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INTRODUCTION TO FUNCTIONAL ANALYSIS WITH A VIEW TO ITS APPLICATIONS IN APPROXIMATION THEORY

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ABSTRACT

These introductory lecture notes cover (without formal proofs) linear vector spaces, normed and inner product spaces, Hilbert spaces, reproducing kernel Hilbert spaces, linear functionals and linear operators. The relationship of these concepts to equivalent concepts used in connection with stochastic processes is indicated. Many examples related to the gravitational potential of the Earth (namely vector spaces of harmonic functions) illustrate the different concepts.

Special weight is put on the use of the Riesz-representation of a bounded linear functional in a reproducing kernel Hilbert space and the application of this representation in connection with approximation procedures.

The linear vector space with inner product is used to describe methods of "best" approximation as projections on a finite dimensional subspace. The collocation method is introduced as a projection method in a reproducing kernel Hilbert space, where the (not necessarily finite-dimensional) subspace is defined through the requirement that the approximation agrees with a finite set of observed (errorfree) values.

The lecture notes contain several exercises, both elementary and advanced ones.

1. Introduction

This paper was prepared in order to serve as lecture notes at the Second International Summer School in the Mountains, held in Ramsau, Austria in 1977. The topic of the summer school was "Geodesy and Approximation Theory" and the lecture notes was used in a 6 hours course which had the purpose of providing the students with a common background in functional analysis.

Nearly all aspects of functional analysis play a role in approximation theory. But because of the limited time available for the presentation, the author was faced with the difficult task of selecting what could be considered the most important part of functional analysis regarded from the standpoint of approximation theory.

As stressed by Shapiro (1971, Chapter 6), the reproducing kernel of a Hilbert space may serve as an important tool in approximation theory and it is already in use in geodesy. Therefore in order to attain at least a presentation of this important topic (following the usual road through linear vector spaces, inner product spaces and Hilbert spaces) it has been necessary to drop nearly all proofs of theorems. (The interested reader will find the necessary proofs in the referenced literature.) However, many examples are given, mostly related to well known results in classical potential theory as known and used in geodesy.

We have very closely followed the presentation of functional analysis as given in the excellent book by P.J. Davis "Interpolation and Approximation", but also the introductory papers by P. Meissl (1975,1976) has been a main source of inspiration for this paper.

As we mainly will deal with objects which are functions defined in a subset of a three-dimensional euclidian space R^3, the letter Ω will be reserved for such a set. Points in Ω will be denoted P,Q and the boundary of Ω is denoted c. The functions will be denoted f,g,h or l and for real numbers we will reserve the letters a,b,c, and for complex numbers w,z. Points in the n-dimensional real euclidian space will be denoted x and y with coordinates $\{x_i\}$ and $\{y_i\}$ with respect to the canonical basis. The sign ■ will denote "end of example".

2. Linear vector spaces

An abstract linear vector space X over the field of real numbers is a set of elements (or vectors) f,g,..., for which two types of operations are possible, namely addition of two elements and multiplication of an element of X with a scalar, which obey the following rules

(a) $f+g = g+f$,

(b) $f + (g+h) = (f+g) + h$,

(c) There exists a unique element $0 \in X$ so that $0+f = f$ for all $f \in X$,

(d) To each $f \in X$ there exists a unique inverse $-f$ so that $f+(-f) = 0$,

(e) $a(f+g) = af + ag$, $a \in \mathbb{R}$,

(f) $a(bf) = (ab)f$ for $a,b \in \mathbb{R}$ and $f \in X$,

(g) $(a+b)f = af + bf$,

(h) $1 \cdot (f) = f$, where 1 is the real number one.

The space could also have been defined as a vector space over the complex numbers, but we will in the following with a few exceptions exclusively regard vector spaces over the real numbers.

An expression

$$\sum_{i=1}^{n} a_i f_i = a_1 f_1 + \ldots + a_n f_n, a_i \in \mathbb{R}, f_i \in X$$

is called a linear combination of the vectors f_i. They are called linearly dependent if there exist constants $a_1, \ldots, a_n$ so that

$$\sum_{i=1}^{n} a_i f_i = 0 \quad ,$$

otherwise the vectors are called independent.

Suppose that in X we can find n linearly independent vectors while any set of n+1 vectors are dependent. Then X is said to have *dimension* n. When no such n exists, X is called an infinite dimensional space.

A set of elements $f_1, f_2, \ldots,$ is said to be a basis for X if they are independent and every $f \in X$ can be expressed uniquely as a linear combination of the f_i. It is obvious that X has dimension n if and only if it has a basis of n elements and that any set of n independent elements constitute a basis.

Example 2.1. The real, three-dimensional, Cartesian Space R^3. The space consists of triples of real numbers. We set $0 = (0,0,0)$ and define addition and multiplication by a scalar in the usual way. The vectors (1,0,0), (0,1,0) and (0,0,1) form a basis for the space.∎

Example 2.2. Linear spaces of functions. Let Ω be a point set in $\mathbb{R}^n$ with elements P,Q. Consider the set X of functions defined on Ω, $f:\Omega \to \mathbb{R}$.

For $f,g \in X$ we define

$$(f+g)(P) = f(P) + g(P)$$

and

$$(af)(P) = a \cdot f(P),$$

and through these definitions X will be a linear space. When Ω contains more than a finite number of points X is of infinite dimension. When Ω is an open set in $\mathbb{R}^n$ will the linear space of functions with domain Ω and having continuous derivatives of up to order m be denoted $C^m(\Omega)$.∎

Example 2.3. The functions with domain Ω and range $\mathbb{R}_+$, the positive real numbers, do not constitute a linear space. An example of an element of such a space is the density distribution of the Earth.∎

Example 2.4. Let Ω be the open set in $\mathbb{R}^3$ outside a sphere with radius R and center at the origin and let X consist of all functions harmonic in Ω and regular at infinity. A finite dimensional subspace of this linear vector space is spanned by all linear combinations of the solid spherical harmonics

$$V_{ij}(P) = \left(\frac{R}{r}\right)^{i+1} \bar{P}_{i|j|}(\sin\phi) \begin{cases} \cos j\lambda & 0 \le j \le i \\ \sin|j|\lambda & -j \le i < 0 \end{cases} \qquad (2.1)$$

where $0 \le i \le I$, (r,ϕ,λ) = (distance from the origin, latitude, longitude) of P and $\bar{P}_{i|j|}$ are the normalized associated Legendre polynomials of degree i and order $|j|$, cf. Heiskanen and Moritz (1967, eq.(1-77a,b)). The dimension of this subspace is $(I+1)^2$. X is infinite dimensional with the functions V_{ij} as a basis $i=0,1,\ldots\infty, |j| \le i$.∎

Let J be an arbitrary index-set, e.g. the set of positive integers. Then, given a set of functions $\{f_i,\ i \in J\}$ like the set defined by eq.(2-1), a linear vector space may be formed by taking all (finite) linear combinations of the given elements. This is called the space *spanned* by the elements, and is denoted span $\{f_i,\ i \in J\}$.

Some properties like harmonicity or continuity of higher order derivatives are generally preserved by the elements of the spanned space, while other, like positiveness, are not preserved.

In many cases we must associate a number with a function extracted from a given class of functions, e.g. its value in a specific point. Such an association is denoted a *functional*. An important class of functionals are the *linear functionals*. Such a functional, L, must fulfil

$$L(af + bg) = a\,L(f) + b\,L(g)$$

for $f,g \in X$ and $a,b \in \mathbb{R}$.

Example 2.5. Let $X = C[a,b]$, the space of functions continuous on the interval $[a,b]$. Then we can define the functional

$$L(f) = \int_a^b f(x)dx, \qquad x \in [a,b],\ f \in X. \blacksquare$$

Example 2.6. The evaluation functional. For $X = C[a,b]$, $x \in [a,b]$ we define

$$L_x(f) = f(x), \qquad \text{for } f \in X.$$

For $X = C(\Omega)$ with Ω defined like in Example 2.4, and $P \in \Omega$

$$L_P(f) = f(P) \qquad \text{for } f \in X.$$

($C(\Omega)$ consists of the continuous mappings from Ω to $\mathbb{R}$). ■

Example 2.7. The functions harmonic in Ω and regular at infinity permit a representation through a convergent series in solid spherical harmonics,

$$f(P) = \sum_{i=0}^{\infty} \sum_{j=-i}^{i} a_{ij} V_{ij}(P) \; ; \qquad (2\text{-}2)$$

cf. Example 2.4.

The mapping which to f associates the coefficient a_{ij} is a lin functional, L_{ij},

$$L_{ij}(f) = \frac{1}{4\pi R^2} \int_\sigma f(P) \cdot V_{ij}(P) \, d\sigma = a_{ij} ; \tag{2-3}$$

cf. Heiskanen and Moritz (1967, eq.1-76).■

Example 2.8. $X = \mathbb{R}^3$. Let $a = (a_1, a_2, a_3)$ be a fixed vector in R^3. Ther linear functional is defined through

$$L(x) = \sum_{i=1}^{3} a_i x_i .$$

Note that all linear functionals in $\mathbb{R}^3$ will have this form.■

Example 2.9. Let X be the space of harmonic functions considered in Ex 2.4 and 2.7. Gravity in a point P, with Cartesian coordinates $(x_1, x_2, x$

$$g_P = \left(\sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\Big|_P^2 \right)^{\frac{1}{2}} \tag{2-4}$$

is a non-linear functional.

The gravity anomaly is as well known associated with a *linear* functional, Δg,

$$\Delta g_P = - \frac{\partial f}{\partial r}\Big|_P - \frac{2}{r} f(P) ; \tag{2-5}$$

cf. Heiskanen and Moritz (1967, eq.(2-154)).

The height anomaly is also a linearized functional

$$\zeta_P = L_P(f)\frac{1}{\gamma} = f(P) \cdot \frac{1}{\gamma} ,$$

where γ is the reference gravity in P.■

The set of linear functionals defined on a linear vector spa will form a linear vector space X^* (the algebraic dual space) with th

$$(L_1+L_2)(f) = L_1(f)+L_2(f), \quad f \in X,$$

$$(a\ L_1)(f) = a\ L_1(f).$$

The vector space X^* will have the same dimension as X.

Example 2.10. The linear space spanned by a random function.

Let there be given a probability space $(H,\mathcal{B},\mathcal{P})$, where H is an arbitrary set (generally denoted Ω), $\mathcal{B}$ is a Boolean σ-algebra of subsets of H and $\mathcal{P}$ is a probability measure. The stochastic or random variables are mappings from H to R. When these mappings are parameterized through a parameter $t \in T$ (an index set), a stochastic process $\{X_t,\ t \in T\}$ is defined provided the joint probabilities are defined, cf. e.g. Parzen (1959) or Lauritzen (1973). The stochastic process is called a random function provided the second order moments are finite, i.e.

$$\int_H X_t^2 d\mathcal{P} < \infty .$$

A linear vector space can be defined as the one spanned by the random function,

$$L\ \{X_t,\ t \in T\} = \text{span}\ \{X_t,\ t \in T\}.$$

Let for example H be the space of regular harmonic functions defined in Example 2.4, let the index set T be the set Ω and let the random function be the evaluation functionals $\{X_t,\ X_t(f) = f(t),\ \text{for } f \in H,\ t \in \Omega\}$. $\mathcal{P}$ may be implicitly defined by requiring all the stochastic variables X_t to be normally distributed with zero mean value and covariance

$$\int_H X_t \cdot X_u d\mathcal{P} = \text{cov}(t,u), \qquad (2\text{-}6)$$

where cov(t,u) is a positive definite function, i.e. for any finite set of values $t_1, \ldots, t_n$ will the matrix $\{\text{cov}(t_i,t_j)\}$ be positive definite. ∎

3. Hilbert spaces

3.1. Normed linear spaces

A linear space X is called a normed linear space if for each element of the space (f) there is defined a real number designated $\|f\|$ so that

$$\|f\| \geq 0 \qquad \text{(positivity)}$$

$$\|f\| = 0 \Longrightarrow f = 0 \qquad \text{(definiteness)}$$

$$\|af\| = |a| \cdot \|f\| \qquad \text{(homogeneity)}$$

$$\|f+g\| \leq \|f\| + \|g\| \qquad \text{(triangle inequality),}$$

$\|\cdot\|$ is a functional known as the norm.

A normed space is also a so-called *metric* space since we to each two elements f,g may associate the distance $d(f,g) = \|f-g\|$.

Example 3.1. The real numbers $-\infty < x < \infty$ with $\|x\| = |x|$. ■

Example 3.2. $C[a,b]$. $\|f\| = \max\limits_{x \in [a,b]} |f(x)|$. ■

Example 3.3. R^n, $\|x\|^2 = \sum\limits_{i=1}^{n} x_i^2$. ■

Example 3.4. The H^s-norms. Regard $X = C^m(\Omega)$. Define

$$D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f, \quad |\alpha| \leq s,$$

$$|\alpha| = \alpha_1 + \ldots + \alpha_n,$$

where α is a so-called multiindex symbol.

Then norms may be defined for positive integer values of s on subsets of $C^m(\Omega)$, $m \geq s$, by

$$\|f\|_s^2 = \sum_{|\alpha| \leq s} \int (D^\alpha f)^2 \, d\Omega.$$

The subsets consist of those elements of $C^m(\Omega)$ for which the norm is finite. The corresponding complete spaces (see section 3.5) are denoted the Sobolev or H^s-spaces, and the norms are therefore denoted the H^s-norms.

The norms may also be defined for non-integer values of s, but their definition is rather technical. Norms corresponding to negative values of s may also be defined,

$$\|f\|_{-s} = \max_{g \in H_s} \frac{|\int fg d\Omega|}{\|g\|_s} .$$

The H^o-space is also known as the L_2-space.■

3.2. Inner product spaces

A linear space X is called an inner product space if there is defined a mapping denoted the inner product $(\cdot,\cdot) : X \times X \to \mathbb{R}$ so that

$$(f,g) = (g,f) \qquad \text{(Symmetry)}$$

$$(f_1+f_2,g) = (f_1,g) + (f_2,g) \qquad \text{(Linearity)}$$

$$(af,g) = a(f,g) \qquad \text{(Homogeneity)}$$

$$(f,f) \geq 0,\ (f,f) = 0 \Longleftrightarrow f \equiv 0 \qquad \text{(Positivity)} .$$

Example 3.5. $X = \mathbb{R}^n$. $(x,y) = \sum_{i=1}^{n} x_i y_i$, and

$$X = C[a,b],\ (f,g) = \int_a^b fg dx.$$■

In an inner product space the *Schwarz inequality* is valid,

$$|(f,g)|^2 \leq (f,f)\cdot(g,g),$$

and defining $\|f\| = (f,f)^{\frac{1}{2}}$ we have

$$|(f,g)|^2 \leq \|f\|^2 \cdot \|g\|^2 \cdot \qquad (3\text{-}1)$$

The equality sign is only valid when f and g are linearly dependent. This enables us to define an angle θ between two (non zero) vectors f,g because

$$-1 \leq \cos\theta = \frac{(f,g)}{\|f\| \cdot \|g\|} \leq 1.$$

When $\theta = 0$ we say that f,g are parallel and when $\theta = \pi/2$ that the two elements are orthogonal, i.e. $(f,g) = 0$.

We are then able to define the *projection* of f on g by

$$Pr_g(f) = (f,g)\frac{g}{\|g\|^2} \tag{3-2}$$

which has the usual property

$$Pr_g(g) = (g,g)\frac{g}{\|g\|^2} = g.$$

3.3. Orthonormal systems

A set of elements $\{f_i,\ i=1,2,\ldots,\}$ in an inner product space is called orthogonal if

$$(f_i,f_j) = 0, \qquad \text{for } i \neq j$$

and orthonormal if also

$$(f_i,f_i) = 1.$$

Example 3.6. The linear elements.

On the interval $[0:\pi]$ we define the space $S^h \subset C^0[0:\pi]$, $(h\,N = \pi)$ to be all functions which are linear over each interval $[(j-1)h,\ j\cdot h]$, and continuous at the nodes $x = jh$. The functions are called linear elements.

The functions $f_j^h(x)$ are the functions in S^h which equal one for $x = jh$ and vanish at all other nodes. These functions constitute a basis for S^h since every member can be written as a combination

$$g^h(x) = \sum_{i=1}^{N} a_i f_i^h(x).$$

The space S^h may be equipped with the H^1-norm (cf. Example 3.4). Here the basis is not orthogonal, but f^h_{j-1} is orthogonal to f^h_{j+1}. ∎

Example 3.7. The Legendre polynomials $P_i(x)$ on $C[-1,1]$ (cf. Heiskanen and Moritz (1967, section 1-11)) are orthogonal, but not orthonormal. Using the inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx,$$

$$\| P_i \|^2 = \int_{-1}^{1} (P_i(x))^2 dx = \frac{2}{2i+1} \;\blacksquare$$

Any finite set of non-zero orthogonal elements is linearly independent. Such sets are therefore extremely important, because they may be used as a basis for X or a subspace of X, cf. Example 2.4. Conversely any finite or infinite sequence of linear independent elements can be used to create an orthonormal set through a simple procedure known as the Gram-Schmidt orthonormalization process. Regard the sequence $\{f_i, i=1,\dots, m\}$. Then we may generate an orthogonal sequence $\{g_i, i=1,\dots, m\}$ by defining

$$g_1 = f_1 / \| f_1 \|$$

$$g_n = (f_n - \sum_{k=1}^{n-1} (f_n, g_k) g_k) / \| f_n - \sum_{k=1}^{n-1} (f_n, g_k) g_k \|$$

$$n = 2, \dots, m.$$

It is easily seen how the process works. The first vector g_1 is created by dividing f_1 by its length (normalizing), whereby g_1 gets the lenght one. The projection of f on a unit vector is

$$Pr_{g_k}(f) = (f, g_k) \cdot g_k \tag{3-3}$$

so g_n is created by subtracting from f_n its projection on the subspace spanned by $\{g_k, k=1,\dots, n-1\}$ and then normalizing this difference vector.

Let $g_1, g_2, \dots,$ be a finite or infinite sequence of orthonormal elements and let f be an arbitrary element. The series

$$\tilde{f} = \sum_{n=1}^{\infty} (f, g_n) g_n \tag{3-4}$$

is called the orthogonal expansion or the Fourier expansion of f. In view of eq. (3-3) we see that $\bar{f}$ is the sum of the projections of f on the system of orthonormal elements. (The question whether $\bar{f}=f$ will be dealt with in sub-section 3.5).

In case the functions g_n were only orthogonal we would have

$$\bar{f} = \sum_{n=1}^{\infty} (f, g_n) \frac{g_n}{\|g_n\|^2} .$$

3.4. Best linear approximation

In a normed linear space we are able to define what we mean by the best linear approximation by a set of elements g_i, i=1,..., n. This approximation is an element $\sum_{i=1}^{n} a_i g_i$ so that

$$\| f - \sum_{i=1}^{n} a_i g_i \| \leq \| f - \sum_{i=1}^{n} b_i g_i \|$$

for every choice of constants b_i, i=1,..., n. Hence, the approximation is best in the sense that the error norm is minimized.

The problem has a solution, cf. Davis (1975, Theorem 7.4.1), but the solution is not unique. But when X is an inner product space, a unique solution can be found.

Let us suppose that the elements g_i are linearly independent. Then we may create a corresponding system of orthonormalized elements g_i^*. It is then easily seen that a best linear approximation is given by

$$\bar{f} = \sum_{i=1}^{N} (f_i, g_i^*) g_i^*, \tag{3-5}$$

i.e. equal to the sum of the projections of f on the elements of an orthogonal basis spanning the same subspace as spanned by the elements g_i, or simply the projection of f on span $\{g_i, i=1,...,n\}$. (For a proof see Davis (1975, Theorem 8.5.1 and Corollary 8.5.2)).

The difference between f and the best approximation $\bar{f}$ is orthogonal on all g_i^* because

$$(f - \sum_{k=1}^{n} (f, g_k^*) g_k^*, g_i^*)$$
$$= (f, g_i^*) - \sum_{k=1}^{n} (f, g_k^*)(g_k^*, g_i^*) = (f, g_i^*) - (f, g_i^*) = 0$$

and must therefore also be orthogonal on all elements g_k . We can use this fact to write up a system of *normal equations* which directly will determine the coefficients of $\bar{f}$ expressed as a linear combination of the functions g_i. We have

$$\bar{f} = \sum_{i=1}^{n} a_i g_i \qquad (3\text{-}6)$$

and

$$(f-\bar{f}, g_i) = (f - a_1 g_1 - \ldots - a_n g_n, g_i) = 0 \text{ , or}$$

$$\begin{aligned} a_1(g_1,g_1) + a_2(g_1,g_2) + \ldots + a_n(g_1,g_n) &= (f,g_1) \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad &\quad \vdots \\ a_1(g_1,g_n) + a_2(g_2,g_n) + \ldots + a_n(g_n,g_n) &= (f,g_n) \end{aligned}$$

and in matrix form

$$\{(g_i, g_j)\}\ \{a_j\} = \{(f, g_i)\}. \qquad (3\text{-}7)$$

3.5. Completeness

It is naturally of extreme interest to know whether $\bar{f}$ as given by eq. (3-6) will converge towards f when n goes to infinity. This will depend on whether X is a *complete* space. Let us first define what we mean by a complete basis.

Let I be a subset of the integer numbers. A set of elements $\{f_i, i \in I, f_i \in X\}$ is called a complete basis if every element $f \in X$ can be approximated arbitrarily closely by finite linear combinations of the elements, i.e. for all $f \in X$ and $\varepsilon > 0$ there exist constants $\{a_i,\ i=1,\ldots,n\}$ so that $\|f - \sum_{i=1}^{n} a_i f_i\| \leq \varepsilon$.

Example 3.8. $X = C[a,b]$. $1, x, x^2, \ldots,$ form a complete basis in X with $\|f\| = \max_{a \leq x \leq b} |f(x)|$. ∎

Example 3.9. $X = C[a,b]$ with $\|f\|^2 = \int_a^b f(x)^2 dx$.
Here again the polynomials $1, x, x^2, \ldots,$ form a complete basis. ∎

A sequence of elements $\{f_i,\ i=1,2,\ldots,\ f_i \in X\}$ where X is a metric space is said to converge to f if $\lim_{n\to\infty} d(f-f_n) = 0$. Let S be a subset of the metric space X, $S \subset X$. The closure $\bar{S}$ of S is defined as all limits of convergent sequences of S. If $S = \bar{S}$ then S is called *closed*. S is called *dense* in X if $\bar{S} = X$. X is called *separable* if there is a countable dense set in it.

Example 3.10. The set

$$S = \text{span}\ \{V_{ij},\ i=0,\ \ldots,\ \infty,\ |j| \leq i\}$$

of finite linear combinations of the solid spherical harmonics, cf. Example 2.4, with the norm

$$||f||^2 = \int_\Omega f^2 d\Omega$$

is not closed (the infinite series are not included). The space is separable.■

Example 3.11. The sequence $\{\frac{1}{n},\ n = 2,\ldots,\ \infty\}$ is not convergent on the open interval $]0,1[$.■

The sequence of Example 3.11 is an example of a so-called *Cauchy sequence*. Such a sequence $\{f_i \in X,\ i=1,\ldots,\ \infty\}$ has the property that for every $\varepsilon>0$ there is an integer N so that $d(f_i,f_j) \leq \varepsilon$ for $i,j \geq N$. The (metric) space X is called complete if every Cauchy sequence has a limit in X.

A complete, normed linear space is called a *Banach space*.

Example 3.12. $X = C[a,b]$ normed by $||f|| = \max_{a\leq x\leq b} |f(x)|$ is complete. But using $||f||^2 = \int_a^b f(x)^2 dx$, X is not complete because the limits of continuous functions need not to be continuous using this norm, cf. Davis (1975, section 8.9, example 9).■

3.6. Hilbert space

A complete inner product space is called a Hilbert space. (We will here use this label for spaces of both finite and infinite dimension). In a Hilbert space, H, several important properties can be proved related to the representation of an element through its orthogonal expansion, cf. section 3.3. Let $\{f_i^*,\ i=1,\ldots,\ \infty\}$ be a sequence of orthonormal elements in H. Then the following statements are equivalent:

(A) $\{f_i^*\}$ is a complete basis in H.

(B) The orthogonal expansion of any element $f \in H$ converges in norm to f,

$$\lim_{n\to\infty} \| f - \sum_{i=1}^{n} (f,f_i^*)f_i^* \| = 0 .$$

(C) Parseval's identity holds:

$$\| f \|^2 = (f,f) = \sum_{n=1}^{\infty} (f,f_n^*)^2 .$$

(D) The extended Parseval's identity holds:

$$(f,g) = \sum_{n=1}^{\infty} (f,f_n^*)\cdot(g,f_n^*) .$$

(E) There is no strictly larger orthonormal system containing $\{f_i^*\}$.

(F) An element $f \in H$ orthogonal on all f_i^* must be the zero element,

$$f \in H \text{ and } (f,f_i^*) = 0 \rightarrow f = 0 .$$

(G) Any element is determined uniquely by its orthogonal expansion, i.e.

$$(g,f_i^*) = (f,f_i^*) \rightarrow g = f.$$

For a proof see Davis (1975, Theorem 8.9.1). (Appropriate changes will have to be made if H is finite dimensional.)

The Hilbert spaces in which (A) - (G) holds will be finite dimensional or separable. We see that in these spaces will f not necessarily be equal to $\bar{f}$ as given through its orthogonal expansion eq. (3.4).

The completeness property (F) will in the following be used to define completeness of a basis or a set in an inner product space.

Example 3.13. The set of infinite sequences $\{a_i\}$ so that $\sum_{i=1}^{\infty} a_i^2 < \infty$, with

$$(a,b) = \sum_{i=1}^{\infty} a_i b_i \quad \text{where } a = \{a_i\},\ b = \{b_i\},$$

is a separable Hilbert space. ■

Example 3.14. $X = L^2[a,b]$ of square integrable functions with $(f,g) = \int_a^b f(x)g(x)dx$ is a separable Hilbert space. (Functions which only are different on a set of zero-measure are identified.)

The orthonormal elements are certain modified Legendre polynomials. ■

Example 3.15. A set of single valued analytic functions defined in an open circular disc B with center at the origin. We define the inner product

$$(f,g) = \iint_B f \cdot \overline{g}\ dxdy,$$

where the overbar denotes complex conjugation. We denote the space $L^2(B)$. (In vector spaces of complex functions we have $(f,g) = \overline{(g,f)}$.)

The functions

$$V_i(z) = \left(\frac{i}{\pi}\right)^{\frac{1}{2}} z^{i-1}, \qquad i=1,2,\dots,\infty,$$

form an orthonormal system,

$$\iint_B V_i(z)\cdot V_j(z)dxdy = \frac{(ij)^{\frac{1}{2}}}{\pi} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r^{i+j-1} e^{i\theta(i-j)} drd\theta = \delta_{ij},$$

and it is also well known that for all single valued analytic functions f in B we have

$$f(z) = \sum_{i=1}^{\infty} a_i z^{i-1} = \sum_{i=1}^{\infty} \left[\left(\frac{\pi}{i}\right)^{\frac{1}{2}} a_i\right] \cdot V_i(z)$$

and that for the elements of this Hilbert space

$$\sum_{i=1}^{\infty} a_i^2 \frac{\pi}{i} < \infty. \ ■$$

Example 3.16. Let Ω designate the set outside a sphere with radius R in R^3 and with center at the origin. The surface is denoted σ.

We will like in Example 2.4 regard the functions harmonic in Ω and regular at infinity. The subset equipped with the inner product

$$(f,g) = \frac{1}{4\pi R^2} \int_\sigma f \cdot g \, d\sigma$$

for which $\|f\| < \infty$ will form a (separable) Hilbert space. (The integral is to be understood as the limit of the integrals over concentric spheres with radii $R + \varepsilon$, $\varepsilon > 0$ for $\varepsilon \to 0$.)

The solid spherical harmonics $V_{ij}(P)$ defined in Example 2.4 are orthogonal. We know that any regular, harmonic function can be expanded as a sequence with respect to these functions,

$$f(P) = \sum_{i=0}^{\infty} \sum_{j=-i}^{i} a_{ij} \cdot V_{ij}(P),$$

but only the functions for which

$$\|f\|^2 = \sum_{i=0}^{\infty} \sum_{j=-i}^{i} a_i^2 < \infty$$

will be elements of the space. The elements will be harmonic functions with square-integrable boundary values.■

Example 3.17. The Hilbert space spanned by a random function..

In Example 2.10 we introduced the linear space spanned by a random function $L(X_t, t \in T)$. On this space an inner product is defined through the covariance function,

$$(X_t, X_u) = \text{cov}(t,u),$$

which easily is extended to an inner product for all elements of $L(X_t, t \in T)$. This space is not complete, but may be enlarged by adding all limits of sequences.■

Example 3.18. A non-separable Hilbert space. (Cf. Meschkowski (1963,p.21-22)). Regard linear combinations of the type

$$\sum_{\nu=1}^{\infty} a_\nu e^{i\lambda_\nu x}$$

where $-\infty<x<\infty$ and a_ν and λ_ν are real numbers for which the series converge uniformly on the real line. As inner product we use

$$(f,g) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\overline{g(t)}dt.$$

The functions $e^{i\lambda x}$ form an orthonormal system,

$$(e^{i\lambda x},\ e^{i\mu x}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{i(\lambda-\mu)t}dt$$

$$= \lim_{\tau\to\infty} \frac{e^{i\tau}-e^{-i\tau}}{2i\tau} = \lim_{\tau\to\infty} \frac{\sin\tau}{\tau} = 0$$

for $\lambda \neq \mu$ where we have put $\tau = (\lambda-\mu)T$. For $\lambda = \mu$ we have

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt = 1 \text{ and hence } (e^{i\lambda x},\ e^{i\mu x}) = \delta_{\lambda\mu}.$$

The space is *not* separable because we have more than a countable number of orthonormal elements in the space. The elements of the space are the so-called almost-periodic functions. ■

3.7. The continuous linear functionals

In the following we will by "functional" understand "linear functional". In a Banach or a Hilbert space the bounded linear functionals have certain important and useful properties. (A functional, L, is bounded if there exists a constant M so that

$$|L(f)| \leq M\cdot\|f\| \qquad (3\text{-}8)$$

holds for all $f \in X$.)

If $\| f_i - f \| \to 0$ then

$$|L(f_i) - L(f)| = |L(f_i - f)| \leq M \cdot \| f_i - f \| ,$$

so

$$|L(f_i) - L(f)| \to 0,$$

i.e. a bounded functional is continuous. It also holds that a continuous functional is bounded.

For bounded functionals a norm may be introduced being equal to the maximal value M for which eq. (3-8) holds for all $f \in X$, i.e.

$$\| L \|_* = \sup_{f \in X} \frac{|L(f)|}{\| f \|} \qquad (f \neq 0) \qquad (3\text{-}9)$$

and thereby

$$|L(f)| \leq \| f \| \cdot \| L \|_* .$$

Example 3.19. $X = C[a,b]$, $\| f \| = \max_{a \leq x \leq b} |f(x)|$.

Define

$$L_x(f) = f(x) \qquad a \leq x \leq b.$$

Then it is easily seen that $\| L_x \|_* = 1$. ■

Example 3.20. Regard the separable Hilbert space of regular harmonic functions defined in Example 3.16. In this Hilbert space will the functionals L_{ij} which to f associates its i,j'th coefficient a_{ij} have a norm equal to 1. ■

The introduction of the norm of the bounded linear functionals makes the linear space of all bounded linear functionals a normed linear space. It is also denoted X^*. (Note that the boundedness of a functional depends strongly on the norm in X, cf.eq. (3-8)).

In a Hilbert space, H, the bounded linear functionals possess a very simple representation. To each $L \in H^*$ there exists a unique element $l \in H$ so that

$$L(f) = (l,f) \qquad (3\text{-}10)$$

for all $f \in H$. The element l is denoted the (Riesz) *representer* of L. Using eq. (3-10) we see

$$|L(f)| \le |(f,l)| \le \|f\| \cdot \|l\| ,$$

i.e. $\|L\|_* \le \|l\|$.

And

$$\|L\|_* = \sup_{f \in H} \frac{|L(f)|}{\|f\|} \ge \frac{|L(l)|}{\|l\|} = \frac{|(l,l)|}{\|l\|} = \|l\| ,$$

i.e. $\|L\|_* \ge \|l\|$ and hence

$$\|L\| = \|l\| . \qquad (3\text{-}11)$$

and furthermore

$$|L(l)| = \|l\|^2 . \qquad (3\text{-}12)$$

We have here constructed an isometric isomorphism between H and H^* making H^* a Hilbert space itself. The inner product of two elements L_1 and L_2 in H^* is defined as the inner product of their representers l_1 and l_2,

$$(L_1, L_2)_* = (l_1, l_2). \qquad (3\text{-}13)$$

So H^* has in this way been equipped with an inner product, and the completeness may easily be proved using eq. (3-11).

4. The reproducing kernel Hilbert space

4.1. Existence of a reproducing kernel

Let now Ω design a point set in R^n, which in the examples below will be the interval $[a,b]$, the disc, B, in the complex plane with $|z| < 1$ or the set in R^3 outside a sphere with radius R and center at the origin. Points in Ω will be denoted P or Q. H will be a Hilbert space of mappings (functions) from Ω to R.

A function of two variables P, Q, K(P,Q) is called a reproducing kernel (for H) when

(a) for all $Q \in \Omega$, $K(P,Q) \in H$ for Q fixed. (This function will also be denoted $K(P,\cdot)$),

(b) for all $f \in H$ and $Q \in \Omega$ holds the reproducing property

$$f(Q) = \{f(P),\ K(P,Q)\}_P, \qquad (4\text{-}1)$$

where the subscript P indicates that the inner product is to be taken with respect to P.

A necessary and sufficient condition for H having a reproducing kernel is that the evaluation functionals

$$L_P(f) = f(P), \qquad P \in \Omega$$

are bounded, i.e. there exists an M so that

$$|L_P(f)| \leq M \cdot \|f\|$$

for all $f \in H$.

When H has a reproducing kernel we easily see, using Schwarz inequality, eq. (3-1) that

$$|f(Q)|^2 = |\{f(P),\ K(P,Q)\}|^2$$

$$\leq \{f(P),\ f(P)\}_P \cdot \{K(P,Q),\ K(P,Q)\}_P = \|f\|^2 \cdot K(Q,Q),$$

i.e. for M we may use K(Q,Q).

Conversely, when M exists, L_P is continuous. There therefore exists a unique element in H so that

$$L_P(f) = f(P) = \{K_P(Q),\ f(Q)\}$$

and we may put $K(P,Q) = K_P(Q)$.

It is easily seen that K(P,Q) is unique and that it is symmetric, i.e. K(P,Q) = K(Q,P).

Example 4.1. R^n with inner product $(x,y) = \sum_{i=1}^{n} x_i y_i$. The reproducing kernel does not exist in the sense defined above (for function spaces), but the identity mapping has very similar properties. ∎

Example 4.2. In $L^2(B)$ of Example 3.15 the reproducing kernel is equal to

$$K(z,\bar{w}) = \sum_{n=1}^{\infty} (\tfrac{n}{\pi}) z^{n-1} \bar{w}^{n-1} = \frac{1}{\pi(1-z\bar{w})^2} \quad . \quad ∎$$

Hilbert spaces with a reproducing kernel have several very nice properties.

Let us regard a sequence f_n in H so that $\lim_{n\to\infty} \|f_n - f\| = 0$. Then

$$|f(Q) - f_n(Q)| = |L_Q(f-f_n)| \leq \|L_Q\|_* \cdot \|f-f_n\| .$$

This means that strong convergence (convergence in norm) implies pointwise convergence in Ω, and even uniformly pointwise convergence in any subset of Ω where $\|L_Q\|$ is bounded by a constant. In a separable Hilbert space with a reproducing kernel are we then permitted to write an equality sign between an element and its expansion with respect to an orthogonal basis, i.e. $\tilde{f} = f$, cf. eq. (3-4).

In such a space with a complete orthonormal basis $\{V_i,\ i=1,\ldots,\infty\}$ the reproducing kernel must be equal to

$$K(P,Q) = \sum_{i=1}^{\infty} V_i(P)\cdot V_i(Q). \tag{4-2}$$

We have for any function in H,

$$f(P) = \sum_{i=1}^{\infty} a_i V_i(P)$$

with $a_i = (f, V_i)$. Hence

$$f(Q) = (f(P), K(P,Q))_P = (f(P),\ \sum_{i=1}^{\infty} V_i(P)\cdot V_i(Q))$$

$$= \sum_{i=1}^{\infty} (f(P), V_i(P))_P \cdot V_i(Q) = \sum_{i=1}^{\infty} a_i V_i(Q),$$

i.e. the function defined by eq.(4-2) has the reproducing property.

Example 4.3. Let Ω be the set outside a sphere in R^3 with radius R and center at the origin. The elements of H are harmonic functions as used in Example 3.16. Let P and Q have spherical coordinates (latitude, longitude, distance from the origin) (ϕ,λ,r), (ϕ',λ',r'), respectively.

The reproducing kernel is here

$$K(P,Q) = \sum_{i=o}^{\infty}\sum_{j=-i}^{i} V_{ij}(P)\cdot V_{ij}(Q)$$

$$= \sum_{i=o}^{\infty}\left(\frac{R^2}{r\ r'}\right)^{i+1}\sum_{j=o}^{i}\overline{P}_{ij}(\sin\phi)\cdot\overline{P}_{ij}(\sin\phi')\big[\cos(j\lambda)\cdot\cos(j\lambda')$$

$$+\sin(j\lambda)\cdot\sin(j\lambda')\big] = \sum_{i=o}^{\infty}(2i+1)\left(\frac{R^2}{r\ r'}\right)^{i+1}P_i(\cos\psi)$$

$$= R^2((r\cdot r')^2 - R^4)/((r\ r')^2 - 2R^2\ r\ r'\ \cos\psi + R^4)^{\frac{3}{2}}$$

where ψ is the spherical distance between P and Q.

The reproducing property,

$$(f(P),\ K(P,Q)) = \frac{1}{4\pi R^2}\int_\sigma f(P)\cdot K(P,Q)d\sigma,$$

is here nothing but the Poisson integral formulae, cf. Heiskanen and Moritz (1967,eq.1-89). ■

Example 4.4. Let Ω be the complement to a smooth simply connected set in R^3 with a smooth boundary and let again the elements of H be functions harmonic in Ω and regular at infinity for which the norm

$$\|f\|^2 = \int_\Omega(\nabla f)^2 d\Omega$$

is finite. The norm is the well known Dirichlet norm. The inner product is

$$(f,g) = \int_\Omega(\nabla f\cdot\nabla g)d\Omega.$$

($\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3})$, and $\frac{\partial}{\partial n}$ will below denote the derivative with respect to the normal n to the boundary σ of Ω.)

In this case the reproducing kernel is the sum of the Green's and Neumann's functions for the set Ω (see Garabedian (1964, section 7.3)),

$$K(P,Q) = G(P,Q) + N(P,Q).$$

Using Green's identity and the fact that $\frac{\partial N}{\partial n}$ and G are zero at the boundary we get

$$(f, K(\cdot,Q)) = \int_\Omega \nabla f \cdot \nabla K(\cdot,Q)d\Omega$$

$$= - \int_\sigma \frac{\partial}{\partial n} K(\cdot,Q)\cdot f d\sigma = - \int_\sigma \frac{\partial}{\partial n} G(\cdot,Q)\cdot f \, d\sigma = f$$

$$= - \int_\sigma K(\cdot,Q)\cdot \frac{\partial}{\partial n} f \, d\sigma = - \int_\sigma N(\cdot,Q)\frac{\partial}{\partial n} f \, d\sigma = f.$$

When Ω is as in Example 3.16, the set of solid spherical harmonics $V_{ij}(P)$ will be an orthogonal basis. The length of an element of the basis is

$$4\pi \cdot \| V_{ij} \|^2 = - \int_\sigma V_{ij}(P)\frac{\partial}{\partial n} V_{ij}(P)d\sigma = \frac{(i+1)}{R} \int_\sigma V_{ij}(P)^2 d\sigma$$

$$= 4\pi \; R(i+1).$$

The functions $V_{ij}(P) \frac{1}{\sqrt{R(i+1)}}$ will therefore form an orthonormal basis, and we have

$$K(P,Q) = \sum_{i=0}^{\infty} \frac{2i+1}{R(i+1)} P_i(\cos\psi)\left(\frac{R}{r\,r'}\right)^{i+1},$$

which like in Example 4.3 may be put equal to a simple closed expression, cf. Tscherning (1975,p.86).■

Example 4.5. The Hilbert space spanned by the random function in Example 3.17. The covariance function cov(t,u) is a reproducing kernel for the Hilbert space, cf. Parzen (1959).■

Generally Hilbert spaces of functions which are solutions to elliptic and certain classes of parabolic partial differential equations will have a reproducing kernel. This is one of the reasons why these spaces are very important in approximation theory (and as indicated in Example 4.5 also in statistical estimation theory.)

4.2. The representer of a continuous linear functional

The representer (l) of an element of $H^*(L)$ has a very simple form. We must have

$$LK(\cdot,Q) = (K(P,Q),\ l(P))_P$$

or

$$l(Q) = LK(\cdot,Q) \qquad (4\text{-}3)$$

and thereby generally

$$L(f) = (f(Q),\ l(Q)) = (f(Q),\ LK(\cdot,Q)).$$

We see that

$$\|L\|_*^2 = (l(Q),\ l(Q)) = L(l) = LLK(\cdot,\cdot), \qquad (4\text{-}4)$$

i.e. we may compute the norm of L by evaluating L two times on the reproducing kernel with respect to its two variables.

In a similar way we easily see that

$$(L_1,\ L_2)_* = L_1L_2K(\cdot,\cdot). \qquad (4\text{-}5)$$

Example 4.6. Regard the Hilbert space defined in Example 4.3. The evaluation functional L_P has the norm

$$\|L_P\|_*^2 = (r^4 - R^4)R^2/(r^2 - R^2)^3$$

$$= (r^2 + R^2)R^2/(r^2 - R^2)^2,$$

and we see that for $r \to R$ will the norm of L_P tend to infinity, i.e. the boundary value functionals are not elements of H^*. ■

The equations (4-4) and (4-5) are very useful in case we want to construct the best linear approximation to a functional L from a set of given functionals L_i.

Using eq. (3-7) we see that the coefficients of the normal equations $(L_i,L_j)_*$ and the right-hand side $(L_i,L)_*$ simply are $L_iL_jK(\cdot,\cdot)$ and $LL_iK(\cdot,\cdot)$.

4.3. Least-squares or least norm collocation

In a Hilbert space with reproducing kernel we may find a unique element which agrees with the values $L_i(f)$ corresponding to a given set of linear functionals $L_i \in H^*$. The unique solution is the element $\tilde{f}$ of H which has the least norm and which fulfills $L_i(f) = L_i(\tilde{f})$, $i = 1,\dots,n$. This approximation $\tilde{f}$ to f is the projection of f on the linear space spanned by the functions $L_i K(\cdot,Q)$,

$$\tilde{f}(Q) = \sum_{i=1}^{n} a_i L_i K(\cdot,Q) \tag{4-6}$$

where the constants a_i are determined by the normal equations

$$\{L_i L_j K(\cdot,\cdot)\}\{a_j\} = \{L_i(f)\}. \tag{4-7}$$

$\tilde{f}$ will be the best approximation to f in the sense of section 3-4 with $f_i = L_i K(\cdot,Q)$, $i=1,\dots,n$.

For a proof of the uniqueness see Tscherning (1975,p.89). The method of determining $\tilde{f}$ is denoted *least-squares or least norm collocation* because the value of the norm, or equivalently of the quadratic form $\|\tilde{f}\|^2$ is minimized.

For approximations $\tilde{f}$, absolute error bounds may be computed, provided $f \in H^*$. For $L \in H^*$ we have

$$\begin{aligned}|L(f)-L(\tilde{f})| &= |L(f-\tilde{f})| = |L(f) - \{LL_iK(\cdot,\cdot)\}^T \\ &\quad\cdot\{L_iL_jK(\cdot,\cdot)\}^{-1}\{L_j(f)\}| = |(L - \{LL_iK(\cdot,\cdot)\}^T \\ &\quad \{L_iL_jK(\cdot,\cdot)\}^{-1}\{L_j\})(f)| \\ &\le \|f\|\cdot\| L - \{LL_iK(\cdot,\cdot)\}^T\{L_iL_jK(\cdot,\cdot)\}^{-1}\{L_j\}\|_* .\end{aligned}$$

By some simple manipulations we get

$$|L(f) - L(\tilde{f})| \le \|f\|\cdot(\|L\|_*^2 - \{L_iLK(\cdot,\cdot)\}^T\{L_iL_jK(\cdot,\cdot)\}^{-1} \cdot\{LL_jK(\cdot,\cdot)\})^{\frac{1}{2}}. \tag{4-8}$$

Example 4.7. Let the Hilbert space H be the one defined in Example 4.3, but with Ω being bounded by a sphere totally enclosed in the Earth. Let the linear functionals $L_i, i=1,\ldots,n$ be functionals associated with gravity anomalies observed in n points at the surface of the Earth, cf. eq.(2-5). A simple computation using eq. (4-4) will show that $\|L_i\| < \infty$, i.e. $L_i \in H^*$.

An approximation ($\tilde{T}$) to the anomalous potential of the Earth (T) may be computed using eq. (4-6) and (4-7). Error bounds can not be computed because T will not be an element of H because T is not harmonic down to the bounding sphere. (Note that $\tilde{T}$ will be a solution to the so-called Bjerhammar problem, cf. Heiskanen and Moritz (1967,p.321).) ■

5. Linear operators

5.1. The linear operator and its adjoint

An operator

$$A : X_1 \to X_2$$

is a mapping from one vector space X_1 (domain) to another vector space X_2 (range). It is called *linear* when

$$A(af + bg) = aA(f) + bA(g) \tag{5-1}$$

for any $f, g \in X_1$ and any real (or complex) numbers a,b.

Example 5.1. The n x m matrix $A = \{a_{ij}\}_{n\times m}$ represents a linear operator from R^n to R^m through

$$\{y_j\} = \{a_{ij}\}\{x_i\} \quad \text{or} \quad y = Ax$$

with $x = \{x_i\} \in R^n$ and $y = \{y_i\} \in R^m$. ■

Example 5.2. When X_2 is the set of real numbers are the linear operators the linear functionals on X_1. ■

Example 5.3. Let X_1 be a Hilbert space H, let X_2 be the dual space H^* and let us denote the duality mapping introduced in section 3.7 by J. This mapping is a linear operator from H to H^* which to an element l associates the linear functional $J(l) = L$, where for $f \in H$, $L(f) = (l,f)$. The mapping is isometric, i.e. $\|l\| = \|J(l)\|_*$.

When X is a Banach space and X^* the dual space of bounded linear functionals, X^* is called *strongly convex* when for any $L_1, L_2 \in X^*$ and $L_1 \neq L_2$ we have

$$\|L_1\|_* = \|L_2\|_* = 1 \Rightarrow \|L_1+L_2\|_* < 2. \qquad (5\text{-}2)$$

When this equation is fulfilled there will as for Hilbert spaces exist an isometric duality mapping $J : X \to X^*$.■

For an element $L \in X^*$ we will write the result of the functional applied on an element $f \in X$ as

$$\langle L,f\rangle = L(f). \qquad (5\text{-}3)$$

Example 5.4. For a Hilbert space we have

$$\langle L,f\rangle = (l,f) = L(f),$$

where l is the representer of L, i.e. $J(l) = L$.■

The mapping

$$\langle\cdot,\cdot\rangle : X^* \times X \to R$$

is called the duality pairing between the two spaces because we to each pair of functions and functionals have associated a real number. Using this pairing an operator A^* can be introduced working from X_2^* to X_1^* which is denoted the adjoint operator. The operator is defined through the condition

$$\langle L, A(f)\rangle_2 = \langle A^*(L),f\rangle_1$$

with $L \in X_2^*$ and $f \in X_1$ and we use the subscripts 1 and 2 to indicate that we deal with the pairings between X_1, X_1^* and X_2, X_2^*. When X_1 and X_2 are Hilbert spaces will A^* define an operator from X_2 to X_1, mapping the representer of $A^*(L)$ in the representer of L. This mapping will also be denoted the adjoint, A^*.

Example 5.5. The transposed, A^T, of the matrix A used in Example 5.1 maps $R^m \to R^n$, and represents the adjoint operator,

$$x = A^T y.$$

The duality pairing is then the usual scalar product,

$$y^T Ax = <y,Ax>_m = <A^T y,x>_n = x^T A^T y. \blacksquare$$

Example 5.6. Let the matrix A map $\mathbb{R}^n$ in $\mathbb{R}^n$. When A is symmetric will the operator and its adjoint be identical. Such an operator is called self-adjoint.■

In many cases it is of advantage to investigate the properties of A^* instead of A. When for example X_2 is a finite dimensional subspace of X_1, A^* will be a mapping from the finite dimensional space X_2^* to X_1^* and it will have a finite number of eigenvalues and eigenvectors.

Example 5.7. When applying collocation for the determination of an approximation to the anomalous potential of the Earth, T, a projection operator Pr_n is constructed which projects T on the n-dimensional subspace spanned by the functions $L_i K(\cdot,Q)$, $i=1,\dots,n$,

$$Pr_n(T) = \{L_i K(\cdot,Q)\}\{L_i L_j K(\cdot,\cdot)\}^{-1}\{L_j(T)\},$$

cf. eq.(4-6) and (4-7).

The adjoint operator Pr_n^* will be a mapping from the n-dimensional subspace of H^* spanned by the linear functionals L_i, $i=1,\dots,n$ and to the space H^*.■

Example 5.8. Bounded linear operators in a reproducing kernel Hilbert space, H.

Let H have the reproducing kernel K(P,Q), $P,Q \in \Omega \subset \mathbb{R}^n$. Then let $A : H \to H$ be a bounded linear operator with adjoint A^*. Then

$$(Af,g) = (f,A^*g)$$

and we may define

$$A(P,Q) = A_P^* K(P,Q),$$

where the subscript P indicates that A^* operates on K(P,Q) regarded as a function of P only.

Using Λ the operator may be expressed by means of the inner product as

$$A(f) = \{f(P),\ \Lambda(P,Q)\}_P$$

because

$$\{f(P),\ \Lambda(P,Q)\}_P = \{f(P),\ A_P^{*}K(P,Q)\}_P$$

$$= \{Af(P),\ K(P,Q)\}_P = Af(P).$$

It is interesting to see that we just as for linear functionals (cf. section 4.2) have a simple representation of this important class of linear operators.■

The set of continuous linear operators from X_1 to X_2 is denoted $L(X_1,X_2)$ and it is a linear vector space. In the space a norm may be introduced by

$$\|A\| = \sup_{\|f\|=1} \|A(f)\| .$$

Hence we are able to compare two operators and to see whether a sequence of operators A_i, $i=1,\dots,\infty$ converges towards an operator A,

$$\lim_{i\to\infty} \|A_i - A\| = 0.$$

5.2. The inverse operator

It is frequently of importance to know whether the inverse operator

$$A^{-1} : X_2 \to X_1,$$

$$A^{-1}A = I \qquad \text{(I is the identity operator in } X_1\text{)},$$

does exist. A necessary and sufficient condition for the existence of A^{-1} is that

$$Af = 0 \Leftrightarrow f = 0$$

and that the range of A is all of X_2.

Example 5.9. Let X_1 be the Hilbert space of functions harmonic outside a sphere and regular at infinity equipped with the norm

$$||f||^2 = \int_\Omega \sum_{|\alpha|=3} \left[\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}\right]^2 d\Omega$$

which contains no zero and first order harmonics. The boundary value operator, which to an element of X_1 associates the gravity anomalies on the sphere will as range X_2 have a Hilbert space of continuous functions at the boundary. The inverse operator is Stokes integral operator, cf. Heiskanen and Moritz (1967,eq.(2-161)).∎

Example 5.10. Let X_1 be the set of bounded functions with support in $\Omega \subset R^3$ and let X_2 be the range of the *Newton potential-operator*

$$A(f) = k\int_\Omega f(P)\cdot \frac{1}{||P-Q||}\, dP,$$

restricted to the complement of Ω, i.e. $A(f)$ is harmonic in $C\Omega$. (k is the gravitational constant).

It is well-known that the inverse operator does not exist. But if we restrict the space X_1 to consist of the harmonic functions with compact support will A^{-1} exist, cf. eg. Weck (1972).∎

Exercises

E 1. Orthonormalize the functions

$$1,\ x,\ x^2,$$

with respect to the inner product

$$\int_{-1}^{1} f\cdot g\, dx = (f,g).$$

The three functions form the orthonormal base in a three-dimensional Hilbert space H, equipped with this inner product. What is the reproducing kernel of this space?

A function

$$g(x) = -2x + 1$$

is an element of H. Compute $\|g\|$ and write down the Fourier expansion of g.

E 2. Let $H^1(\Omega)$ be the Hilbert space of functions - harmonic outside a sphere with radius R and regular at infinity equipped with the norm

$$\|f\|^2 = \frac{1}{4\pi}\int_\Omega (f^2 + (\nabla f)^2)d\Omega.$$

Compute the norm $\|V_{ij}\|^2$ of the solid spherical harmonics used in example 2.4. (Hint: use Example 4.4.)

E 3. (E 2 continued.) What is the reproducing kernel of $H^1(\Omega)$?

E 4. Regard a separable Hilbert space (H) of continuous functions $f : \Omega \to R$, which has a reproducing kernel. Suppose that there in Ω exists an ε-net of points P_i, $i=1,\ldots,\infty$. (An ε-net has the property that for any point P and any ε a point P_i exists so that the distance from P to P_i is less than ε). Prove that the functions $K(P_i,Q)$, $i=1,\ldots,\infty$ form a complete set in H or equivalently that the evaluation functionals L_{P_i} form a complete set in H^*. (Hint: use the definition of completeness given in section 3.6.)

E 5. In a Hilbert space of regular harmonic functions the reproducing kernel is

$$K(P,Q) = \sum_{i=2}^{\infty} \frac{1}{(i-1)^2}\left(\frac{R^2}{r\,r'}\right)^{i+1} P_i(\cos\psi).$$

Compute the norm of the Δg-functional

$$L(T) = -\frac{\partial T}{\partial r}\Big|_P - \frac{2}{r}\,T(P),$$

for $r = \sqrt{2}\cdot R$. (Hint: use eq.(4-4).)

E 6. A function f is an element of the Hilbert space H defined in E 1 and $\|f\| = \|g\|$ (also defined in E 1). The values of the function are known for x = 0 and $x = \frac{1}{2}$,

$$L_o(f) = 1 \quad \text{and} \quad L_{\frac{1}{2}}(f) = 0.$$

Compute an approximation $\tilde{f}$ to f using least norm collocation (i.e. so that $L_o(\tilde{f}) = 1$, $L_{\frac{1}{2}}(\tilde{f}) = 0$ and $\|\tilde{f}\| = \text{min.}$). Compute using eq. (4-8) the maximal error which may occur for x = 0 and $x = -\frac{1}{2}$.

E 7. An approximation $\tilde{T}$ has been computed using collocation. Compute $\|T\|$ as expressed through the observations $L_i(T)$ and the reproducing kernel.

E 8. Let V_i, $i=1,\ldots,\infty$ be an orthonormal basis for a (separable) Hilbert space, H. Prove that the sequence of functions

$$f_n = \sum_{i=1}^{n} V_i, \qquad n=1,\ldots,\infty$$

does not converge towards an element of H. Then prescribe a norm for the linear vector space

$$S_o = \text{span}\{V_i,\ i=1,\ldots,\infty\}$$

for which the sequence will converge towards an element in the completion, $\bar{S}_o$.

E 9. Let T have the expansion

$$T(P) = \sum_{i=2}^{\infty} \sum_{j=-i}^{i} a_{ij} V_{ij}(P)$$

with respect to the orthonormal base in example 2.4. Let there be given a (new) Hilbert space of regular harmonic functions with reproducing kernel

$$K(P,Q) = \sum_{i=2}^{\infty} \sigma_i \left(\frac{R^2}{r\, r'}\right)^{i+1} P_i(\cos\psi)$$

with

$$\sigma_i = \sum_{j=-i}^{i} a_{ij}^2 .$$

Prove that T is not an element of this space. (A proof can be found in Tscherning (1977).)

E 10. Let there be given an ε-net of points (see E 4) on the *surface* of the Earth, which we suppose is spherical. Suppose that in each of these points P_i, $i=1,\ldots,\infty$ the gravity anomaly is given, so that

$$L_i(T) = \Delta g(P_i).$$

For each set of i points $n=1,\ldots,n$ an approximation $Pr_n(T)$ is constructed as explained in Example 5.7. Prove that $Pr_n(T)$, $n=1,\ldots,\infty$ will converge pointwise towards T provided $T \in H$ and that H contains no zero and first order solid spherical harmonics. (Hint: use E 4 and the properties of the corresponding "continuous" boundary value problem.)

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