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THE USE OF DENSITY INFORMATION WHEN APPLYING COLLOCATION FOR THE PREDICTION OF DEFLECTIONS OF THE VERTICAL.

C.C. Tscherning, Geodætisk Institut,
Gamlehave Alle 22, 2920 Charlottenlund, Danmark.

Abstract. The method of collocation can be applied for the determination of approximations to the anomalous (gravitational) potential of the Earth and thereby for the prediction of for example deflections of the vertical.

Two alternatives for the use of density information when applying collocation are discussed. The one requires the adoption of a method for the unique assignment of a density value to an arbitrary harmonic function. The other is the traditional technique of removing and restoring the masses. It is argued, that the latter must be preferred, because it makes it possible to use the available density information without having to solve an additional number of equations arizing from the known density values.

Results from an application of the remove-restore technique in a mountanous area of New Mexico, U.S.A., are given. Here is observed a 40% decrease in the standard deviation of the difference between observed and predicted values as compaired to the standard deviation obtained when no density information was used.

Key words: collocation, mass density, deflections of the vertical.

1. Introduction

The anomalous potential of the Earth, T, is equal to the difference between the gravitational potential of the Earth, W, and a

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normal or reference potential, U, i.e. T = W-U.

We will in the following suppose that T is a harmonic function outside the surface of the Earth, and that it fulfils certain regularity conditions at infinity. This means that we in U have included the rotational part of W, the potential of the Atmosphere, the Moon etc. and that there are no zero and first order harmonic coefficients in the expansion of T as a series in solid spherical harmonics.

We may consider U as being generated by a density reference model (see e.g. Moritz (1973)). Then T can be considered as being generated by an anomalous density distribution, ρ, i.e.

$$T(P) = k \int_{\text{Earth}} \rho(Q) / \|P-Q\| \, dR^3, \tag{1}$$

where P and Q are points in R^3, $\|P-Q\|$ is the distance between the points and k is the gravitional constant. The total mass of ρ is zero, i.e. it takes on positive as well as negative values. The equation (1) is the basis for using gravity field information in geophysical investigations, but it is naturally also the basis for using density information when computing approximations to T or to quantities dependent on T such as deflections of the vertical.

We will in this paper only consider one method for the determination of approximations to T, namely the method of least squares or least norm collocation. The use of density information in connection with this method, however, illustrates well how density infortion can be used in connection with other techniques. In section 2 we will give a brief introduction to collocation and describe two possible methods for the use of density information. In section 3 we describe the results of a test computation, where deflections of the vertical were predicted in a mountaineous area of New Mexico, U.S.A. Finally in section 4 we discuss the requirements on density information which are needed in order to push forward further applications and developments.

2. The method of collocation and the use of density information.

Let us regard a certain function (in our case T) for which we have obtained observations x_i, $i = 1, .., n$, We will suppose that these observations are errorfree *) and that they can be related to the function through a set of linear functionals, L_i, $i = 1, ..., n$, i.e. $L_i\ (T) = x_i$. The free air gravity anomalies, the geoid undulations and the deflections of the vertical, for example, are related to T through such linear(ized) functionals.

Collocation is then a method for the determination of an approximation to the function, so that it agrees with the values of the observations. Denoting the approximation to the anomalous potential by $\tilde{T}$ we must have

$$L_i(\tilde{T}) = L_i(T) = x_i,\ i = 1, ..., n. \qquad (2)$$

Let us require the approximation to be an element of a linear vector space of functions, which we here will require to be harmonic outside the surface of the Earth or another suitable surface. We will also require this vector space to have an inner product (and thereby a norm) so that the space is a so-called reproducing kernel Hilbert space. We will denote this space H, and we will suppose that the linear functionals occurring in (2) are elements of the space dual to H, and that they are linearely independent.

The reproducing kernel K(P,Q) is a function of two variables, which here will be points in the area of harmonity. For either P or Q fixed will the reproducing kernel be an element of the space, i.e. here a harmonic function. This is also the case if we apply a linear functional on K(P,Q), i.e. $L_i K(\cdot, Q)$ is an element of H,(where the dot indicates that L_i is applied on K(P,Q) with respect to the first variable).

In case H had been of finite dimension, n, the approximation could have been obtained by determining a set of cofficients a_i, so that

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*) This does not mean that errors can not be taken into account, see Moritz (1972).

$$\tilde{T}(P) = \sum_{i=1}^{n} a_i h_i(P), \text{ and } a_i \text{ is given by}$$

$$L_i(\tilde{T}) = L_j\left(\sum_{i=1}^{n} a_i h_i\right) = \sum_{i=1}^{n} a_i L_j(h_i) = x_j\ ,\ j=1,\ldots,n. \quad (3)$$

h_i, i=1,...,n is basis for H.

Now, it is well known, that such approximations may be very "unstable", and show unwanted oscillation in between the data points. It is generally better to choose H to be of very high dimension (in our case of infinite, but countable dimension) and then require T to be smooth or equivalently to have the least possible norm. This is the method of least norm or least squares collocation, (see Tscherning (1978) for more details). It can then be shown that $\tilde{T}$ will be an element of a n - dimensional subspace of H spanned by the functions $L_i(K(\cdot,Q))$, so we get from eq. (3)

$$L_j(\tilde{T}) = L_j\left(\sum_{i=1}^{n} a_i L_i K(\cdot,Q)\right) = x_j \qquad \text{or}$$

$$\sum_{i=1}^{n} a_i L_i L_j K(\cdot,\cdot) = x_j\ , \quad (4)$$

which is a system of n equations with n unknowns.

If the reproducing kernel is chosen to be equal to the so-called empirical covariance function of the anomalous potential,

$$\mathrm{cov}(T(P),\ T(Q)) = K(P,Q),$$

then we have the equations

$$\tilde{T}(Q) = \sum_{i=1}^{n} a_i \mathrm{cov}(T(Q),\ L_i(T)) \quad (5)$$

with

$$\left\{\mathrm{cov}(L_i(T),\ L_j\ (T))\right\}_{n\mathrm{x}n} \left\{a_j\right\}_n = \left\{x_i\right\}_n \tag{6}$$

The North-South and the prime vertical components of the deflection of the vertical (ξ, η) are related to T through linearized functionals

$$\begin{aligned} \xi_P &= -\frac{1}{r\gamma}\ \frac{\partial T}{\partial \varphi}\Big|_P\ , \\ \eta_P &= -\frac{1}{r\gamma\cos\varphi}\ \frac{\partial T}{\partial \lambda}\Big|_P\ , \end{aligned} \tag{7}$$

where the subscript P indicates that they refer to a specific point P. (φ,λ,r) are the spherical coordinates of P (latitude, longitude, distance from the origin) and γ is the reference gravity. Predictions are then obtained by applying the functionals on $\tilde{T}$, i.e.

$$\tilde{\xi}_P = -\frac{1}{r\gamma}\ \frac{\partial\tilde{T}}{\partial\varphi}\ ,\ \tilde{\eta}_P = -\frac{1}{r\gamma\cos\varphi}\ \frac{\partial\tilde{T}}{\partial\lambda}\ , \tag{8}$$

or e.g.

$$\tilde{\xi}_P = \sum_{i=1}^{n} a_i \mathrm{cov}(L_i(T),\ \xi_P)\ , \tag{9}$$

when expressed with covariance functions.

In order to use density information for the determination of $\tilde{T}$ there are then two alternatives. One alternative is to regard the density values as beeing equal to a linear functional applied on T. The linear functional is given by the wellknown Poisson-equation

$$\rho(Q) = \frac{-1}{4\pi k}\left(\frac{\partial^2 T}{\partial x^2}\Big|_Q + \frac{\partial^2 T}{\partial y^2}\Big|_Q + \frac{\partial^2 T}{\partial z^2}\Big|_Q\right)\ , \tag{10}$$

where T now is regarded as a function in all of R^3.

The problem is now to choose an appropriate reproducing kernel Hilbert-space of functions defined in R^3. This can be done as described in Tscherning (1973, 1977) by requiring ρ, the density distribution, to fulfil a simple mathematical condition. Seen from a computational standpoint, the problem is to compute the quantities $L_i L_j K(\cdot,\cdot)$ or the corresponding cross covariances between density values and others quantities. Such covariance functions are discussed in Tscherning (1977) and Jordan (1978). (In case the density variations are considered to be sufficiently well represented by the variations of the topographic heights, then the auto-and cross covariances between theese heights and gravity field dependent quantities are needed, see e.g. Heiskanen and Moritz (1967, Chp. 7)).

Now, this alternative is unsatisfactory seen from a computational standpoint, because the system of equations to be solved will be very big. It is also, seen from a theoretical standpoint, somewhat unsatisfactory, because we will have to introduce restrictions on the function ρ, which are not necessarily justified geophysically.

The other alternative is closely related to the traditional techniques of removing and restoring the masses, which are applied when solving the geodetic boundary value problem, see e.g. Heiskanen and Moritz (1967, Chp. 8). Instead of determining an approximation to T, an approximation $T^c = T - T_M$ is determined. T_M is a harmonic function computed from a model of the known densities. T^c will be a smooth function (compaired to T) if the model represents the topography well locally, see e.g. Tscherning and Forsberg (1978, Table 1).

The smoothness implies that fewer values are needed to represent the function T^c than to represent T.
Or, $T = T^c + T_M$ is easier approximated using the density information than without using it.

The observations used for constructing $\tilde{T}^c$ must naturally be the original observations minus the effect of the densities, i.e.

$$x_i^c = x_i - x_{iM} , \qquad (11)$$

where $L_i T_M = x_{iM}$. Similary, the predicted quantities will be equal to

$$L\,(\tilde{T}) = L_c(\tilde{T}^c)+L(T_M) \qquad (12)$$

This simple method do not have the drawbacks of the first alternative and has therefore been used in the test described in the following section.

3. Test of prediction of deflections of the vertical.

From eq. (7) and (12) we see that deflections are predicted as

$$\tilde{\xi}_P = -\frac{1}{\gamma r}\left[\frac{\partial\tilde{T}^c}{\partial\varphi}\Big|_P + \frac{\partial T_M}{\partial\varphi}\Big|_P\right] = -\frac{1}{r\gamma}\frac{\partial\tilde{T}^c}{\partial\varphi}\Big|_P + \xi_M \qquad (13)$$

$$\eta_P = -\frac{1}{\gamma r\cos\varphi}\left[\frac{\partial\tilde{T}^c}{\partial\lambda}\Big|_P + \frac{\partial T_M}{\partial\lambda}\Big|_P\right] = -\frac{1}{\gamma r\cos\varphi}\frac{\partial\tilde{T}^c}{\partial\lambda} + \eta_M\,,$$

where ξ_M and η_M are the part of the deflections which originate from the model of the densities.

The technique of removing and restoring the masses has been tested in a 30' x 30' area in New Mexico, U.S.A., with strongly varying deflections due to a mountain chain running North-South through the area. The full details describing the best can be found in Tscherning and Forsberg (1978), and we shall only repeat the most essential here. Two approximations to T was computed using the method of stepwise least squares collocation. As data was used a set of potential cofficients, 1°equal area mean free-air gravity anomalies, point free-air gravity anomalies (less than 100 values) and density values in the form of topographic heights, (spaced 30" apart in the 30' x 30' area). The terrain was considered to be isostatically compensated at a depth of 30 km.. One of the approximations was computed without using the density information. The two approximations were then used for the prediction of 69 pairs of deflections of the vertical in the area. The standard-deviations of the differenc between observed and predicted values are shown in the table.

Table of standard deviations of observed and predicted deflections of the vertical.

	ξ	η
observed	± 1".84	5".42
predicted <u>not</u> using density info	1".35	1".49
predicted using density information	0".71	1".01

It is obvious, that the use of density information gives a substantial improvement in the results. (The results do, however, contain a considerable bias varying with the different used depths of isostatic compensation, see Tscherning and Forsberg (1978) for details).

4. Future requirements.

In the above described test we used density information in a very unsofisticated way, namely in the form of topographic heights, compensated isostatically. In case geological or geophysical (seismic) information had been available, this information could have be used easily. The problem is to have such data made available for geodetic purposes in a computer-readable form.

Pioneering work of collecting results of seismic refraction measurements is published by Lee and Tayler (1966). But much more detailled information is needed especially in areas where the geological features causing the gravity variations are hidden under sediments, see e.g. Wassouf (1975). Here a preliminary analysis of geodetic and seismic data could be used in order to determine the location of the different density variations.

Let us finally touch a small, but important problem. What should be used as a density reference model generaling the normal or referencepotential, U. Models developed by Moritz (1973) can be used, but can we for example use the reference ellipsoid as an external boundary for the reference densities or should the masses be included in a ellipsoid with a semimajor axis about 1-2 km shorter as indicated by Romanowicz and Lambeck (1977).

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