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USE OF COLLOCATION FOR PREDICTING GEOID UNDULATIONS AND RELATED QUANTITIES OVER LARGE AREAS *

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Abstract: The use of the method of collocation for the determination of an approximation to the anomalous (gravitational) potential of the Earth (T) is briefly described and results obtained when predicting geoid undulations and related quantities are reviewed.

The method requires in principle that a set of linear equations is solved, with the number of unknowns equal to the number of observations. This number will depend on (1) the required precision of the predictions, (2) the size of the area regarded and (3) the variation of the anomalous potential in the area. The number of gravity observations needed in a 1°-square is estimated to be at least 100 if geoid undulations with a standard deviation of less than ± 0.25 m are required. The system of equations needed to be solved when regarding a large area, like e.g. Southern Europe will therefore be very difficult to handle.

It is therefore proposed, that collocation should be used for the approximation of the harmonic function $T^c = T - T_M$, where T_M is a harmonic function generated by known mass density anomalies. This will reduce the needed number of observations to about 25 % of the original number.

It is also proposed to use either the method of stepwise collocation or the combined integral-formulae- collocation method in order to further reduce the computational effort. Finally the practical organization of the computations using these two techniques is discussed.

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1. Introduction

The need for gravity field information (geoidal undulations, deflections of the vertical, gravity anomalies) has increased in the recent years. Traditionally the geodesists and the geophysicists have been both the main suppliers and users of this information. The geodesists have used geoid undulations and deflections of the vertical (in order to reduce distances and angles to the ellipsoid) and gravity for the computation of potential differences in levelling. The geophysicists have used the information in order to obtain knowledge of the internal constitution of the Earth.

Today gravity field information is needed for the precise computation of the orbits of spacecrafts, in inertial navigation or surveying, in oceanography and in mapping (in order to obtain orthometric heights from ellipsoidal heights). Fortunately new datatypes have become available, so that there is a good chance that the new requirements can be fulfilled. Where we had earlier only gravity, astronomical latitude and longitude and geopotential differences, we now have low degree potential coefficients and geoid undulations, the latter determined either through satellite radar altimetry or through doppler satellite determined geocentric coordinates combined with orthometric heights.

This implies, that the task of computing the geoid undulations in an area can not be separated from the task of computing related quantities like deflections of the vertical. The real task must be the determination of (a numerical representation of) the gravitational potential of the Earth, W, or equivalently of the anomalous potential T = W-U, where U is a normal or reference potential. We will in the following suppose that U includes the potential of the atmosphere, and has been chosen so that T is harmonic in space and is regular at infinity. Having determined such a numerical representation (which we will denote $\tilde{T}$), all related quantities such as geoid undulations or gravity anomalies can be computed using well known formulae (see section 5).

Traditional techniques, using for example integral formulae, can in a certain sense be thought of as giving a numerical representation of T. Evaluating Stokes integral formulae with a fixed set of mean gravity anomalies, Δg_i, $i = 1, \ldots, N$, we have

$$\tilde{T}(P) = \sum_{i=1}^{N} S(\psi_i, r)\, \Delta g_i \,, \tag{1}$$

where ψ_i is the spherical distance between P and the center of the i'th block, r is the distance from the Earth's center and $S(\psi, r)$ is the Stokes function in space. However, the use of this type of representation requires data to be available in a regular grid. The (gridded) data are frequently obtained by a prediction or interpolation technique (see e.g. Strange and Fury (1977)).

Using an approximation technique like collocation this step may be bypassed, because data neither needs to be spaced in a regular manner nor to be of only one kind. However, the approximation techniques generally require a set of linear equations to be solved.

The number of equations will depend on size of the area regarded, the wanted precision and the magnitude of the variation of the anomalous potential in the area.

We will in following discuss the use of the method of collocation for the determination of $\tilde{T}$. (This does not mean that other techniques can not be used.) In section 2 we will give a brief description of the method and review results of predictions. The method requires in principle that a set of linear equations with a dimension equal to the number of observations is solved. In section 3 we describe how the number of needed observations can be decreased by taking known density variations (mainly the topography) into account. In section 4 we describe how the system of equations may be broken up into several smaller systems such as it is done when using the method of stepwise collocation or the combined integral formulae - collocation approach. Finally we discuss the prediction of geoid undulations and related quantities like deflections of the vertical and give some suggestions about the practical organization of the computations.

Throughout the paper we will have one example in mind, namely the computation of an approximation to T in the Southern-European-Mediterranean are. This has an extend of approximately $35^{\circ} \times 20^{\circ}$ or 700 1°-squares. We will also suppose, that we want to be able to compute geoid undulations with a standard deviation of ± 0.25 m, because this is the precision with which we may expect ellipsoidal heights to be given in the future, using e.g. the U.S. Global Positioning System (see Anderlee (1978)).

2. Collocation

An extensive literature exist describing the theoretical background for and practical implementation of collocation when used for the determination of an approximation to the anomalous potential. Extensive lists of references can be found in Moritz (1978) and Tscherning (1978).

There exist two approaches to collocation, which have been used in connection with the estimation of gravity field dependent quantities. Krarup (1969) proposed that $\tilde{T}$ should be constructed, so that it was an element of a reproducing kernel Hilbert space consisting of functions harmonic down to a sphere totally enclosed in the Earth. As reproducing kernel he proposed that a mathemetically simple kernel should be used.

$\tilde{T}$ should then be the element of the Hilbert space which had the least norm and which agreed exactly with the measurements, if these were error-free (exact collocation). In case the observations were not error-free, the square of the norm plus the mean square variation of the errors should be minimalized (smoothing collocation).

A somewhat different approach has been considered by Moritz (1970, 1972) as an extension or generalization of his technique of least squares prediction. In this approach it is required to work in spherical approximation, i.e. so that $\tilde{T}$ becomes a harmonic function in the set outside a sphere approximating the Earth. $\tilde{T}$ will furthermore be an element of a reproducing kernel Hilbert space, which had the empirical covariance function as its reproducing kernel. Besides that, $\tilde{T}$ must fulfil the same requirements as mentioned above, namely having minimum norm and agreeing with observed values. The use of the empirical covariance function has the theoretical advantage, that $\tilde{T}$ is optimal in a least squares sense.

Let T in spherical approximation have an expansion as a series in fully normalized solid spherical harmonics, i.e.

$$T(P) = \frac{GM}{r} \sum_{i=2}^{\infty} \left(\frac{R}{r}\right)^i \bar{P}_{ij}(\sin\varphi) \left[\bar{a}_{ij} \cos j\lambda + \bar{b}_{ij} \sin j\lambda\right] \qquad (2)$$

where R is the radius of the sphere approximating the Earth, GM is the product of the mass of the Earth and the gravitational constant, r is the distance of P from the center of the Earth, φ is the latitude, λ is the longitude and $\bar{P}_{ij}$ are the fully normalized associated Legendre polynomials. Then the empirical covariance function is a function of two points in space, P,Q. If r, r' are the distances from the Earth's center of P,Q, respectively, and ψ is the spherical distance between the points, then

$$\mathrm{cov}(P,Q) = \sum_{i=2}^{\infty} \sigma_i^2 \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\psi), \qquad (3)$$

where P_i are the Legendre polynomials and

$$\sigma_i^2 = \sum_{j=o}^{i} (\bar{a}_{ij}^2 + \bar{b}_{ij}^2) (\frac{GM}{R})^2 \quad , \tag{4}$$

are the so-called degree-variances.

The covariance function (3) is the empirical covariance function of the anomalous potential, and we will sometimes write cov(P,Q) = cov(T(P),T(Q)). From this covariance function we may derive covariance functions of other quantities which are related to T through linear functionals (in the following denoted L_i , L_j or L). Let these quantities be equal to $L_i(T)$ and $L_j(T)$. Then $cov(L_i(T), L_j(T)) = L_i L_j cov(P,Q)$, i.e. the linear functionals applied on the basic function cov(P,Q). This is the so-called law of covariance propagation.

From eq. (3), (4) we see that a complete knowledge of the covariance functions requires a complete knowledge of T. However, the covariance function may be estimated as described for example in Tscherning and Rapp (1974). Here is also described how the estimated values can be used for the construction of analytic representations of the covariance function. The considered and finally chosen representations are all relatively simple reproducing kernels. So we have here in a certain sense a compromise between the two above mentioned approaches to collocation.

Moritz has also proposed that other quantities than $\tilde{T}$ should be estimated as a part of the procedure, where $\tilde{T}$ is estimated. Such parameters could for example express a change of coordinate system or datum shift. However, this simply means that the Hilbert space has been extended with a finite dimensional subspace, which generally is equipped with a very simple norm.

Let us suppose that our observations are related to T through a set of linear functionals,

$$L_i(T) = m_i \quad , \qquad i=1,\ldots,n \quad , \tag{5}$$

and that they are error-free. The approximation $\tilde{T}$ will then be given as

$$\tilde{T}(P) = \sum_{i=1}^{n} a_i L_i cov(P,Q) = \sum_{i=1}^{n} a_i cov(T(P), L_i(T)) \tag{6}$$

where the constants a_i are solutions to a system of linear equations

$$\{\mathrm{cov}(L_i(T),\ L_j(T))\}\ \{a_i\} = \{m_j\}\ . \tag{7}$$

This set of equations are denoted the normal equations.

When $\tilde{T}$ or equivalently the constants $\{a_i\}$ have been determined, quantities dependent on T can be estimated. The quantities must be related to T through a linear functional L, and we then get the predicted quantity by applying L on $\tilde{T}$, i.e.

$$L(\tilde{T}) = \sum_{i=1}^{n} a_i \mathrm{cov}(L(T), L_i(T))\ . \tag{8}$$

Within this framework the mean square error of prediction may be computed for an estimated quantity $L(\tilde{T})$,

$$\sigma^2(L(\tilde{T})) = C_{LL} - \{C_{Li}\}^T\ \{C_{ij}\}^{-1}\ \{C_{Lj}\}\ , \tag{9}$$

where we have put $C_{LL} = L(L(\mathrm{cov}(P,Q)))$, $C_{Li} = L_i(L(\mathrm{cov}(P,Q)))$ and $C_{ij} = L_i(L_j(\mathrm{cov}(P,Q)))$.

Note, that this error has no probabilistic interpretation, but expresses the mean square error of prediction. (The mean value is taken over a sphere with center at the Earth's center and passing through the point of evaluation, if such a point is associated with L. This is for example the case, if $L(T) = \Delta g(P)$, the gravity anomaly in the point P). However, numerical investigations have shown, that eq. (8) gives a meaningful estimate of the actual error, see e.g. Tscherning (1975).

In the general case the observations can not be considered errorfree. Let us suppose that the errors are normally distributed with a covariance matrix $\{D_{ij}\}$. The equations of "smoothing" collocations, which must be used in this case, are only slightly different from the equations of "exact" collocation. In fact, if we in eq. (7) and (9) for $\{C_{ij}\} = \{\ \mathrm{cov}(L_i(T), L_j(T)\}$ substitute $\{\bar{C}_{ij}\} = \{\ \mathrm{cov}(L_i(T), L_j(T)) + D_{ij}\}$ than $\tilde{T}(P)$ and $\sigma^2(L(T))$ are given again by eq.(6) and (9), respectively.

The method of collocation has been extensively tested by several investigators, before it was used for solving a specific prediction problem. Each investigator has modified the method depending on the problem. We will describe some of these modifications in section 4.

Tscherning (1975, 1975a, 1978a), Lachapelle (1975, 1977) and Tscherning and Forsberg (1978) have used the method for the prediction of gravity anomalies, deflections of the vertical and geoid undulations and have compared predicted results with observed values. Rapp (1974, 1977) has used the method for the prediction of mean gravity anomalies from geoid undulations (obtained by satellite radar altimetry). The results, of which some are summarized in Table 1, confirm the usefulness of the method.

Table 1. Survey of some prediction results using the collocation and the combined integral-formulae/collocation method.

No.	Source and area	Observations**				Difference predicted-observed	
		type*	spacing	mean	std.v.	mean	std.v.
1	Tscherning(1975a) 2^ox2^oarea, Denmark	Δg	15'	15.0	13.6	0.0	4.5
		ξ	1^o	-1.5	2.3	-0.2	0.7
		η	1^o	0.7	1.8	-0.2	0.8
2	Tscherning(1975) 2^ox2^oarea,Ohio/USA	Δg	7!5	-7.0	18.5	0.4	7.5
		ξ	not used	1.4	1.4	1.2	1.1
		η	not used	1.7	5.0	-0.6	1.6
3	Tscherning/Forsberg (1978)+Tscherning (1978a)	Δg	6'	-16.7	19.4	1.0	5.0
		ξ	1^o	- 1.6	1.8	-0.6	1.2
	1^ox1^oarea,New Mexico/ USA	η	1^o	- 3.0	5.4	1.0	1.6
4	As no 3, but with	Δg	6'	-16.7	19.4	1.0	3.0
	isostatically compensated terrain	ξ	not used	- 1.6	1.8	-1.1	0.7
		η	not used	- 3.0	5.4	1.8	1.0
5	Lachapelle(1976) Canada	Δg	12'				
		ξ	1^o	- 1.6	2.3	0.2	1.0
		η	1^o	- 7.4	3.7	0.1	1.0
6	Lachapelle(1977) Canada	Δg	8'				
		ξ	not used	- 1.4	5.1	-0.1	1.4
		η	not used	-2.6	7.8	-0.4	1.6
		N	not used	-30.8	9.4	0.0	1.2

* Δg-values are in units of mgal, ξ,η-values in arcsec and N-values in m.
** The observations used for prediction include potential coefficient sets and mean gravity anomalies.

Eq. (9) can in principle be used in order to study the precision of predictions computed using a given data-configuration, see Tscherning (1975). The difficulty encountered in practical applications is related to the problem whether a chosen representation of an empirical covariance function really will represent the actual behaviour of the gravity field in an area. However, from computational experiments, where observed values are predicted from other observed values, insight have and can be gained concerning the precision which may be obtained using a given data spacing (cf. Table 1). The computational experiment carried out by Tscherning and Forsberg (1978) shows for example, that deflections of the vertical may be predicted with a standard deviation of 30 % ($\approx \pm 1''.4$) of the root mean square variation of the observations ($\pm 4''.5$) using a spacing of free-air gravity anomalies of 6', i.e. 100 per 1°-square. Unfortunately we can not make a similar computation for geoidal undulations in a local area, because they are not yet available as "independent" observations to a precision better than about 1 m.

Anderlee (1978) has estimated that positions in the future may be obtainable with a very high precision using the Global Positioning System. Ellipsoidal height differences should be obtainable over continental distances with a precision of ± 10 cm. Hence, where levelling is available within the same precision, geoid undulation (differences) may be computed with a standard deviation of ± 0.15 cm. Let us suppose, that such values are available with a spacing of 1° in between the points. In order to predict geoid undulations "everywhere", i.e. also in points at 50 km distance with a standard deviation of ± 0.25 cm, we must be able to compute geoid undulation differences over such distances with a precision of ± 20 cm. This means again that deflections of the vertical must be available with a standard deviation of about $\pm 1''$. ($1''$ deflection correspond to 25 cm geoid undulation over 50 km).

These very loose calculations show that for a variation of the deflections of $\pm 4''.5$ will minimally 100 gravity anomalies be needed per 1°-square in order to obtain the required precision. Hence for a large area, like the Southern European Sub-continent will totally about 70000 gravity anomalies be needed. (The variation of the deflections of the vertical in this area is (considered locally) not smaller than $\pm 4''.5$). It is then naturally possibly to combine this information with existing observations of deflections of the vertical and geoid undulations (obtained at sea by satellite radar altimetry and on land as differences between ellipsoidal and orthometric heights). However, these data will not reduce the needed total number of observations in any substantial manner.

Considering this huge amount of observations, it is therefore necessary to find methods which will reduce the size of the normal equations. This will be the subject of the next two sections.

3. The use of density information

In section 2 we have seen, that one of the factors, which determined the size of the system of normal equations, was the variation of the gravity field. One of the main reasons for the local gravity field variations are, as it is well-known, the local topography and its isostatic compensation.

We may, using known density information, compute a harmonic function, T_M, representing the potential of the masses. Hence $T^c = T-T_M$ is a harmonic function, and we may use the method of collocation for the computation of an approximation to T^c, which we will denote $\tilde{T}^c$. Hence $\tilde{T} = \tilde{T}^c+T_M$.

The observations used for constructing $\tilde{T}^c$ must be the original observations minus the effect of the densities, i.e.

$$m_i^c = m_i - m_{iM} , \tag{10}$$

where $L_i T_M = m_{iM}$. Similarly the predicted quantities will be equal to

$$L(\tilde{T}) = L(\hat{T}^c) + L(T_M). \tag{11}$$

From Tscherning and Forsberg (1978, Table 1) and Tscherning (1978a, Table 1) we see, that the root mean square variation of the "reduced" quantities is about half of the variation of the "unreduced" quantities. (Reduced gravity anomalies, for example, had a variation of ± 10 mgal, corresponding to a variation of ± 20 mgal for the unreduced quantities.)

This means that the spacing of the observations can be reduced to the double (linear) distance, i.e. only 25% of the original number of observations are needed.* For detaills describing a possible procedure for computing $L(T_M)$ see e.g. Tscherning and Forsberg (1978). In a computational test in the same publication it is shown, that this has made it possible to predict deflections of the vertical with a standard deviation of $\pm 1''$ using reduced gravity anomalies (variation ± 10 mgal) spaced 6' apart, as opposed to ± 1.5 using the unreduced observations (see Table 1).

* This can be seen from Tscherning (1975, Fig. 5a-5e, 6), when the data-spacing is not to big, i.e. not bigger than ψ_c, the zero correlation distance.

4. Methods for reducing the size of the normal equation matrix

Let us suppose, that we have used the known density variations for the construction of a function T_M, which represents the local gravity field variations well. The variations of the reduced observations m_i^c will then be reduced considerably as discussed in section 3. However, the variation will not be below the gravity field variations, which are observed in flat areas like North-West Europe, i.e. between 100-200 $mgal^2$ for the gravity anomalies in a 1^o-square. This still leaves us with a need for solving big systems of equations. We will here discuss two methods for reducing the size of the normal equation matrix.

4.1. Stepwise collocation

Let us suppose that we have as observations two sets of data. The first set being a set of potential coefficients complete up to degree J and having no errors. Then it is easily seen, that the normal equation matrix may be split in four sub-matrices, where one sub-matrix is a diagonal matrix, see Fig. 1.

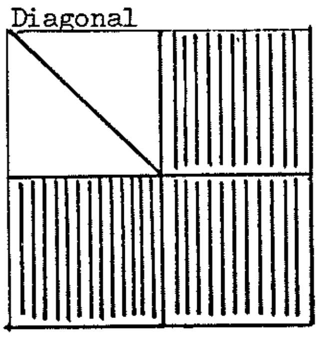

Fig. 1.

Solving first for the unknown's corresponding to the potential coefficients, it is seen that this correspond to representing $\tilde{T}$ as a sum of two harmonic functions, $\tilde{T}_1$ and $\tilde{T}_2$. We will have

$$\tilde{T}_1 = \frac{GM}{r} \sum_{i=2}^{J} \left(\frac{R}{r}\right)^i \bar{P}_{ij}(\sin\varphi)\left[\bar{a}_{ij} \cos j\lambda + \bar{b}_{ij} \sin j\lambda\right] \qquad (12)$$

and

$$\tilde{T}_2 = \sum_{i=1}^{n} b_i \, \mathrm{cov}_2(L_i(T), T(P)) \qquad (13)$$

with

$$\{cov_2(L_i(T),\ L_j(T)\}\ \{b_i\} = \{L_j(T-\tilde{T}_o)\}\ , \tag{14}$$

i.e. the right hand side of the normal equations is equal to the original observations minus the effect of $\tilde{T}_o$. A new covariance function will have to be used, which is related in a simple manner to the original one:

$$cov_2(P,Q) = \sum_{i=J+1}^{\infty} \sigma_i \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\psi). \tag{15}$$

Now let us suppose, that we may further divide the observations in groups, consisting of the observations which in some way are related to different wavelengths, such as 1^o-mean gravity anomalies and point related data. Here a further subdivision of the normal equation matrix can be made, but the change in covariance is not as obvious as before. Using as a reference field $\tilde{T}_1+\tilde{T}_2$, where $\tilde{T}_2$ has been constructed using 1^o-mean gravity anomalies, new "observations" can be computed,

$$m_{i,3} = m_i - L(\tilde{T}_1+\tilde{T}_2). \tag{16}$$

These quantities can be used for the computation of an empirical covariance function in a local area, and an analytic reproducing kernel can be sought, which represent this function. Such a representation $cov_3(P,Q)$ will e.g. be like eq. (15), but with $J \approx 200$ when 1^o-mean gravity anomalies have been used, see Tscherning (1974, 1976). Using these reduced observations, an approximation $\tilde{T}_3$ can be constructed, so that

$$\tilde{T} = \tilde{T}_1+\tilde{T}_2+\tilde{T}_3\ . \tag{17}$$

The covariance function $cov_3(P,Q)$ will generally have a zero-correlation distance ψ_o which for gravity anomalies is not very big, e.g. 15' - 30'. This indicates, that $\tilde{T}_2$ may be constructed disregarding data in a distance ψ_o from the area of interest. This is the background for that the technique called stepwise collocation can be used as a technique for reducing the size of the normal equation matrix. Hence an approximation to T in an area can be constructed as several local approximations, partly using the same observations in "overlapping" areas, see section 6.

4.2. The combined integral formulae - collocation method

When using the integral formulae of Stokes and Vening-Meinesz (or the corresponding formulae developed by Molodensky), the difficult problems arize in the inner zones, near the point of evaluation, where the kernels have a singularity. This is why it as explained above, is necessary to predict gravity anomalies in a dense grid near the point of evaluation. Such a prediction generally involves predicting free-air gravity anomalies from free-air gravity anomalies.

Instead of simply adopting a gravity anomaly predicting procedure one may use collocation, but with data in a limited area in order to construct an approximation $\tilde{T}_2$. (A first order approximation $\tilde{T}_1$ using a set of known potential coefficients is generally used as reference field). In the zone outside the area an integration is then done of (mean) gravity values now referring to the field $\tilde{T}_1+\tilde{T}_2$. This function is called $\tilde{T}_3$, and we have $\tilde{T} = \tilde{T}_1+\tilde{T}_2+\tilde{T}_3$ as above in 4.1.
In the procedure considered originally by Lachapelle (1977, 1978) the data configuration was changed completely for each new point. If the shape of the area is bounded by lines of equal latitude and longitude, this should not be necessary. The approximation $\tilde{T}_2$ (which is based on the most detailled data), can then be valid for an area of size $1^\circ \times 1^\circ$, contingently surrounded by a border area of width 30', see Fig. 2.

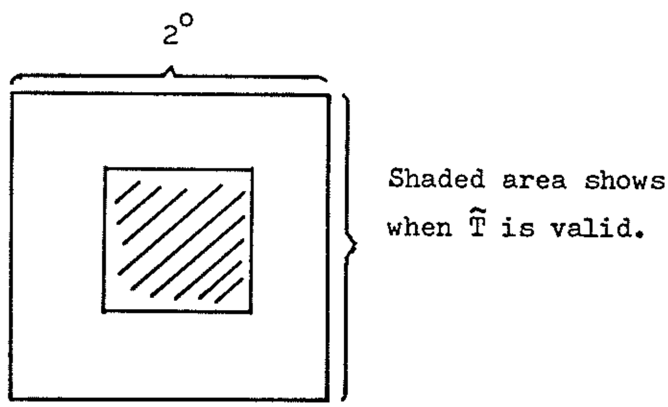

Fig. 2.

The drawback of this method is that total error estimates can not easily be computed using eq. (9). Error estimates may however be computed for both of the discussed methods by comparing predictions with observed values not taken into account when computing $\tilde{T}$.

5. The prediction of geoid undulations and related quantities

Using any of the two methods discussed in section 4, a local approximation to T will be given as a sum

$$\tilde{T} = T_M + \sum_{k=1}^{K} \tilde{T}_k^c . \tag{18}$$

We will in the following suppose that T_M and its related quantities are computed separately, i.e. we drop the superscript c in the following. This means, for example, that we from the potential coefficients $\bar{a}_{ij}$ and $\bar{b}_{ij}$ have subtracted the corresponding coefficients related to T_M. (These coefficients may be computed as described e.g. in Balmino et al. (1973)).

A height anomaly (ζ), a gravity anomaly (Δg) or a pair of deflection components are computed using the associated linear functionals. They are in spherical approximation as follows

$$L_\zeta(T) = T(P)/\gamma(P) \tag{19}$$

$$L_{\Delta g}(T) = -\left.\frac{\partial T}{\partial r}\right|_P - \frac{2}{r}\, T(P) \tag{20}$$

$$L_\xi(T) = -\frac{1}{r\gamma}\left.\frac{\partial T}{\partial \varphi}\right|_P \tag{21}$$

$$L_\eta(T) = -\frac{1}{r\gamma\cos\varphi}\left.\frac{\partial T}{\partial \lambda}\right|_P . \tag{22}$$

The geoid undulation (N) is then given by (19) evaluated at the (mean) Earth surface, i.e. r = R.

For an arbitrary linear functional L we have (with K = 3)

$$L(\tilde{T}) = L(\tilde{T}_1) + L(\tilde{T}_2) + L(\tilde{T}_3), \tag{23}$$

so we will write the different quantities in a similar way, e.g.

$$\tilde{N} = \tilde{N}_1 + \tilde{N}_2 + \tilde{N}_3.$$

Let now $\tilde{T}_1$ be given by eq. (13). For quantities at the surface of the (mean) Earth (where we may put $\gamma = GM/R^2$) we have the simple equations

$$\tilde{N}_1 = L_\zeta(\tilde{T}_1) = R \sum_{i=2}^{J} \sum_{j=o}^{i} \bar{P}_{ij}(\sin\varphi)[\bar{a}_{ij}\cos j\lambda + \bar{b}_{ij}\sin j\lambda)] \quad (24)$$

$$\tilde{\xi}_1 = L_\xi(\tilde{T}_1) = - \sum_{i=2}^{J} \sum_{j=o}^{i} \frac{\partial}{\partial\varphi} \bar{P}_{ij}(\sin\varphi)[\bar{a}_{ij}\cos j\lambda + \bar{b}_{ij}\sin j\lambda] \quad (25)$$

$$\tilde{\eta}_1 = L_\eta(\tilde{T}_1) = - \frac{1}{\cos\varphi} \sum_{i=2}^{J} \sum_{j=o}^{i} \bar{P}_{ij}(\sin\varphi)[\bar{b}_{ij}\cos j\lambda - \bar{a}_{ij}\sin j\lambda] \quad (26)$$

$$\tilde{\Delta g}_1 = L_{\Delta g}(\tilde{T}_1) = - \frac{GM}{R^2} \sum_{i=2}^{J} (i-1) \sum_{j=o}^{i} \bar{P}_{ij}(\sin\varphi)[\bar{a}_{ij}\cos j\lambda + \bar{b}_{ij}\sin j\lambda] \quad (27)$$

When collocation is used for the computation of $\tilde{T}_k$ we have

$$\tilde{T}_k(P) = \sum_{i=1}^{M_k} \mathrm{cov}_k(T(P),\ L_i(T))b_{ki} \quad (28)$$

where M_k is the number of observations used and

$$\{b_{ki}\} = \{\mathrm{cov}_k(L_i(T),L_j(T)) + D_{ij}\}^{-1}\{m_j - \sum_{n=1}^{k-1} L_j(\tilde{T}_n)\}. \quad (29)$$

Then we have for example

$$\tilde{\Delta g}_k = \sum_{i=1}^{M_k} \mathrm{cov}_k(L_{\Delta g}(T),L_i(T))b_{ki}\ . \quad (30)$$

Using the combined integral-formulae-collocation method we have

$$\tilde{T}_3(P) = \frac{R}{4\pi} \iint_{\sigma_3} (\Delta g - \tilde{\Delta g}_1 - \tilde{\Delta g}_2)S(\psi,r)d\sigma, \quad (31)$$

where $S(\psi,r)$ is Stokes function, and ψ the spherical distance.

At the surface of the (mean) Earth we then have

$$\tilde{N}_3 = \frac{R}{4\pi\gamma} \iint_{\sigma_3} (\Delta g - \tilde{\Delta g}_1 - \tilde{\Delta g}_2) S(\psi) d\sigma \qquad (32)$$

$$\left.\begin{matrix} \tilde{\xi}_3 \\ \tilde{\eta}_3 \end{matrix}\right\} = \frac{1}{4\pi\gamma} \iint_{\sigma_3} (\Delta g - \tilde{\Delta g}_1 - \tilde{\Delta g}_2) \left\{\begin{matrix} \frac{\partial S(\psi)}{\partial \varphi} \\ \frac{\partial S(\psi)}{\partial \lambda} \cdot \frac{1}{\cos\varphi} \end{matrix}\right\} d\sigma \, . \qquad (33)$$

The set of integration σ_3 is a set outside (or overlapping) the area with the most detailled data, e.g. the set outside the shaded area shown on Fig. 2. (See Lachapelle (1977) for computational details).

In all the equations we have used the assumption already stated in section 1, namely that T is regular at infinity. This is used in eq. (13), where the summation first starts with i=2. This means that we have presupposed, that the value of GM is very precisely known, and that the reference coordinate system has its origin in the Earth's center of mass. This is, however, not valid assumptions at the level of precision which we are discussing here ($\approx$ 25 cm). A correction to GM as well as to datum shift parameters may in principle be estimated using collocation see e.g. Tscherning (1976a). The values of GM is, however, of global character, so the estimation of this quantity may not give the correct result.

In Lachapelle (1977, 1978) Doppler derived geoid undulations (N_c) have been compared with undulations computed based on potential coefficients (eq.(5.7)). $N_o = N_c - N_1$ can in this case be interpreted as a correction to the semi-major axis of the reference ellipsoid which is closely connected to the value of GM. Depending on the set of potential coefficients used, the correction will vary $\pm$35 cm within an area like Canada (cf. Lachapelle (1978, table 3)). Much bigger variations are observed when different regions are compared.

A considerable improvement in the global geodetic constants and reference systems is therefore required in order to enable the computation of detailled geoids over large areas in the future.

The errors are, however, of a systematic character. This means that a relatively higher precision is obtained for geoid undulation or deflection differences in local areas.

Let us finally note, that all equations until now have been written down in spherical approximation, which is still sufficient, when we only want to compute geoid undulations with a precision of ± 25 cm. We may, however, without any problems work with more rigorous equations.

An important consequence is that we will have to give up the empirical covariance function, which will be substituted by some mathematically defined reproducing kernel as originally proposed by Krarup (1969). This should not in practice cause any difficulties.

6. The practical organization of the computations

Let us suppose that we have access to all available gravity observations, deflections of the vertical and geoid undulations through a data base. This base should also contain detailled height and density information, in order to permit the computation of T_M.

For both methods considered in section 4, the smallest reasonable unit of area is a 1^o-block surrounded by a $\frac{1}{2}^o$-border zone. In the corresponding 2^o-block will namely at least 400 observations be needed in order to obtain a $\pm 1''$ accuracy as discussed above.

A first step must therefore be to extract, spaced as regulary as possible, a set of 2^o-blocks of free-air gravity anomalies from the data base, contingently supplemented with deflections of the vertical or geoid undulations. (The whole set of 2^o-blocks must cover the area, i.e. in our case are about 700 blocks needed. The mean square variation of the data in each block must then be computed. If the variation is below ± 10 mgal, then no more data is needed. If it is above ± 10 mgal, then the effect of the known masses (T_M) must be subtracted, and a new variation must be computed. If this variation still is above ± 10 mgal a new data spacing must be determined using e.g. Tscherning (1975, Fig. 5e). (One may naturally end up in a situation,where the required precision can not be obtained).

The observations ($L_i(T)$ or $\tilde{L}_i(T^c)$) must then be used for the computation of a preliminary approximation to $\tilde{T}$. However, some observations must be left out, so that the precision of the predictions can be determined. If the precision is as required, then we are finished. Otherwise more data must be used, and the process repeated.

When using stepwise collocation, an intermediate approximation $\tilde{T}_1$ must be constructed. Note, that this approximation may be constructed, so that it is valid for several 1^{o}-areas.

Let us finally note, that the computational speed is of the same magnitude or smaller compared with the one reported for integral formulae methods. The speed of existing programs may however be improved following the proposals by H. Sünkel (1978) for rapid covariance computation.

7. Conclusions

Using collocation for the computation of geoid undulations and other gravity field dependent quantities we have several advantages compared with for example integral formulae techniques:

(1a) Not only one quantity is available, but any gravity field related quantity may be predicted.

(1b) The method permits that all types of data can be used, so the highest precision can be obtained.

(2) The approximation $\tilde{T}$ will represent the used data exactly, if their values have very small errors.

(3) The approximations can be used as the basis for the computation of more precise approximations, see Tscherning (1978).

(4) Error estimates can be computed.

(5) Observation errors can be taken into account.

(6) Parameters may be estimated (not discussed here, see Moritz (1972)).

(7) Computer programs exist, extensively tested and documented, see Tscherning (1974, 1978).

However, the precision needed in the future can only be obtained,if the basic geodetic constants (GM and semi-major axis) and the Earth's center of mass becomes more precisely known.

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