be an element of a linear vector space of functions, which we will here require to be harmonic outside the surface of the Earth or another suitable surface. We will also require this vector space to have an inner product (and thereby a norm) so that the space is a so-called *reproducing kernel* Hilbert space. We will denote this space H, and we will suppose that the linear functionals occurring in equation (2) are elements of the space dual to H, and that they are linearly independent.

The reproducing kernel $K(P, Q)$ is a function of two variables, which here will be points in the area of harmonicity. For either P or Q fixed the reproducing kernel will be an element of the space, i.e. here a harmonic function. This is also the case if we apply a linear functional on $K(P, Q)$, i.e. $L_i K(\cdot, Q)$ is an element of H (where the dot indicates that L_i is applied on $K(P, Q)$ with respect to the first variable).

If H were of finite dimension, n, it would be spanned by n harmonic functions, h_i, $i = 1, \ldots, n$. The approximation could then have been obtained by determining a set of coefficients a_i, so that

$$\tilde{T}(P) = \sum_{i=1}^{n} a_i h_i(P),$$

and a_i is given by

$$L_i(\tilde{T}) = L_j\left(\sum_{i=1}^{n} a_i h_i\right) = \sum_{i=1}^{n} a_i L_j(h_i) = x_j, \qquad j = 1, \ldots, n, \tag{3}$$

where $L_j(h_i)$ are known quantities.

It is well known that such approximations may be very 'unstable', and show unwanted oscillation in between the data points. It is generally better to choose H to be of very high dimension (in our case of infinite, but countable dimension), and then require T to be

* This does not mean that errors cannot be taken into account, see Krarup (1969) or Moritz (1972).

smooth in the sense that it has the least possible norm. This is the method of least norm or least squares collocation (see Tscherning 1978 for more details). It can then be shown that $\tilde{T}$ will be an element of a n-dimensional subspace of H spanned by the functions $L_i(K(\cdot, Q))$. So we get from equation (3)

$$L_j(\tilde{T}) = L_j\left(\sum_{i=1}^{n} a_i L_i K(\cdot, Q)\right) = x_j$$

or

$$\sum_{i=1}^{n} a_i L_i L_j K(\cdot, \cdot) = x_j, \tag{4}$$

which is a system of n equations with n unknowns.

If the reproducing kernel is chosen to be equal to the so-called empirical covariance function of the anomalous potential,

$$\operatorname{cov}(T(P), T(Q)) = K(P, Q),$$

then we have the equations

$$\tilde{T}(Q) = \sum_{i=1}^{n} a_i \operatorname{cov}(T(Q), L_i(T)) \tag{5}$$

with

$$\{\operatorname{cov}(L_i(T), L_j(T))\}_{nxn} \{a_j\}_n = \{x_i\}_n. \tag{6}$$

The free-air gravity anomalies and the geoid undulations are related to T through linearized functionals

$$\Delta g_P = -\left.\frac{\partial T}{\partial r}\right|_P - \frac{2}{r} T(P) \tag{7}$$

$$\zeta_P = T(P)/\gamma(P)$$

where the subscript P indicates that they refer to a specific point P, r is the distance from the Earth's centre and γ is the reference gravity. *Predictions* are then obtained by applying the functionals on $\tilde{T}$, e.g.

$$\tilde{\Delta g}_P = -\left.\frac{\partial T}{\partial r}\right|_P - \frac{2}{r} \tilde{T}(P) \tag{8}$$

or

$$\tilde{\Delta g}_P = \sum_{i=1}^{n} a_i \operatorname{cov}(L_i(T), \Delta g_P), \tag{9}$$

when expressed with covariance functions.

In order to use density information for the determination of $\tilde{T}$ there are then two alternatives. One alternative is to regard the density values as being equal to a linear functional applied to T. The linear functional is given by the well-known Poisson equation

$$\rho(Q) = \frac{-1}{4\pi k}\left(\left.\frac{\partial^2 T}{\partial x^2}\right|_Q + \left.\frac{\partial^2 T}{\partial y^2}\right|_Q + \left.\frac{\partial^2 T}{\partial z^2}\right|_Q\right), \tag{10}$$

where T now is regarded as a function in all of R^3.

The problem is then to choose an appropriate reproducing kernel Hilbert-space of functions defined in R^3. This can be done as described in Tscherning (1974, 1977) by requiring ρ, the density distribution, to fulfil a simple mathematical condition. Seen from a computational standpoint, the problem is to compute the quantities $L_i L_j K(\cdot, \cdot)$ or the corresponding cross covariances between density values and other quantities. Such covariance functions are discussed in Tscherning (1977) and Jordan (1978). (If the density variations are considered to be sufficiently well represented by the variations of the topographic heights, then the auto- and cross-covariances between these heights and gravity field dependent quantities are needed, see, e.g. Heiskanen & Moritz (1967, chapter 7).

However, this alternative is unsatisfactory from a computational standpoint, because the system of equations to be solved will be very big. It is also somewhat unsatisfactory from a theoretical standpoint because we will have to introduce restrictions on the function ρ, which are not necessarily justified geophysically.

The other alternative is closely related to the traditional techniques of removing and restoring the masses, which are applied when solving the geodetic boundary value problem, see, e.g. Heiskanen & Moritz (1967, chapter 8) and which have been considered in connection with prediction of gravity field dependent quantities by, e.g. Wilcox (1974) and Lachapelle (1975). Instead of determining an approximation to T, an approximation to $T^c = T - T_M$ is determined. T_M is a *harmonic* function computed from a model of the known densities. T^c will be a smooth function (compared to T) if the model represents the topography well locally, see Table 1 and Tscherning & Forsberg (1978, table 1).

The smoothness implies that *fewer values* are needed to represent the function T^c than to represent T, or, $T = T^c + T_M$ is more easily approximated using the density information than without using it.

Table 1. Variation of free-air and topographic-isostatically reduced gravity anomalies in different areas (mean values subtracted).

Location	Area size	Source	Variation of Δg Free-air (mgal)	topogr.-isost. red. (mgal)
New Mexico, USA	1° × 1°	Tscherning & Forsberg (1978)	± 19	± 10
West-Greenland	2° × 1°	Olsen (1971)	± 38	± 21
USA	15° × 45°	Heiskanen & Vening-Meinesz (1958)	± 59	± 24
Canada	5° × 5° *	Heiskanen & Vening-Meinesz (1958)	± 41	± 18
India	25° × 30°	Heiskanen & Vening-Meinesz (1958)	± 80	± 39
East Africa	10° × 9°	Heiskanen & Vening-Meinesz (1958)	± 46	± 24
The Alps	2° × 3°	Heiskanen & Vening-Meinesz (1958)	± 72	± 26
At sea	Worldwide	Heiskanen & Vening-Meinesz (1958)	± 36	± 29

This can be expressed mathematically or statistically through the equation expressing the upper bound for the errors and the mean square prediction error, respectively. The main factor in the equation for the mean square prediction error is the mean square variation of the predicted quantity. So if as seen in Table 1 we are able to reduce this variation by at least 40–50 per cent by taking known mass densities into account, then the mean square prediction error will be reduced by approximately the same amount.

The observations used for constructing $\tilde{T}^c$ must naturally be the original observations minus the effect of the densities, i.e.

$$x_i^c = x_i - x_{iM} \tag{11}$$

where $L_i T_M = x_{iM}$. Similarly, the predicted quantities will be equal to

$$L(\tilde{T}) = L(\tilde{T}^c) + L(T_M). \tag{12}$$

This simple method does not have the drawbacks of the first alternative and has therefore been used in the test described in the following section.

3 Test of prediction of free-air gravity anomalies

From equations (7) and (12) we see that gravity anomalies are predicted as

$$\widetilde{\Delta g}_P = -\left.\frac{\partial \tilde{T}^c}{\partial r}\right|_P - \frac{2}{r}\tilde{T}^c(P) - \left.\frac{\partial T_M}{\partial r}\right|_P - \frac{2}{r} T_M(P) = \widetilde{\Delta g}^c + \Delta g_M \tag{13}$$

where Δg_M are the parts of the free-air gravity anomaly which originate from the model of the densities.

The technique of removing and restoring the masses has been tested in a $1° \times 1°$ area in New Mexico, USA, with strongly varying gravity anomalies due to a mountain chain running north–south through the area. Details describing a similar test, where deflections of the vertical were predicted, can be found in Tscherning & Forsberg (1978), and we shall only repeat the essential here. Two approximations to T were computed using the method of stepwise least squares collocation. The data used were a set of potential coefficients, 1°-equal area mean free-air gravity anomalies, point free-air gravity anomalies (98 values spaced 6′ apart in the $1° \times 1°$ area) and density values in the form of topographic heights. The terrain was considered to be isostatically compensated at a depth of 30 km. One approximation was computed without using the density information. The two approximations were then used for the prediction of 113 values of free-air gravity anomalies in the $30' \times 30'$ block in the middle of the area. The standard-deviations of the difference between observed and predicted values are shown in Table 2.

Table 2. Standard deviation of the observed and the difference between observed and predicted gravity anomalies.

	Standard deviation
Observed free-air anomalies	± 19
Difference *not* using density information	± 5
Difference using density information	± 3

It is obvious, that the use of density information gives a substantial improvement in the results, even when we as here use density information in a very unsophisticated way. (Namely in the form of topographic heights, compensated isostatically.) In case geological or geophysical (seismic) information had been available, this information could easily have been introduced in the density model.

Let us finally notice that a similar improvement in the prediction results can be expected in general. The improvement will depend on the magnitude of the variation of the topographic-isostatically reduced gravity anomalies as compared to the variation of the free-air gravity anomalies. Inspecting Table 1 we see that the improvement will be between 40 and 50 per cent.

4 Conclusions

The idea of using known mass density anomalies when predicting gravity anomalies is not new. (An extensive bibliography is included in Wilcox (1974).) The older techniques have, however, in the author's opinion lacked a sound mathematical basis, and it is hoped that this paper will help in providing such a basis. The basic problem, namely the lack of observational data, is naturally still unsolved. But the method of collocation provides (together with other approximation techniques) a tool for combining data in a consistent way, which will give us an optimal use of the existing data.

Most gravity data are today available in computer-readable form and so are geoid undulations derived from satellite radar altimetry. Topographic heights and ocean depths are becoming available for some areas of the world, but a coordinated effort seems to be lacking. Pioneering work of collecting results of seismic refraction measurements is published by Lee & Taylor (1966). But much more detailed information is needed especially in areas where the geological features causing the gravity variations are hidden under sediments, see, e.g. Wassouf (1975). Here a preliminary analysis of geodetic and seismic data could be of great help in determining the location of the different density variations.

Let us finally touch on a small, but important problem. What should be used as a density reference model generating the normal or reference potential, U. Models developed by Moritz (1973) can be used, but is it advisable to use the reference ellipsoid as an external boundary for the reference densities or is it more appropriate to include masses in an ellipsoid with a semimajor axis about 1–2 km shorter as indicated by Romanowicz & Lambeck (1977)?

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