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PREDICTION OF DEFLECTIONS OF THE VERTICAL

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Abstract: The method of collocation may be used for the computation of an approximation $\tilde{T}$ to the anomalous potential of the Earth, T, from, e.g., potential coefficients, gravity anomalies, geoid undulations, and deflections of the vertical. Predictions of deflections of the vertical (ξ, η) are then obtained (in spherical approximation) as

$$\xi = -\frac{\partial \tilde{T}}{\partial \phi}\frac{1}{\gamma r}, \quad \eta = -\frac{\partial \tilde{T}}{\partial \lambda}\frac{1}{\gamma r \cos\phi},$$

where (φ, λ, r) are the spherical coordinates of and γ is the reference gravity at the point of prediction.

Excellent prediction results have been obtained in areas with a smoothly varying topography. In areas with strongly varying topography it is necessary to subtract the effect of the isostatically compensated terrain in order to get acceptable results. (Otherwise a very dense gravity network must be established.)

The method of collocation requires that the observed quantities can be regarded as values of linear functionals applied on T. It is therefore necessary to use a rigorously computed terrain or anomalous mass potential (T_m) and then determine an approximation $\tilde{T}^c = \tilde{T} - T_m$ using collocation. Deflections of the vertical can then be predicted using (*) with $\tilde{T}^c + \tilde{T}_m$ substituted for T.

This technique has been tested in a 30′ by 30′ area in New Mexico with a locally strongly varying topography, where defelections were predicted from gravity anomalies (spaced 6′ apart). The use of topographical information resulted in a 40-percent decrease in the root mean square error of prediction (to ±0.″7 for ξ and ±1.″0 for η).

This result (and results obtained by other investigators) shows that the use of the method of collocation for the prediction of deflections of the vertical has the potential of giving results comparable to results obtained using the integral formula of Vening-Meinesz or Molodensky, but with less observed gravity data.

Introduction

Let us as usual denote the gravitational potential of the Earth by W and let a certain first-order approximation be denoted U. Then we denote the anomalies or disturbing potential T, where

$$T = W - U \tag{1.1}$$

We will suppose that T is a harmonic function in space, and that the zero and first-order harmonic coefficients are zero. In a point P in space the deflection of the vertical is the direction difference between the gradient of W and the gradient of U. In so-called spherical approximation we have for the north-south component of the deflection of the vertical

$$\xi = -\frac{\partial T}{\partial \phi}\bigg|_P \cdot \frac{1}{r \cdot \gamma} \tag{1.2}$$

and for the prime vertical component

$$\eta = -\frac{\partial T}{\partial \lambda}\bigg|_P \cdot \frac{1}{r \cdot \gamma \cdot \cos \phi}, \tag{1.3}$$

where $\gamma = |\nabla U|$ is the reference gravity in P, φ is the latitude, λ is the longitude, and r is the distance from the origin.

Deflections of the vertical are applied for the reduction of horizontal or vertical angles (Heiskanen and Moritz 1967, section 5–5), for the correction of inertial navigation or inertial surveying errors (Levine and Gelb 1969; Bernstein and Hess 1976), for the determination of T (to which they are related through equation (1.2) and (1.3)) and thereby for geoid determination. They also contain information about density anomalies within the Earth.

On the other hand, the density anomalies determine the disturbing potential, which again determine the deflections of the vertical. So we should not in practice distinguish between the determination of the deflections, of T or of the density anomalies.

The deflection of the vertical may be determined through astronomical observations or in the future possibly through inertial techniques. (See, e.g., Heller 1977; Lyon 1977.) Astronomical observations are time consuming (and the weather may cause trouble). In polar regions they are nearly impossible to carry out. Inertial technique is not fully developed and will anyway require reference points with known deflections. Methods which enable the determination of deflections from other data types as, e.g., gravity or density anomalies, are therefore extremely important.

In principle two techniques are available. One technique uses the integral formulae relating the disturbing potential to gravity anomalies, geoid heights, or density anomalies. The deflection components are then obtained using equations (1.2) and (1.3). The integral formulae of Vening-Meinesz (Stokes), Molodensky, and Poisson are examples of this technique. It is characteristic for this technique that only one type of quantity will appear under the integral sign.

The other technique may use a possibly heterogeneous set of observations for the determination of an approximation to the disturbing potential or to a scalar or vector valued function representing one or both components of the deflection of the vertical or the geoid undulations. These functions are then regarded as mappings from the reference ellipsoid to a (generally 1- or 2-dimensional) real vector space. The approximation will be determined as the element of a specific linear vector space of functions which represent the observations in an "optimal" manner. An example of such a technique is the method of least-squares collocation (Krarup 1969), which enables error-free observations to be reproduced exactly, and methods which determine a set of point masses or surface densities generating the approximation to T.

Predictions or *interpolated values* are then obtained using equations (1.2), (1.3) when an approximation $\tilde{T}$ to T has been determined or by a direct evaluation of the function representing the deflection components.

The techniques are also used combined. Gravity anomalies, for example, may be predicted so that a sufficiently dense set of gravity anomalies are available before carrying out the integration. Or an integration is first performed using incomplete data (i.e., missing data are put equal to zero, or all mass densities are put equal to a constant value). The (generally very smooth) differences between observed deflections or gravity anomalies and values computed using the integral formulae are then used as "reduced" observations in a prediction procedure.

Examples of such techniques are the method of astrogravimetric levelling, the method of interpolating topographic-isostatic reduced deflections of the vertical (Elmiger 1969, Lachapelle 1976) and the combined integral formula-collocation method (Lachapelle 1976).

Results obtained using different techniques are collected in table 1. It is seen that both techniques give excellent results in areas with a smoothly varying topography (Rice 1952, Tscherning 1973, 1975). In areas with a strongly varying topography are fine results reported using the integral formula procedure when gravity observations and terrain data are available in a dense grid near the point of computation (Pick and Pola 1967; Tengström 1971). The combined integral formula and prediction/interpolation techniques seems to give comparable results using a smaller number of observations. However, the observations were deflections of the vertical or their horizontal derivatives (Badekas and Mueller 1968; Biro 1967; Merry and Vanicek 1974; Bosch and Wolf 1974; Grafarend and Offermanns 1976), i.e., observations of a type, which we here consider as not being available.

A combined technique, using topographic heights and known densities together with the more easily obtainable gravity anomalies and the probably scattered deflections and geoid undulations determined from Doppler-derived geocentric coordinates, would be a more satisfactory technique. Obviously, the collocation method could be used. However, certain conditions have to be fulfilled before this method can be used. This is what we will discuss in section 2.

In section 3 we describe how the collocation method has been used for deflection prediction with topographic-isostatic reduced data in an area of New Mexico. And finally in section 4 we compare the technique with other techniques and discuss possible applications in northern Greenland.

TABLE 1.—*Some results obtained using different techniques for the computation of deflections of the vertical.*

Data type		Area	Spacing data (a)	(b)	Source		Method[a]	r.m.s.v.[b] ξ_{obs}	ξ_{dif}	η_{obs}	η_{dif}
Terrain	(a)	Switzerland			Bosch/Wolf	(1974)	Int.	±8″9	±5″6	±3″3	±1″4
		Switzerland			Elmiger	(1969)	Int.	16.4	4.3	9.2	3.3
		BRD			Heitz	(1969)	Int.	4.7	3.7	3.3	2.8
		New Mexico	30″		This paper		Int.	1.8	1.3	5.4	2.2
Gravity	(a)	Central U.S.A.			Rice	(1952)	V.M.	3.0	0.5	1.6	0.7
		Canada	8′		Lachapelle	(1977)	Col.	5.1	1.4	7.8	1.6
		Ohio	15′		Tscherning	(1973)	Col.	1.5	1.1	5.0	1.3
		Denmark	15′		Tscherning	(1975)	Col.	2.3	1.4	1.4	0.8
		New Mexico	6′		This paper		Col.	1.8	1.3	5.4	1.6
Gravity	(a)	Canada			Merry/Vaníček	(1974)	V.M.+I.	5.0	2.1	5.0	1.7
Deflections	(b)	Canada	12′	1°	Lachapelle	(1976)	Col.+V.M.	2.3	1.0	3.7	1.0
		Denmark	15′	1°	Tscherning	(1975)	Col.	2.3	0.7	1.4	0.8
		New Mexico	6′	30′	This paper		Col.	1.8	1.2	5.4	1.5
Terrain	(a)	Seamount			Fischer	(1975)	Int.+A.	ca. 20.0	1.2	ca. 15.0	1.2
		Switzerland			Elmiger	(1969)	Int.+I.	16.4	1.4	9.2	1.4
Deflections 31	(b)	BRD		15′	Grafarend/ Offermanns	(1975)	K.-W.	4.7	0.3	3.3	0.3
		Switzerland		2′5	Bosch/Wolf	(1974)	Pred.	8.9	0.6	3.3	0.6
Terrain	(a)	Central U.S.A.		1′	Szabo	(1962)	V.-M.	3.0	0.7	1.6	0.7
Gravity	(b)	ČSR		2″	Pick/Pola	(1967)	Mol.	?	0.7	?	0.7
		New Mexico	30″	6′	This paper		Col.	1.8	0.7	5.4	1.0
Deflections Gravity gradients terrain	(a) (b)	Ohio	(30′)	30″	Badekas/Mueller	(1968)	I.	1.5	0.4	5.0	0.4

[a] Method legend: Int. = Integration of masses, V.M. = Vening-Meinesz Integral, Col. = collocation, I. = Polynomial Interpolation, K.-W. = Kolmogorow-Wiener Prediction, Pred. = Linear estimation or prediction, Mol. = Molodensky's Integral, A. = Adjustment with parameters.

[b] ξ_{obs}, η_{obs}: observed values; ξ_{dif}, η_{dif}: difference between observed and computed deflections of the vertical. Mean values were subtracted in all cases.

Collocation and the Use of Terrain Data

To understand better how terrain data can be used in connection with collocation let us for a moment recall how this kind of data is taken into account by the methods for the solution of the boundary value problem. Using Stokes integral formula the gravity anomalies must be "reduced" to the geoid in such a manner that all masses outside the geoid are removed or "shifted." This presupposes a complete knowledge of the density distribution between the Earth's surface and the geoid.

Using the modern theory of Molodensky, as modified by Pellinen (1962, 1968), not T itself, but

$$T^c = T - T_m \tag{2.1}$$

is determined. Here T_m is a *harmonic* function generated by a model of the known terrain and its isostatic compensation. This has no influence on the area of harmonicity. However, experience has shown that T^c will be a more smooth quantity than T, when T_m is generated by a model of the local terrain (table 1).

When applying collocation the terrain may be taken into account in exactly the same manner. Let us then recall the main principles of the method of collocation.

Because the Laplace operator is a *linear* operator, then T will be an element of a linear vector space of functions harmonic in the set of points outside the surface of the Earth, Ω. Subsets of such a space may be equipped with an inner product, and may become a so-called reproducing kernel Hilbert space. The reproducing kernel is a mapping

$$K: \Omega \times \Omega \rightarrow R$$

with values $K(P, Q)$, $P, Q \in \Omega$. K will for either P or Q fixed be a harmonic function. In such a Hilbert space we may find suitable approximations $\tilde{T}$ to T. When we have given observation (x_i), which are related to T through linear functionals (L_i) then we may require that $\tilde{T}$ agrees with these values, i.e.,

$$L_i(\tilde{T}) = L_i(T) = x_i, \qquad i = 1, \ldots, n \tag{2.2}$$

The only condition is that the linear functionals are elements of the space dual to the reproducing kernel Hilbert space (or that the linear functionals applied two times on $K(P, Q)$ with respect to both variables, $L_i(L_i K(P, Q))$, is a finite number).

It is in principle not necessary that T itself is an element of the Hilbert space. This means that the area of harmonicity, for example, may have a sphere as boundary. We will in the following presuppose that we use such a *Bjerhammar-sphere*, which will have a radius $R(< 6{,}371$ km).

Many approximations will exist which fulfill the collocation condition, equation (2.2). We may select one by requiring it to have the least possible norm. (This will generally correspond to a condition of maximal smoothness.)

This approximation $\tilde{T}$ is then

$$\tilde{T}(P) = \sum_{i=1}^{n} a_i L_i(K(\cdot, P)) \tag{2.3}$$

with the constants a_i determined as the solution to the normal equations

$$\{ L_i L_j K(\cdot,\cdot) \} \{ a_i \} = \{ x_j \} \tag{2.4}$$

Note, that the condition that the functionals L_i are elements of the dual to the reproducing kernel Hilbert space assures that the diagonal of the normal equation coefficient matrix consists of finite numbers. The condition for the solvability of the normal equations is that the linear functionals are linearly independent regarded as elements of the dual space.

Exactly the same mathematical model can be used for the approximation of T^c as long as we know that T_m is a harmonic function. It is not even required that T_m is a specially good representation of the potential of the terrain and density anomalies. Putting

$$\tilde{T}^c = \tilde{T} - T_m, \tag{2.5}$$

we have

$$\tilde{T}^c = \sum_{i=1}^{n} a_i L_i (K(\cdot, P)) \tag{2.6}$$

$$\{L_i L_j K(\cdot,\cdot)\} \{a_i\} = \{x_j - L_j(T_m)\} = \{x_j^c\} \cdot \tag{2.7}$$

For "observed" deflections (ξ, η), height anomalies (ζ) or free-air gravity anomalies (Δg) (all referring to a geocentric reference system) we have

$$\xi^c = -\frac{\partial T^c}{\partial \phi}\frac{1}{r\gamma} = \xi + \frac{\partial T_m}{\partial \phi}\frac{1}{r\gamma} \tag{2.8}$$

$$\eta^c = -\frac{\partial T^c}{\partial \lambda}\frac{1}{r\gamma \cos\phi} = \eta + \frac{\partial T_m}{\partial \lambda}\frac{1}{r\gamma \cos\phi} \tag{2.9}$$

$$\zeta^c = \frac{T^c}{\gamma} = \zeta - \frac{T_m}{\gamma} \tag{2.10}$$

$$\Delta g^c = -\frac{\partial T^c}{\partial r} - \frac{2}{r} T^c = \Delta g + \frac{\partial T_m}{\partial r} + \frac{2}{r} T_m \tag{2.11}$$

For potential coefficients C_{ij}^c, S_{ij}^c, we have similar equations. However, the contribution from T_m cannot easily be computed.

The choice of an appropriate reproducing kernel Hilbert space (and thereby of $K(P, Q)$) is not without both theoretical and practical difficulties (Tscherning 1976, 1977). In practice reproducing kernels which approximate the so-called empirical covariance functions has proved very useful (Tscherning and Rapp 1974; Tscherning 1975a).

In case T had been an element of the Hilbert space, maximal errorbounds could have been computed for the predicted quantities knowing the norm of T. Now, as this will generally not be the case, we will have to look for other possibilities for estimating the error of prediction. We can here use the connection between the method of collocation (as carried out in a reproducing kernel Hilbert space) and the method of least-squares prediction or collocation as developed by H. Moritz (1972). Here the reproducing kernel is equivalent to the covariance function of the anomalous potential, and everywhere we above have written $L_i\, L_j\, K(\cdot, \cdot)$ we must write cov $(L_i\,(T), L_j\,(T))$, (i.e., cov $(\Delta g_P, \Delta g_Q)$ if $L_i\,(T) = \Delta g_P$ and $L_j\,(T) = \Delta g_Q$). So if we use reproducing kernels which approximate (empirical) covariance functions, results from the theory of least-squares collocation can be used.

Within this theory the mean square error of prediction $\sigma^2(L)$ for a quantity $L(T)$ will be

$$\sigma^2\,(L) = LLK(\cdot\,,\,\cdot) - \{LL_iK(\cdot\,,\,\cdot)\}^T\,\{L_iL_jK(\cdot\,,\cdot)\}^{-1}\,\{LL_jK(\cdot\,,\,\cdot)\} \quad (2.12)$$

This equation gives reasonable error estimates (Tscherning 1975a) and is useful, for example, when evaluating the quality of the predictions which may be determined from a given set of observations.

The main term in equation (2.12) is $LLK(\cdot\,,\,\cdot)$, which can be interpreted as the mean square variation of the quantity $L(T)$ (e.g., 1,800 mgal² for a free-air gravity anomaly at the Earth's surface).

Alone from this error equation, it is obvious that it is necessary to remove as much as possible of the gravity field variations before starting to compute predictions. The local variations we hope to reduce by subtracting the effect of T_m. The removal of global or regional variations is called removal of trend in the statistical theory of linear prediction.

This trend removal can be made a mathematically rigorous process within the framework of collocation, and we get the method of *stepwise* collocation (Tscherning 1973 and 1974). The use of the terrain potential T_m will make no difference here. We just have to use in each step the reduced observations

$$x_i^c = x_i - L_i\,(T_m) = x_i - x_{im}.$$

Until now we have regarded T_m as a known quantity. However, at some point we have to choose a (realistic) model of the terrain and its compensation. Different types of problems will occur depending on the chosen model.

Generally the Airy-Heiskanen isostatic model is excellent. The so-called indirect effect is relatively small (which means that ζ^c is small) and the produced field very smooth (the global variation of the isostatic gravity anomaly Δg^c is only of the order of ± 20 mgal). Furthermore, the model is physically realistic, easily allowing modelling of known density anomalies in the crust (e.g., sedimentary basins).

To assure the harmonicity of T_m, it is necessary that exactly the same masses are used in each computation of L_i (T_m). This is best done by taking into account the whole terrain of the Earth. Much data exist which have been reduced this way, and expansions in spherical harmonics of T_m have been published (Lachapelle 1975a).

When working in a small area it is not necessary to remove the terrain of the whole Earth. Just the local terrain may be removed. In a small area (extend less than 2°) we should be able to ignore the effect on C_{ij}, S_{ij} for $i < 30$ (however, this is a point which must be further investigated). In order to ensure the harmonicity of T_m, the same piece of terrain must be used in each evaluation. Hence, e.g., the local isostatic anomalies only calculated out to the Hayford zone 0_2 (166.7 km) cannot be used, because the terrain taken into account is different for each point considered.

The actual method of calculation of T_m, Δg_m, etc. are well known and well described in the literature. The classical method is to subdivide the area around the computation point in "cylinder"-compartments, estimate mean heights of each compartment, and sum up the contribution from each sector. An increasingly used and more computer-oriented method represents the terrain as arrays of rectangular boxes of varying size. Normally each quantity (e.g., Δg_m) has to be calculated directly. But in small areas, where plane approximation can be used, many different quantities may easily be derived from one calculation of, e.g., T_m, using the fast Fourier transform. Unfortunately time has not permitted us to make investigation into this last subject.

Let us finally note that attempts have been made to construct reproducing kernel Hilbert spaces in which the terrain heights or density anomalies could be considered as observed values, i.e., represented by linear functionals, which were elements of the dual space. (Grafarend 1970; Tscherning 1974a, 1976a; Jordan 1977). The drawback of regarding, e.g., the heights as observed quantities is that the dimension of the normal equations will be very big.

Test of Collocation in a Small Area in New Mexico

Results of tests, where the method of collocation has been used for the prediction of deflections of the vertical from *heterogeneous* data have been published in Tscherning (1973, 1975) and Lachapelle (1975, 1976, 1977). These tests have shown that satisfactory results (prediction error $< \pm 1''$) can be obtained in areas with a smoothly varying topography.

The root-mean square variation (r.m.s.v.) of topographic-isostatically reduced deflections of the vertical (Reinhart 1968; Elmiger 1969; Bosch and Wolf 1974, see table 1), are of the same magnitude as the r.m.s.v. observed in areas with a smooth topography. So we should in theory be able to predict deflections of the vertical in mountainous areas having the same magnitude of the prediction error as occurring in areas with a smooth topography using the same spacing of the observation data.

We naturally wanted to see this confirmed through a computation test with "real" data from an area with a strongly varying topography. Such a data set was kindly provided to us by R. Fury, National Geodetic Survey (NGS).

The data provided consisted of:

(A) 68 pairs of astro-geodetic deflections referring to NAD 1927 all from a 30′-square with south-west corner having longitude $\lambda = -106°30'$, latitude $\phi = 32°$. These deflections are a part of a dataset used in Morrison (1977).

(B) Topographic heights from the same 30′-square spaced 30″ apart ($\approx$1 km).

(C) About 200 point gravity values from the 1°-square with south-west corner having $\phi = 32°$ and $\lambda = -107°$.

A set of datum shift parameters for the area (NAD1927–NWL9D) was also kindly provided by NGS. They were used for the transformation of the astrogeodetic deflections to a geocentric reference system.

Furthermore R. H. Rapp, Ohio State University, provided us with a set of 1° mean free air gravity anomalies and 1° mean topographic heights.

As seen from the sketch map (fig. 1) the 1° × 1°-square is quite flat, except for the Organ/San Andres mountains which go approximately north-south in the middle of the square, and raise 700 to 1,500 m from the surrounding plateau. Supplementary heights were read from an old 1:500,000 aeronautic map, which was the only one we could get owing to lack of time. Consequently rather large errors, especially in Δg_m are to be expected compared to the "true" values, but this only affects the method in such a way that we must expect our model T^c to be less "smooth" than the "actual" T^c.

We first used the method of stepwise collocation with unreduced data. The approximation $\tilde{T}$ was in this case determined as the sum of three harmonic functions,

$$\tilde{T} = \tilde{T}_0 + \tilde{T}_1 + \tilde{T}_2 \tag{3.1}$$

$\tilde{T}$ is simply given through a set of potential coefficients. We chose here the coefficient set known as Goddard Earth Model 8 (GEM8) (Wagner et al. 1976). $\tilde{T}_1$ was then constructed using equations (2.3) and (2.4) with a set of 1° mean free air gravity anomalies from which the contribution of GEM8 had been subtracted. The gravity anomalies covered an 11°-square, with the above-mentioned 1°-square in in the middle. As a reproducing kernel we used the (covariance) function recommended in Tscherning and Rapp (1974), but with the first 20-degree-variances put equal to zero,

$$K_1(P, Q) = \sum_{i=21}^{\infty} \frac{a_1 R_1^2}{(i-1)(i-2)(i+24)} \left(\frac{R_1^2}{rr'}\right)^{i+1} P_i(\cos\psi), \tag{3.2}$$

where R_1 is the radius of the Bjerhammar-sphere (= 6,369.88 km), a_1 = 425.28 mgal², r, r' the distances of P, Q, respectively, from the origin, ψ the spherical distance between P and Q, and P_i the i'th Legendre polynomial. (See (table 2) for values of some of the covariance functions which may be derived from this reproducing kernel.)

In the following step, $\tilde{T}_2$ was determined using two slightly different datasets:

(a) 98 point gravity values all within the above mentioned 1°-square spaced approximately 6′ apart,

(b) the same 98 gravity values and four pairs of deflection components situated as near as possible to the corners of the 30′ square.

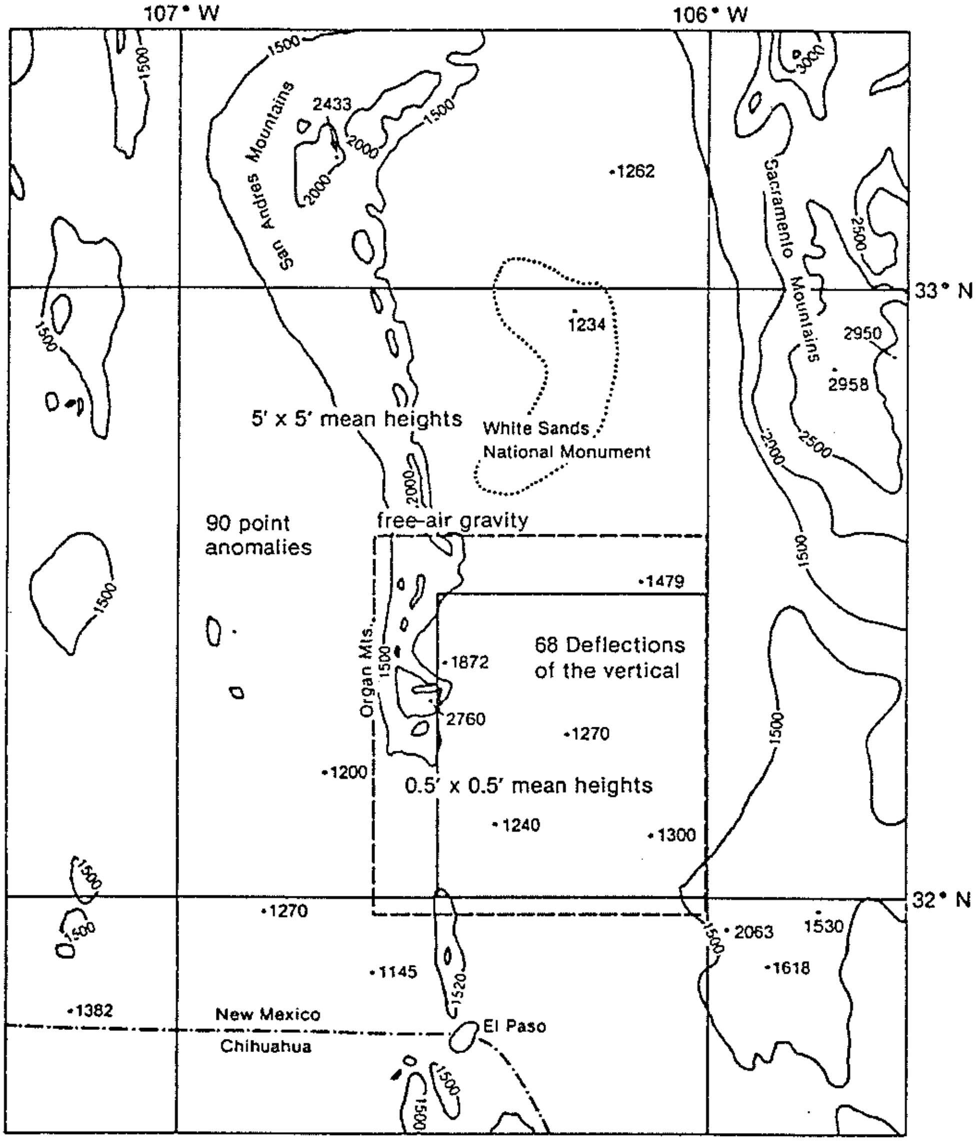

FIGURE 1.—White Sands missile range test area. Scale 1:1,000,000, contour interval 500 m. Data used in the collocation are indicated.

TABLE 2.—*Mean values and root mean square variations of observed and topographic/isostatic reduced gravity anomalies and deflections of the vertical.*

	Δg (mgal)		ξ (arcsec)		η (arcsec)	
	mean	r.m.s.v.	mean	r.m.s.v.	mean	r.m.s.v.
Observed	−16.7	19.4	−1.58	1.84	−3.05	5.42
Compensation depth 30 km	29.9	10.6	0.96	1.49	3.84	2.78
Compensation depth 45 km	40.2	10.4	0.62	1.50	3.49	2.71
Topography alone	161.9	10.8	−1.23	2.07	1.93	3.33

Using the 98 gravity anomalies from which the effect of $\tilde{T}_0 + \tilde{T}_1$ had been subtracted, the *empirical* covariance function was estimated, see figure 2. This function was then approximated (using the value for $\psi = 0$ and the first zero-point) with an analytic expression

$$K_2(P, Q) = \sum_{i=201}^{\infty} \frac{a_2 R_2^2}{(i-1)(i-2)(i+24)} \left(\frac{R_2^2}{rr'}\right)^{i+1} P_i(\cos\psi) \tag{3.3}$$

where $R_2 = 6.369.5$ km and $a_2 = 200$ mgal². The graphs of the derived covariance function of the gravity anomalies

$$\mathrm{cov}\,(\Delta g_P, \Delta g_Q) = \sum_{i=201}^{\infty} \frac{a_2(i-1)}{(i-2)(i+24)} \left(\frac{R_2^2}{rr'}\right)^{i+2} P_i(\cos\psi) \tag{3.4}$$

and the covariance function of the ξ-components of points on the same meridian $(\lambda_P = \lambda_Q)$

$$\mathrm{cov}\,(\xi_P, \xi_Q) = \sum_{i=201}^{\infty} \frac{a_2(\gamma_P \gamma_Q)^{-1}}{(i-1)(i-2)(i+24)} \left(\frac{R_2^2}{rr'}\right)^{i+2} \frac{\partial^2}{\partial\phi_P \partial\phi_Q} P_i(\cos\psi) \tag{3.5}$$

are shown in figure 2 and figure 3, respectively.

Using the datasets labelled (a) and (b) two approximations were constructed again using equations (2.3) and (2.4), but now with $K_2(P, Q)$ as a reproducing kernel. The two functions $\tilde{T}_a = \tilde{T}_0 + \tilde{T}_1 + \tilde{T}_{2a}$ and $\tilde{T}_b = \tilde{T}_0 + \tilde{T}_1 + \tilde{T}_{2b}$ were then used for the prediction of 68, respectively 64 remaining pairs of deflection components. The results are given in table 3. The maximal error amounted to 8″ (for a station near the mountain chain). Note that the mean value of the difference between observed and predicted deflections is relatively big. This is because we have not used any point gravity values outside the 1°-square. Earlier computational experiments indicate that this is the main reason for the biased results.

For these computations a system of computer programs described in Tscherning (1977a) was used.

Exactly the same kind of computations was then carried through, but now with the reduced data, equations (2.5)–(2.7). The results are also given in table 3. The maximal error now amounted to 5″ for the station mentioned above.

As terrain model we chose the topography and its Airy-Heiskanen isostatic compensation of the "square" sector 30°30′N to 34°N, 104°30′E to 108°E using 30

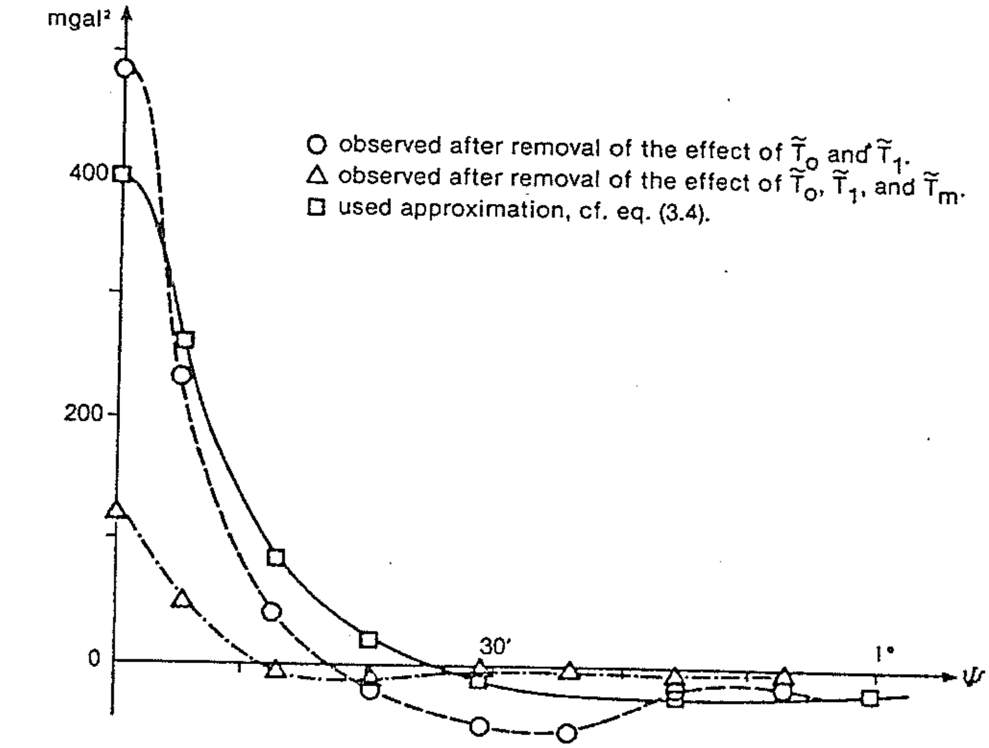

FIGURE 2.—Gravity anomaly covariance functions.

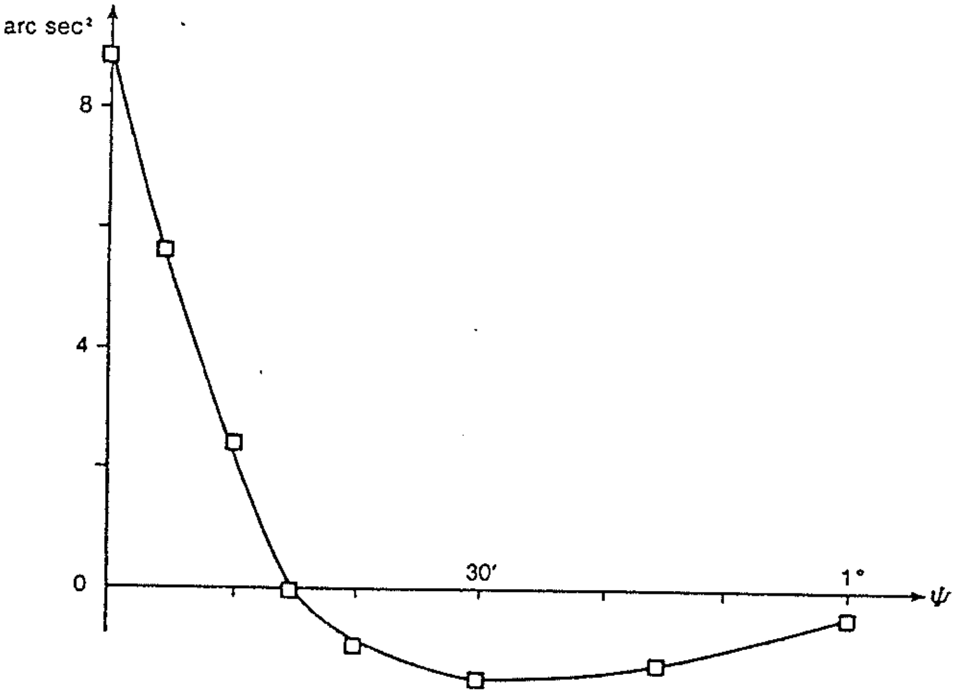

FIGURE 3.—Used covariance function for the ξ-component of the deflection of the vertical in a N–S azimuth.

TABLE 3.—*Results of predictions using* $\tilde{T}_a$, $\tilde{T}_b$, $\tilde{T}_a^c$

	ξ (arcsec)		η (arcsec)	
	mean	r.m.s.v.	mean	r.m.s.v.
Original data	−1.58	±1.84	−3.05	±5.42
Prediction error $\tilde{T}_a$	−1.24	1.35	1.49	1.49
Prediction error $\tilde{T}_b$	−0.65	1.19	0.99	1.47
Prediction error $\tilde{T}_a^c$	−1.09	0.71	1.79	1.01

km as condensation depth. The sector was digitized in compartments 0.5′ by 0.5′, 5′ by 5′, or 15′ by 15′. Δg_m, ξ_m and η_m were then calculated with the terrain and its compensation represented by rectangular boxes. using a general-purpose geodetic terrain correction program, which calculates the effect of a box using the cumbersome analytical expressions (Ehrismann et al. 1966) or more handy approximative formulas, depending on the accuracy wanted. To reproduce the terrain in the vicinity of the computation point, the terrain data around the station was interpolated using a two-dimensional bicubic spline and then integrated using the interpolated heights. This procedure has given very satisfactory results concerning the influence of the "inner zones." But, unfortunately, data lying outside the innermost 30′ square were only reduced using 5′ by 5′ and 15′ by 15′ mean height data, because of lack of detailed terrain data. However, owing to the mostly rather smooth local terrain, the error in neglecting the variation in the inner zone for these stations is estimated to be below a few mgal.

In addition to the production of point values of Δg_m, ξ_m and η_m also 1° by 1° mean values Δg_m were estimated. However, the influence on the GEM8 coefficients were set to zero.

The graph of empirical covariance function of the terrain corrected gravity anomalies is shown in figure 2. It would maybe have been reasonable to change the kernel $K_2(P, Q)$, but this was not done, because the change mainly would have been a multiplication with a scalar factor. (This had the advantage that the reduced normal-equation matrix could be used again with a new right-hand side.)

The bias we have seen for the predicted values is naturally a severe problem in areas where gravity is not available in neighbouring areas (because of ice, sea, or national boundaries). When comparing solution T_a with T_b we saw how the introduction of a few deflections made the bias decrease. It is reasonable to expect that the use of Doppler-determined geoid undulations will have the same effect, and computation experiments with the purpose of confirming this are in progress.

We will not here quote any values of used computer time, as the program system used has not been optimized for deflection prediction as was the FORTRAN-program published in Tscherning (1974). In this publication (table 2) are given different values of the total processing time for the collocation step only (computation of $L_i(T_m)$ cannot be done using this program system). Using this program system one pair of deflection components was predicted in less than 0.5 second, using a slightly bigger number of observations.

Discussions of the Results

The application of collocation for the prediction of deflections of the vertical has many advantages when compared with the integral formulae solutions to the boundary value problem:

(1) Different data-types can be used,
(2) data need not be spaced in a regular grid,
(3) all approximations are explicitly defined,
(4) prediction errors may be estimated.

The only drawback seems to be that a set of normal equations will have to be solved. But note that the dimension of the sytem of normal equations can be substantially reduced using the stepwise collocation method (Tscherning 1974).

Now, collocation is not the only method which have the above-mentioned advantages. If we give up the requirement that error-free observations must be exactly reproduced (eq. (2.2)), then we can use different approximation methods for the determination of $\tilde{T}^c$. An especially well suited method is the method of least-squares approximation in a finite dimensional vector space. Such a vector space may, for example, consist of all linear combinations of harmonic functions generated by n unit point-masses, h_i, $i = 1, \ldots, n$. Then

$$\tilde{T}^c(P) = \sum_{i=1}^{n} a_i h_i(P). \tag{4.1}$$

and the coefficients are determined by a least-square principle from a set of observations $x_i^c = L_i(T^c)$, $i=1, \ldots, N$, i.e.,

$$\begin{aligned} &\{L_i(T^c) - x_i^c\}^T \{P_{ij}\} \{L_i(T^c) - x_i^c\} \\ &= \left\{ \sum_{j=1}^{n} a_i L_i(h_j) - x_i^c \right\}^T \{P_{ij}\} \left\{ \sum_{i=1}^{n} a_i L_i(h_j) - x_i^c \right\} \\ &= \min_{a_j}, \end{aligned} \tag{4.2}$$

where P_{ij} are "weights."

This type of technique has been discussed in Fischer (1974, 1975) and Fischer and Wyatt (1974) and seems to be very promising for the prediction of deflections at sea, where big volumes of known density (seawater) can be used. Results obtained on land areas by Wassouf (1975), who used a slightly different method, indicate that the method also may give good results on land.

The main disadvantages of using equations (4.1) and (4.2) compared with the method of collocation seem to be that the prediction error cannot be expressed through a simple equation such as equation (2.12), and that the approximation may oscillate between data points. However, it is these two disadvantages which have made us here choose to use collocation for prediction purposes.

The results using the Vening-Meinesz or Molodensky integral formulae in mountainous areas reported, e.g., in Pick and Pola (1967) or Tengström (1971) seems all to be based on a dense gravity network and a very detailed topographic mapping near the points of computation. Here it really seems that collocation using topographic-isostatic reduced data could be used with advantage.

Generally it may be too optimistic to expect a decrease in the r.m.s.v. of the gravity anomalies to ±10 mgal and of the deflections to ±2″ when removing the effect of T_m as seen in the test described in section 3. (However, note that we here are discussing the variations in a very limited area.) In the Alps these variations have been reduced to about ±4″ (table 1).

Let us then suppose, that this reduction has been done, and that we want to predict deflections of the vertical with an r.m.s. error of ±1″. This corresponds to a value of 25 percent for the ratio (ρ) between the r.m.s.v. and r.m.s. prediction error. From Tscherning (1975a, fig. 5d) we can then see that this will require a spacing of gravity values with a distance of about 5′ or 10 km (if we can presuppose that the reduced gravity anomalies have the same covariance functions as used in the referenced paper). In the present investigation we have seen that the reduced gravity anomalies becomes slightly more uncorrelated (fig. 2). So a data spacing of between 3′ or 4′ seems to be necessary in alpine areas. This corresponds very well to the results reported by Bosch and Wolf (1975), who predicted deflections of the vertical with an r.m.s. error of ±0″.6 using *deflections* spaced 2′.5 apart.

So we may here conclude that deflections of the vertical may be predicted using collocation with errors of the same magnitude as seen using the integral formula technique, but using less (geodetically) observed data.

The Danish Geodetic Institute is in cooperation with the Greenland Geological Survey, now executing a combined topographic-geological mapping of northern Greenland. An alpine mountain chain of moderate height is crossing the area which is being mapped. Here deflections of the vertical can be expected to have an r.m.s.v. of ±3″, when reduced by the effect of the topography. This should be possible in practice, because it is planned to produce digital maps of the area. There will be a minor problem because of the Greenland Ice-Shield, but it is hoped that the ice thickness recordings will be continued so that also this area is covered.

If the deflections of the vertical are required to be known in this area with an r.m.s. error of ±1″ then (using Tscherning (1975a, fig. 5d)) we see that gravity observations must be spaced with a distance smaller than 15 km ≈ 8′. This will also assure that geoid undulations obtained by Doppler techniques can be interpolated with an accuracy better than ± 1 m even if they are spaced 150 km apart. The maximal distance which can be accepted is about 25 km, which is the approximate value of the first zero-point of the empirical covariance function of the reduced gravity anomalies, i.e., the distance in which the anomalies may be considered as uncorrelated.

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