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Andreas Vogel (Ed.)

Model Optimization in Exploration Geophysics 2

Proceedings of the 5th International Mathematical Geophysics Seminar held at the Free University of Berlin, Feb 4-7, 1987

Editorial Advisers
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REPRINT

Friedr. Vieweg & Sohn   Braunschweig/Wiesbaden

Quasi-harmonic Inversion of Gravity Field Data

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Abstract

Density distributions (ρ) and the outer potential are in a one-to-one correspondence if ρ is supposed to be a solution to a partial differential equation similar to the Laplace equation. Examples of such equations are

$$\Delta(\rho/p(r)) = 0,$$

where Δ is the Laplace operator and $p(r)$ is a polynomial only dependent on the distance from the origin, r. We denote the functions, which are solutions to the equation as quasi-harmonic.

Using $p(r) = r^n$, n an integer, spherical harmonic expansions of the gravity potential and local gravity data have been inverted into density anomaly distributions. The local results are obtained using least-squares collocation. This method not only permits the computation of density anomaly values, but also of error estimates of the values. This may be useful when planning gravity surveys for geophysical exploration purposes.

However, the use of quasi-harmonic density anomaly models with $p(r) = r^n$ gives results with maximal and minimal values at the surface (sphere) bounding the densities. This problem may be solved if general polynomials are used.

1. Introduction

It is well known that there is no unique correspondence between the exterior gravity potential and the set of density distributions which produce the potential. However, restrictions may be put on the set of density distributions, so that a one-to-one correspondence is obtained. An example of such a restriction is the condition that the density distribution, ρ, must satisfy the following equation

$$\Delta(\rho/p(r)) = 0, \tag{1}$$

where $p(r)$ is a polynomial, r the distance from the Earth's center, and Δ the Laplace operator. We will denote functions which fulfil eq.(1) quasi-harmonic.

Also other types of partial differential operators may be used, such as the bi-harmonic operator (Skorvanek, 1981). The use of this operator makes it necessary to use two types of boundary data, e.g. both the value of the potential and of the density at the boundary, in order to have a one-to-one correspondence.

In the practical world of exploration geophysics, uniqueness is obtained by using density distributions which are elements of a finite dimensional inner product linear vector space, or of subsets of such spaces consisting of only the positive functions. This is feasible, because we do not know the complete exterior potential, but only the values of a finite number of functionals applied on the potential.

In the finite dimensional setting, density estimates are obtained using least-squares or minimum norm principles or a combination of the two. Which principle is used depends on whether the number of observations is larger than or smaller than the dimension of the space and on how linear dependent the base functions and the observation functionals are, respectively.

There is a close connection between the quasi-harmonic condition for uniqueness and the finite dimensional, under-determined, minimum norm solution method.

Suppose we use as base functions a set of non-overlapping indicator functions covering the Earth. Those completely inside the Earth have cubic shape, with side length $1/n$, while those intersecting the boundary must be the intersection of the cube and the Earth. Suppose also that we have as data m observations of the potential regularly distributed at the Earth's surface, where m is smaller than the number of blocks. An approximation to the density is then obtained as a linear combination of the indicator functions, by requiring this function to minimalize the integral of its square and simultaneously agree with the m observations. Letting n and m go to infinity will give us a density distribution which we in the limit may expect becomes a harmonic function; see Sanso et al., (1986). (A complete proof of this conjecture is probably not difficult).

In practice, the use of infinite dimensional spaces, such as the set of functions fulfilling eq. (1) imposes no special difficulties. This is clearly seen in the case where the exterior potential is given through its expansion in (solid) spherical harmonics. This is shown in section 2.

If the data consist of values of (linear) functionals (gravity anomalies, for example) then an approximation may be constructed using least-squares or minimum norm collocation. These methods are based on the use of covariance functions or the equivalent reproducing kernel. These functions are product-sums of solid sperical harmonics. So the results obtained in section 2 relating these functions to density distributions may also be used when constructing cross-covariance functions between density (anomaly) values and quantities derived from the exterior gravity field. This is described in section 3. The results from section 2 and 3 are illustrated by global, regional and local examples. For the local examples we have also been able to compute estimates of the errors of the computed density values.

The examples illustrates some of the problems associated with the use of quasi-harmonic density functions. In the concluding section 5, we give some hints on their possible solution

2. Quasi-harmonic inversion of solid spherical harmonic series

From now on we will suppose that we work in spherical approximation. This is only possible if we work with anomalous quantities. We will suppose that we from our observations have subtracted the contribution from a normal potential, generated by a density reference model. The construction of such a consistent model is discussed in Tscherning and Sünkel (1981). We will also suppose that the contribution of masses outside the reference ellipsoid have been eliminated. In this situation, the anomalous gravity potential may be expanded in a series in solid spherical harmonics,

$$T(\varphi,\lambda,r) = \sum_{i=2}^{\infty} \frac{GM}{r} \left(\frac{R}{r}\right)^i \sum_{j=0}^{i} \bar{P}_{ij}(\sin\varphi)[\bar{C}_{ij}\cos j\lambda + \bar{S}_{ij}\sin j\lambda] \qquad (2)$$

Here φ is the latitude, λ the longitude, r the distance from Earth's center, GM the product of the gravity constant and the mass of the Earth, R the mean Earth radius (6371 km), $\bar{P}_{ij}$ the fully-normalized Legendre functions, and $\bar{C}_{ij}$, $\bar{S}_{ij}$ the coefficients of the series. The summation starts from 2 as a consequence of having subtracted the normal potential. In spherical approximation r is calculated as the sum of R and the height above the ellipsoid and φ is used as if it was equal to the geocentric latitude.

Let us now consider the simple case where the polynomial is equal to r^n, n an integer. Then the density anomaly function is expanded in internal solid spherical harmonics,

$$\rho(P) = \rho(\varphi,\lambda,r) = r^n \sum_{i=2}^{\infty} \left(\frac{r}{R}\right)^i \sum_{j=0}^{i} \bar{P}_{ij}(\sin\varphi)[\bar{c}_{ij}\cos j\lambda + \bar{s}_{ij}\sin j\lambda] \quad (3)$$

The potential of this anomalous density is

$$T(Q) = T(\varphi',\lambda',r') = G\int_0^R \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \frac{\rho(\varphi,\lambda,r)}{((r')^2+r^2-2rr'\cos\psi)^{\frac{1}{2}}} r^2\cos\varphi d\varphi d\lambda dr) \quad (4)$$

where ψ is the spherical distance between points P and Q with coordinates (φ, λ, r) and (φ', λ', r'), $\cos\psi = \sin\varphi'\sin\varphi' + \cos\varphi\cos\varphi'\cos(\lambda-\lambda')$, and $l = ((r')^2 + r^2 - 2rr'\cos\psi)^{\frac{1}{2}}$ is the distance between the two points. The inverse distance, 1/l, may be expanded in a product sum of internal and external fully normalized solid spherical harmonics, (Heiskanen and Moritz, 1967, eq. (1-83')),

$$\frac{1}{l} = \sum_{i=0}^{\infty} \frac{1}{2i+1} \frac{R^i}{(r')^{i+1}} \cdot \frac{r^i}{R^i} \sum_{j=0}^{i} \bar{P}_{ij}(\sin\varphi)\bar{P}_{ij}(\sin\varphi') \cdot$$

$$[\cos j\lambda \cdot \cos j\lambda' + \sin j\lambda \cdot \sin j\lambda'] \quad (5)$$

Inserting this in eq.(4) and using the orthogonality

$$\frac{1}{4\pi}\iint \bar{P}_{ij}(\sin\varphi)\bar{P}_{nm}(\sin\varphi) \begin{Bmatrix}\cos j\lambda\,\cos m\lambda \\ \sin j\lambda\,\sin m\lambda\end{Bmatrix} \cos\varphi d\varphi d\lambda = \begin{cases} 1 & i=n,\ j=m \\ 0 & i\neq n \text{ or } j\neq m\end{cases} \quad (6)$$

then

$$T(\varphi',\lambda',r') = 4\pi G \int_0^R \sum_{i=2}^{\infty} \frac{1}{(2i+1)} \frac{r^{2i+2+n}}{R^i(r')^{i+1}} dr \sum_{j=0}^{i} \bar{P}_{ij}(\sin\varphi') \times$$

$$[\bar{c}_{ij}\cos j\lambda' + \bar{s}_{ij}\sin j\lambda']$$

$$= G \sum_{i=2}^{\infty} \frac{4\pi}{(2i+1)(2i+n+3)} \frac{R^{i+3+n}}{(r')^{i+1}} \sum_{j=0}^{i} \bar{P}_{ij}(\sin\varphi') \times$$

$$[\bar{c}_{ij} \cos j\lambda + \bar{s}_{ij} \sin j\lambda] \quad (7)$$

A comparison between the coefficients of the series (7) and (2) then gives the following relation

$$GMR^i \cdot \left.\begin{matrix} \bar{C}_{ij} \\ \bar{S}_{ij} \end{matrix}\right\} = \frac{G\ 4\pi\ R^{i+3+n}}{(2i+1)(2i+3+n)} \left\{\begin{matrix} \bar{c}_{ij} \\ \bar{s}_{ij} \end{matrix}\right. \quad (8)$$

or

$$\left.\begin{matrix} \bar{c}_{ij} \\ \bar{s}_{ij} \end{matrix}\right\} = \frac{M}{4\pi R^3} \frac{(2i+1)(2i+3+n)}{R^n} \left\{\begin{matrix} \bar{C}_{ij} \\ \bar{S}_{ij} \end{matrix}\right. \quad (9)$$

The corresponding equation for a general polynomial $p(r)$ is easily derived from this equation. Note that for $n = 0$ and $i = 0$ then $\bar{c}_{00}$ is equal to $M/(\frac{4}{3}\pi R^3)$, which is the mean density of the Earth.

Using (9) the coefficients of a finite series expansion for the gravity potential (Rapp, 1981; Wenzel, 1985) minus a series expansion for the attraction of the isostatically compensated topography (Sünkel, 1986) may be used to compute the coefficients of the series (3). Standard software (Tscherning et al., 1983) for evaluating the exterior potential may be used also for the evaluation of these series. An artificial value $r^* = \frac{R^2}{r}$ for the radial distance is simply used instead of r.

3. Quasi-harmonic inversion of general gravity field data

Suppose we have N observations related to the anomalous gravity potential through N linear functionals

$$y_i = L_i(T) + e_i, \quad i = 1, \ldots\ldots, N. \quad (10)$$

Here e_i is the noise or error. Based on these observations an approximation $\tilde{T}$ to T may be constructed using minimum norm or least squares collocation. $\tilde{T}$ will be an element of a reproducing kernel Hilbert space with known reproducing or covariance kernel. Let the inner product of two functionals L_i and L_j be donoted C_{ij}. This is also called the covariance of the quantities $L_i(T)$ and $L_j(T)$. The covariance between an arbitrary functional L and L_i will be denoted C_{Li}.

$\tilde{T}$ is obtained by the requirement that eq. (10) is fulfilled and that $\tilde{T}$ and the error vector have minimum norm. (The error is supposed to be normally distributed with covariance matrix $\{D_{ij}\}$). Then an estimate $\tilde{L(T)}$ = $L(\tilde{T})$ is obtained as

$$L(\tilde{T}) = \{C_{Li}\}^T \{C_{ij} + D_{ij}\}^{-1} \{y_j\} \tag{11}$$

If $\tilde{T}$ is an element of the same Hilbert space as T then upper bounds for the error may be calculated. ($\|T\|$ must be known as well). If the so-called empirical covariance function is used, then mean-square error estimates may be obtained,

$$\sigma^2(L(\tilde{T}) - L(T)) = C_{LL} - \{C_{Li}\}^T \{C_{ij} + D_{ij}\}^{-1} \{C_{Lj}\} \tag{12}$$

It should now be obvious that we may obtain estimates of the density and of its error, if we can construct the covariance between a density value and a value of the anomalous potential. This is easily done for isotropic reproducing kernels or covariance functions. An example of such a covariance function is the already mentioned empirical covariance function or reproducing kernel. It is related to T through the coefficients $\bar{C}_{ij}$ and $\bar{S}_{ij}$,

$$K(P, Q) = \sum_{i=2}^{\infty} \sigma_i^2 \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\psi) \tag{13}$$

with

$$\sigma_i^2 = \left(\frac{GM}{R}\right)^2 \sum_{j=0}^{i} (\bar{C}_{ij}^2 + \bar{S}_{ij}^2), \tag{14}$$

the so-called degree-variances. In fact, all isotropic reproducing kernels are on the same form as eq. (13), where all σ_i^2 are positive or zero.

For quasi-harmonic density distribution the kernels (or covariance functions) are on a similar form. For $p(r) = r^n$ we have

$$K_\rho(P, Q) = \sum_{i=2}^{\infty} (r\, r')^n \tau_i^2 \left(\frac{rr'}{R^2}\right)^i P_i(\cos\psi). \tag{15}$$

The corresponding kernel related to the exterior potential is obtained from K_ρ (P, Q) using eq.(4) two times. We then find eq. (9) again (used two times), and from this

$$\tau_i^2 = \sigma_i^2 \left(\frac{M}{4\pi R^3}\right)^2 \frac{(2i+1)^2(2i+3+n)^2}{R^{2n}} \tag{16}$$

The cross-covariance function between the density anomaly in a point P and the anomalous potential in a point Q is then

$$K(T(P), \rho(Q)) = \sum_{i=2}^{\infty} \sigma_i \tau_i \frac{(r')^i}{r^{i+1}} P_i(\cos\psi); \quad (17)$$

see Tscherning,(1976,1977) for further details. Examples of auto - and cross - covariance functions are shown in Figs.(1) - (3). Using these results in eq. (11) and (12) we see that we are able to compute quasi-harmonic density anomalies, use density anomalies as observation, and compute error estimates of the computed values.

4. Applications

The use of eq. (9) with spherical harmonic expansions up to a finite degree is straightforward. We have used GPM2 (Wenzel, 1985) to varying degree, having subtracted the expansion of the topograpic-isostatic reduction potential, tic86 (Sünkel, 1986). In Fig. (4) is shown a global map of the density variation at the Earth's surface using the expansion up to degree 60. In Figs.(5) - (9) are shown density anomalies for depths equal to 0, 30 and 300 km and for maximal degree 60., 120 and 180 and a values of n = 0 for an area around India. Similar maps with values of n = -1, 1 and 2 have also been produced, but they were not different from the maps obtained with n = 0.

The method of least-squares collocation was used to compute residual density estimates for an area around Helsingør - Helsingborg in Denmark and Sweden, respectively; see Figure (11). Here precomputed gravity potential approximation was used (Tscherning and Forsberg, 1986), which was based on point data with a 6' spacing. Also data from a local area in Jylland was used around a known salt-dome; see Figs.(12) - (14). A map of the error of the estimated density anomalies at zero depth is shown in Fig. (15). The auto - and cross - covariance functions between gravity and density anomalies at zero depth and at depth 500 and 1000 m are these shown in Figs.(1) - (3). In all calculations the FORTRAN-program GEOCOL (Tscherning, 1986) was used with covariance function models as specified in the Appendix.

The resulting density anomalies seem in all cases to have the correct magnitude. However the deeper situated densities are just a smooth mirror of the densities at the Earth's surface. This is a key problem in the future use of quasi-harmonic density functions.

5. Conclusion

The use of quasi-harmonic density anomaly functions gives an easy way from gravity field data to density values (see also Eisfeller et al., 1985). The use of simple weight functions $p(r) = r^n$ is not satisfactory, since these functions are similar to harmonic functions which attain their maxima and minima at the boundary. This means, for example, that using $p(r) = r^n$, density anomaly functions, which first are numerically small and then increase, are not permitted. This behavior is, however, easily enforced using polynomials which behave the same way. Also functions, which are piecewise polynomials may be used. In this manner discontinuities in radial direction may be modelled. First when we want to model lateral discontinuities are the advantages of the quasi-harmonicity then lost. In this case the method of mixed-collocation (Sansò and Tscherning, 1982) should be used.

While the practical usefulness of the quasi-harmonic inversion method still has to be proved, there is no doubt that the ability to compute error estimates is useful. One may be able to see the influence of missing data, e.g. due to the occurrence of a water-covered area; see Fig. (15). And this may be used to make the decision whether it is worthwhile to use a sea-bottom gravimeter in order to acquire supplementary data.

Appendix

Covariance function models used.

The density anomaly maps Figs.(11)-(14) are calculated as the sum of a regional contribution derived from GPM2 minus TIC86 to degree 180 using eq.(3) and (9) plus a local contribution from residual gravity anomalies. Fig. 11 is based on only residual anomalies. The anomalies have a covariance function where the degree-variances from degree 0 to 180 are equal to the error degree-variances of GPM2 multiplied by a scale factor α. The coefficients from 181 to infinity are given through a model

$$\sigma_i^2 = \frac{A}{(i-1)(i-2)(i+B)} \left(\frac{R_B}{R}\right)^{2i+2} \qquad (18$$

The density anomalies in Fig. 11 have been computed using $\alpha = 0.5$, $B=4$, $R-R_B = 3.5$ km, and A is determined so that the gravity anomaly variance is equ to 225 mgal². The density anomalies in Figs.(12)-(14) have been computed usi $\alpha=0.5$, $B=4$, $R-R_B=2.5$km, and A is determined so that the gravity variance is mgal².

References

Eisfeller,B., K.Hehl, G.W.Hein and H.Landau: On the Incorporation of Geophysical data in an Integrated Model. In preparation, 1986.

Rapp, R.H.: The Earth's gravity field to degree and order 180 using SEASAT altimeter data, terrestrial gravity data, and other data. Reports of the Department of Geodetic Science and Surveying No. 322, The Ohio State University, Columbus, Ohio 1981.

Sanso ,F., R.Barzaghi and C.C.Tscherning: Choice of Norm for the Density Distribution of the Earth. Geoph.Jour.Royal Astr. Soc., Vol. 87, pp. 123-141, 1986.

Sanso, F. and C.C.Tscherning: Mixed Collocation: A proposal. Quaterniones Geodasiae, Vol. 3, no. 1, pp. 1-15, 1982.

Skorvanek, M.: The inverse gravimetric problem for the Earth. Proc. 4th Int. Symp. "Geodesy and Physics of the Earth", Karl-Marx-Stadt, May 12-17, 1980. Veroeff. Zentralinstituts Physik der Erde, Nr. 63, pp. 464-475, 1981.

Suenkel, H.: Global Topographic-Isostatic Models. In: H.Suenkel (Ed.), Mathematical and Numerical Techniques in Physical Geodesy. Lecture Notes in Earth Sciences Vol. 7, pp. 417-462, Springer Verlag, 1986.

Tscherning, C.C.: A Mass Density Covariance Function consistent with the Covariance Function of the Anomalous Potential. Bolletino di Geodesia e Scienze Affini, Vol. XXXV, No. 2, pp. 161-172, 1976a.

Tscherning, C.C.: Models for the Auto- and Cross Covariances between Mass Density Anomalies and First and Second Order Derivatives of the Anomalous Potential of the Earth. Proceedings 3rd Int. Symposium "Geodesy and Physics of the Earth", Weimar, October, 1976, pp. 261-268, Potsdam, 1977.

Tscherning, C.C.: GEOCOL - A FORTRAN-program for Gravity Field Approximation by Collocation. Technical Note, Geodætisk Institut, 6th ed., Oct. 1986.

Tscherning, C.C., R.H.Rapp and C.C.Goad: A Comparison of Methods for computing Gravimetric Quantities from High Degree Spherical Harmonic Expansions. Manuscripta Geodaetica, Vol. 8, pp. 249 - 272, 1983.

Tscherning,C.C. and R.Forsberg: Geoid determination in the Nordic countries from gravity and height data. Bol. di Geodesia e Sc. Aff., (in print), 1986.

Tscherning, C.C. and H. Suenkel: A Method for the Construction of Spheroidal Mass Distributions consistent with the harmonic Part of the Earth's Gravity Potential. Manuscripta Geodaetica, Vol. 6, pp. 131-156, 1981.

Wenzel,H.-G.: Hochaufloesende Kugelfunktionsmodelle fuer das Gravitationspotential der Erde. Wiss. Arb. Fachrichtung Vermessungswesen der Universitaet Hannover, (in print), 1985.

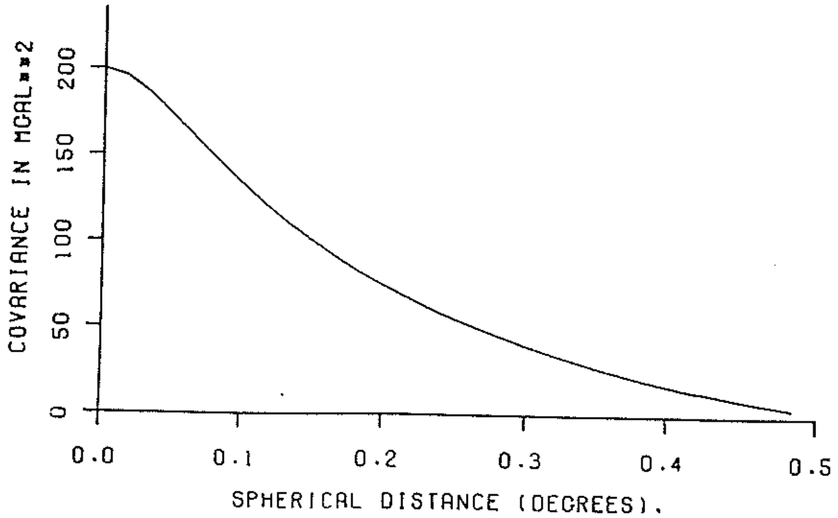

Fig.1 Gravity anomaly covariance function

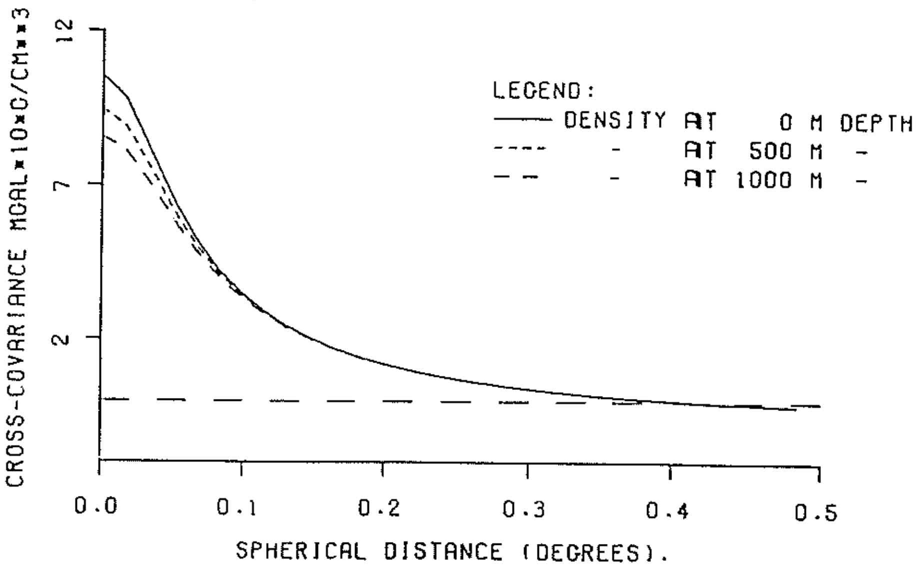

Fig. 2 Cross-covariance function between gravity anomalies and density anomalies at depth 0 m, 500 m and 1000 m.

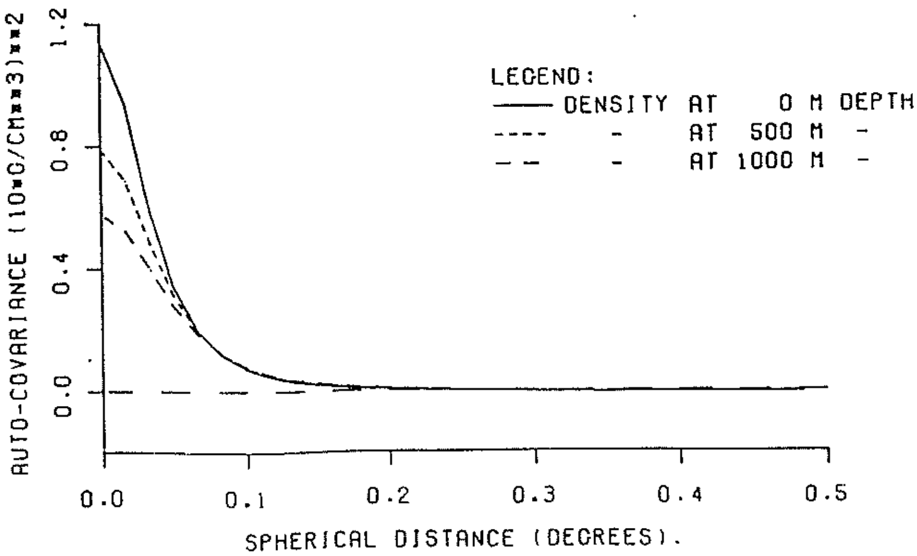

Fig. 3 Covariance function of density anomalies at depth 0 m, 500 m and 1000 m.

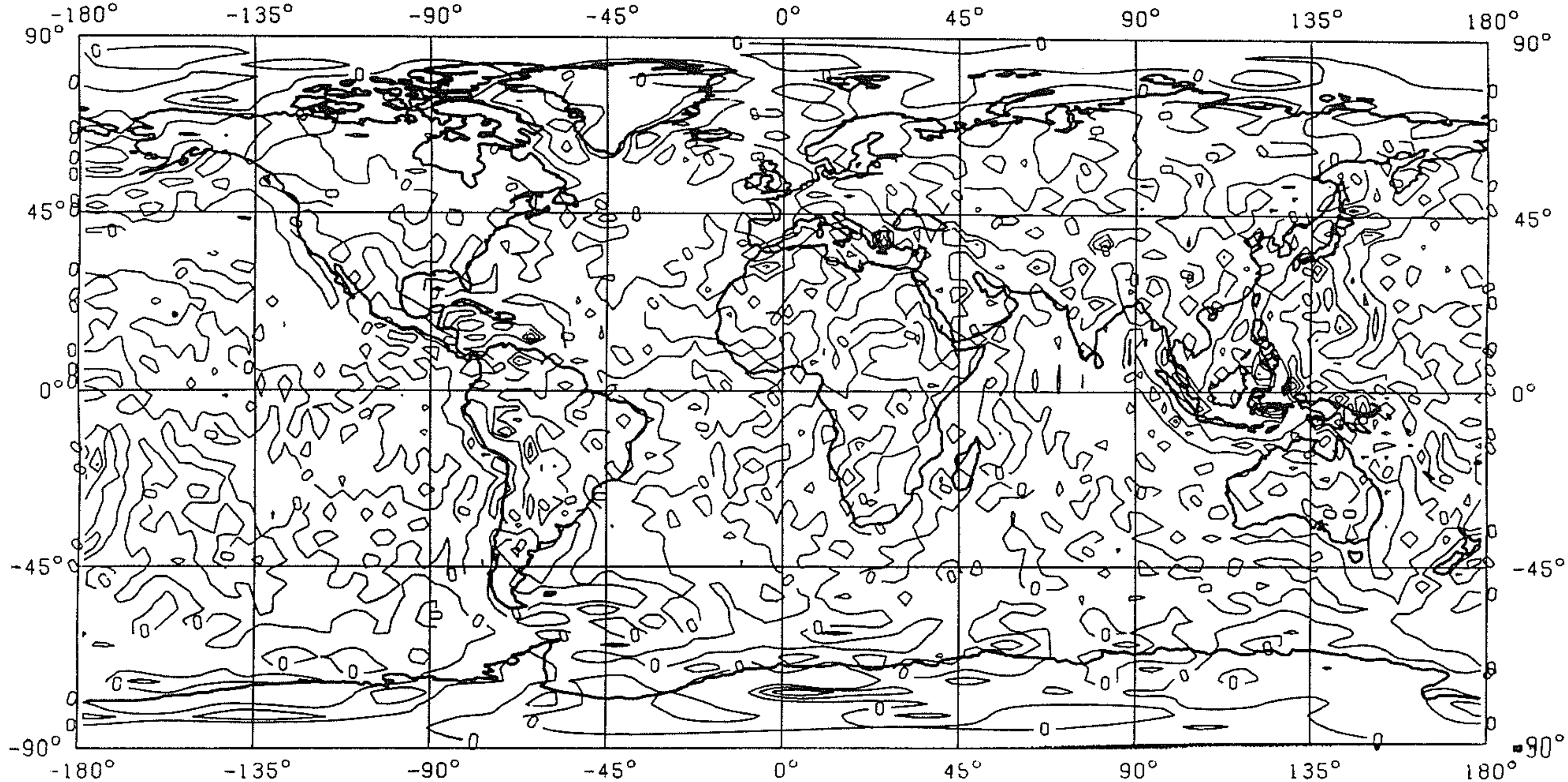

Fig. 4a Harmonic density anomalies from the coefficients of GPM2 minus TIC86 to the degree 60 at depth 0. Contour interval 5×10^{-3} g/cm^3. Only zero and positive values shown.

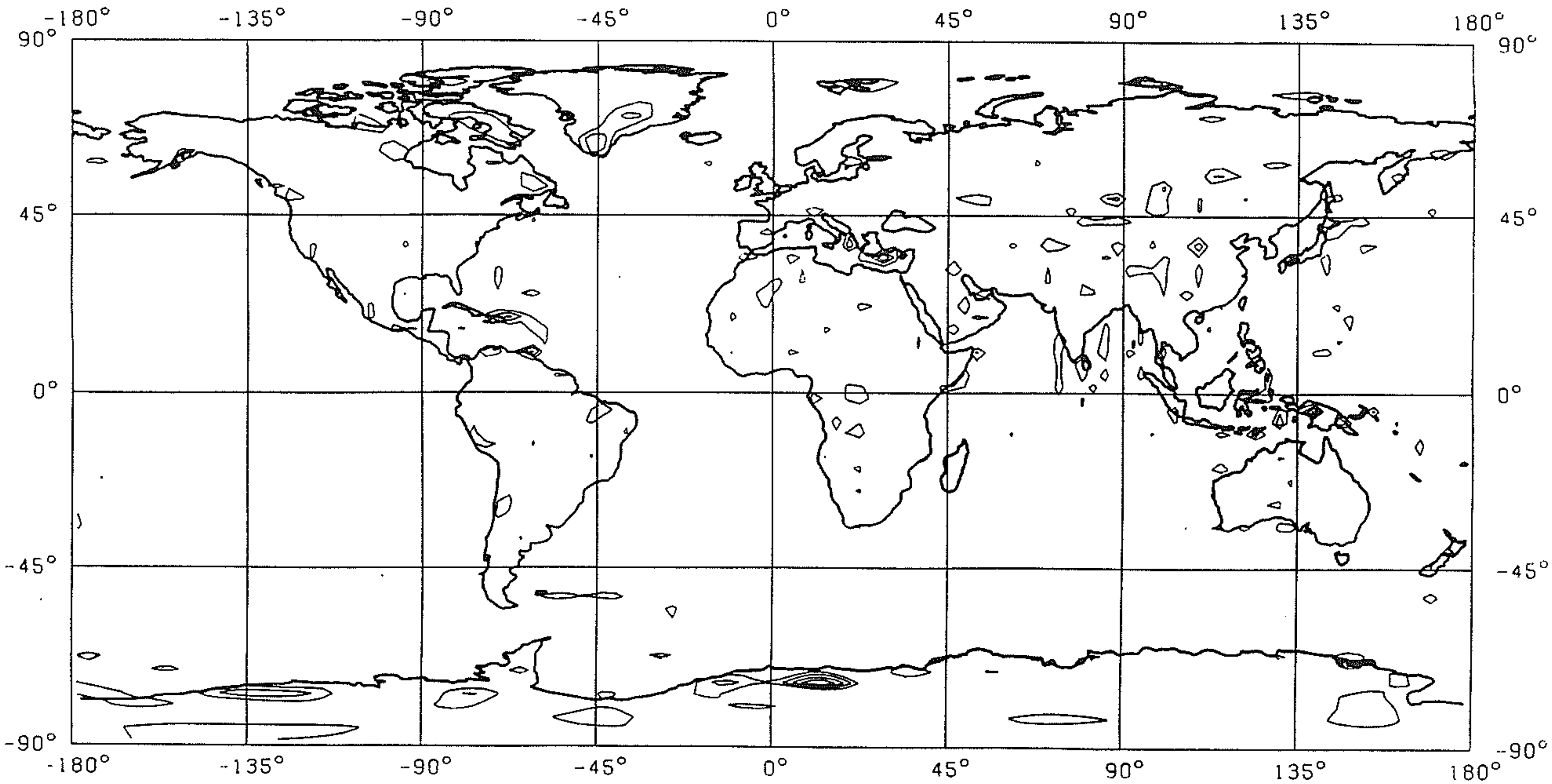

Fig. 4b Harmonic density anomalies from the coefficients of GPM2 minus TIC86 to the degree 60 at depth 0. Contour interval $5x10^{-3}$ g/cm^3. Only negative values shown.

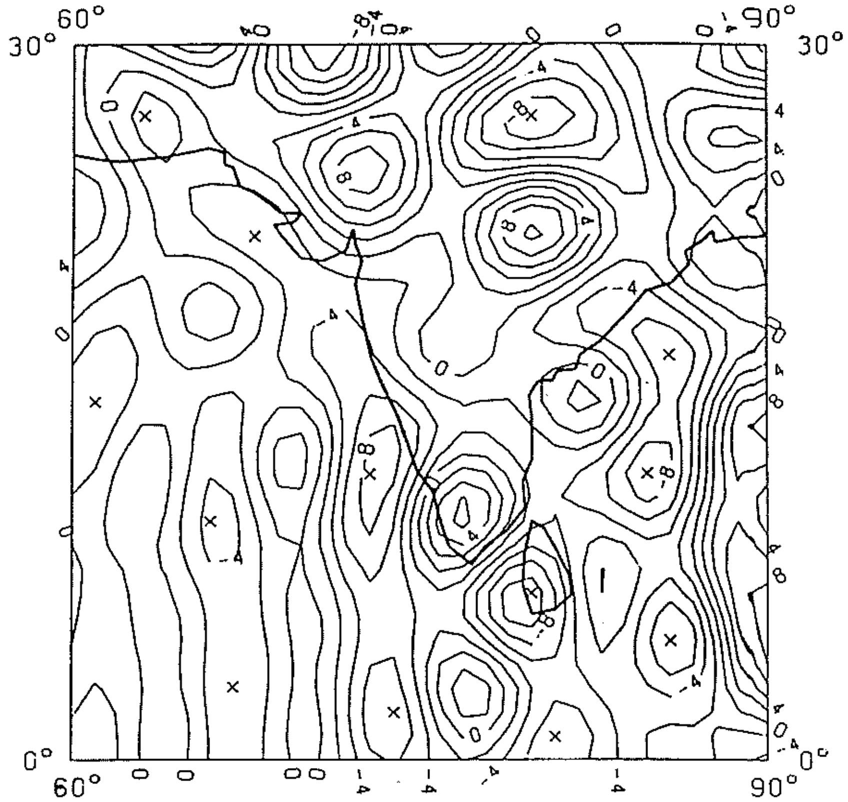

Fig. 5 Harmonic density anomalies from GPM2 minus TIC86 to the degree 60 at depth 0. Contour interval 2×10^{-3} g/cm³.

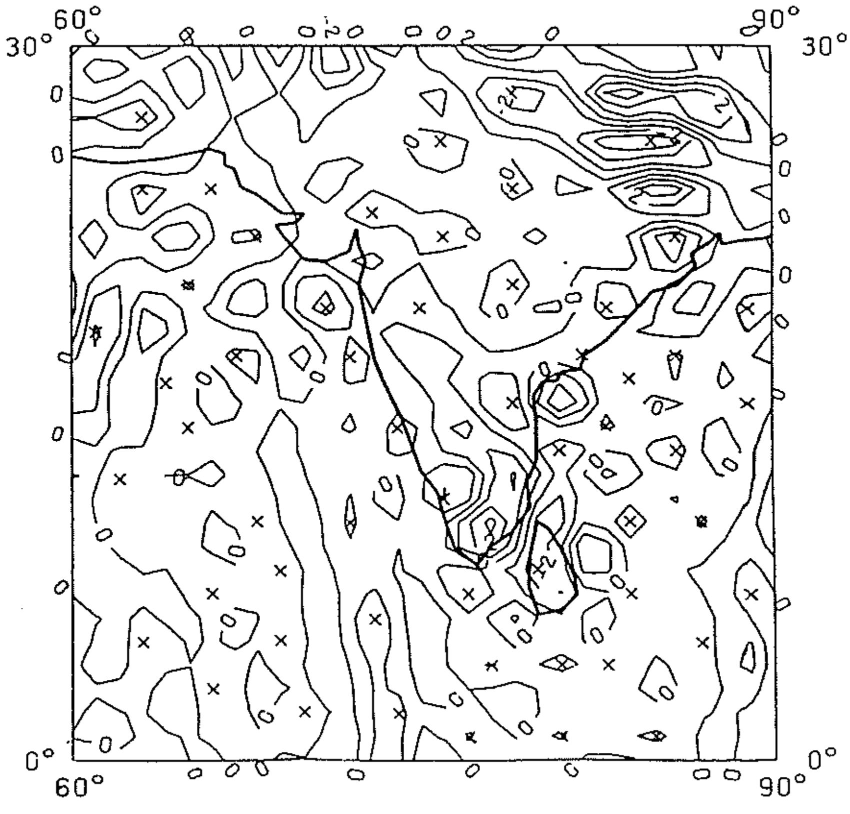

Fig. 6 Harmonic density anomalies from GPM2 minus TIC86 to the degree 120 at depth 0. Contour interval 10^{-2} g/cm³.

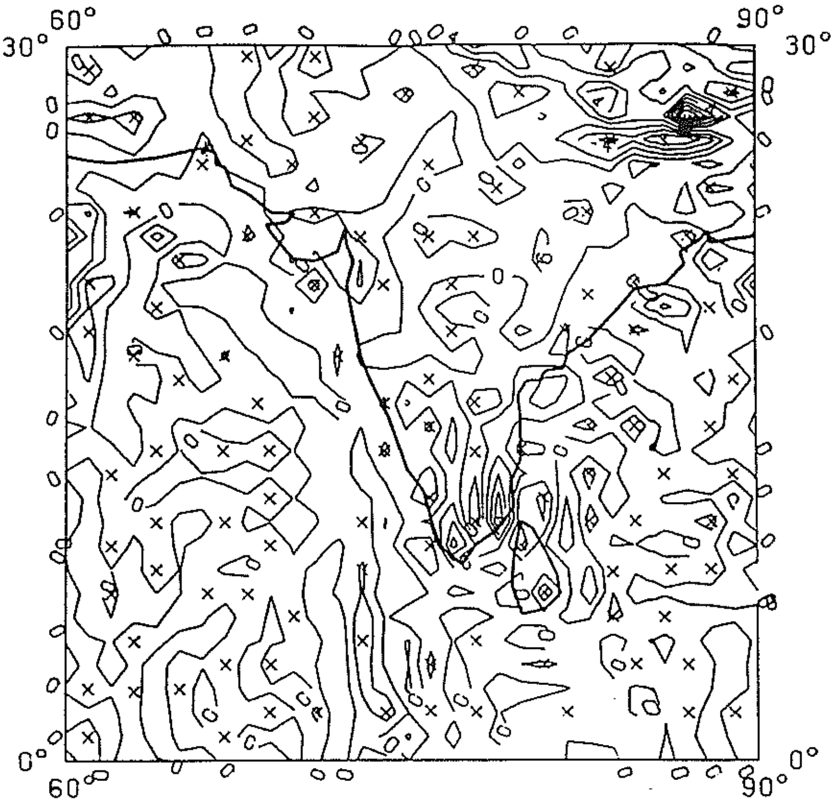

Fig. 7 Harmonic density anomalies from GPM2 minus TIC86 to the degree 180 at depth 0. Contour interval 2×10^{-2} g/cm³.

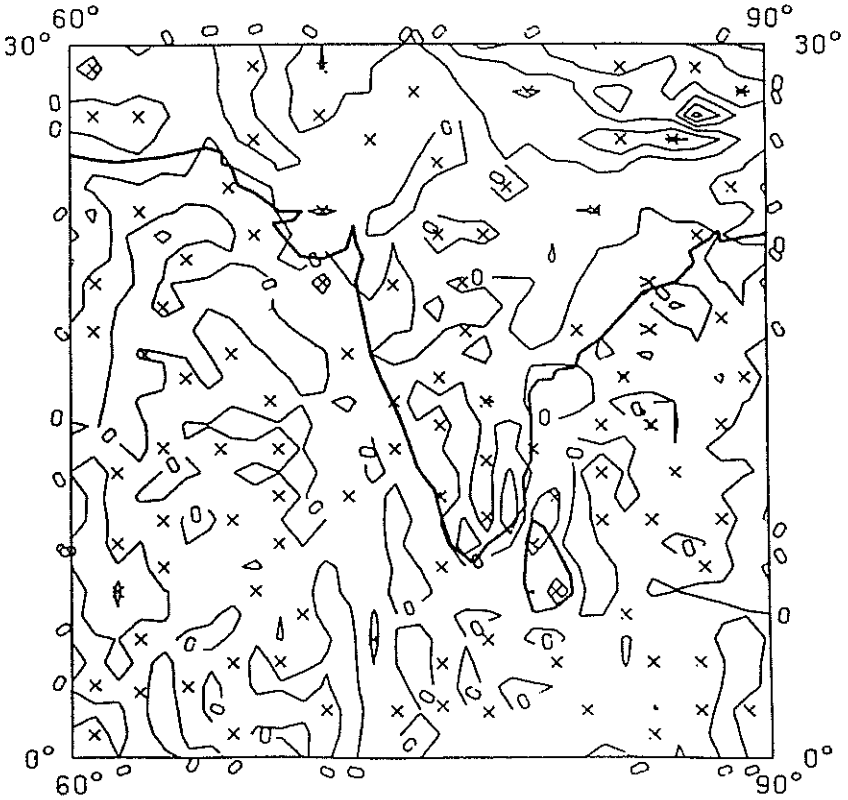

Fig. 8 Harmonic density anomalies from GPM2 minus TIC86 to the degree 180 at depth 30 km. Contour interval 2×10^{-2} g/cm³.

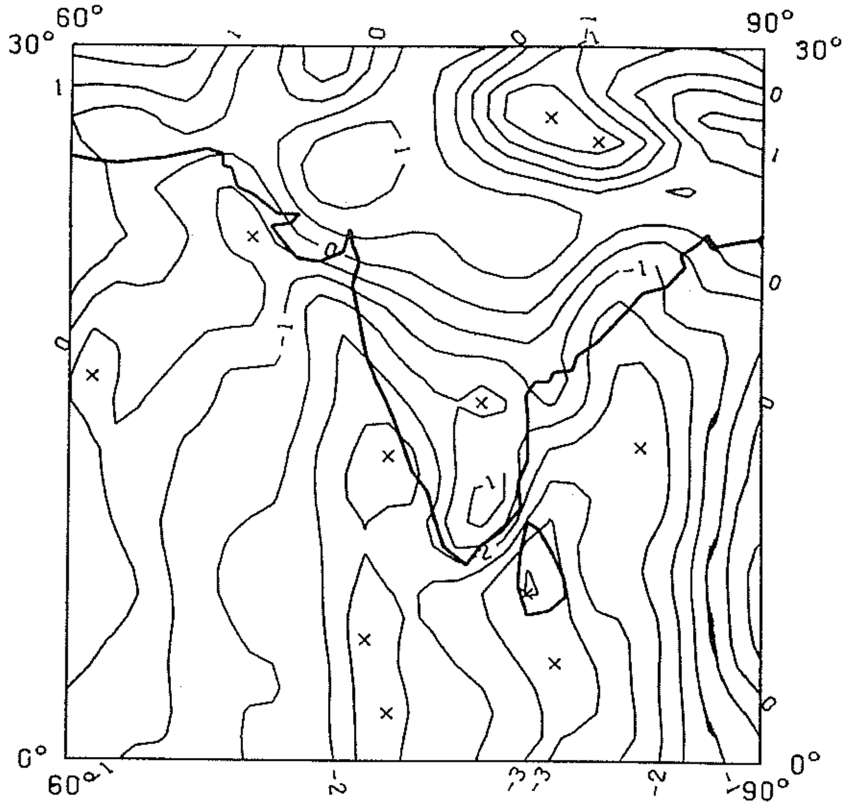

Fig. 9 Harmonic density anomalies from GPM2 minus TIC86 to the degree 180 at depth 300 km. Contour interval 5×10^{-4} g/cm³.

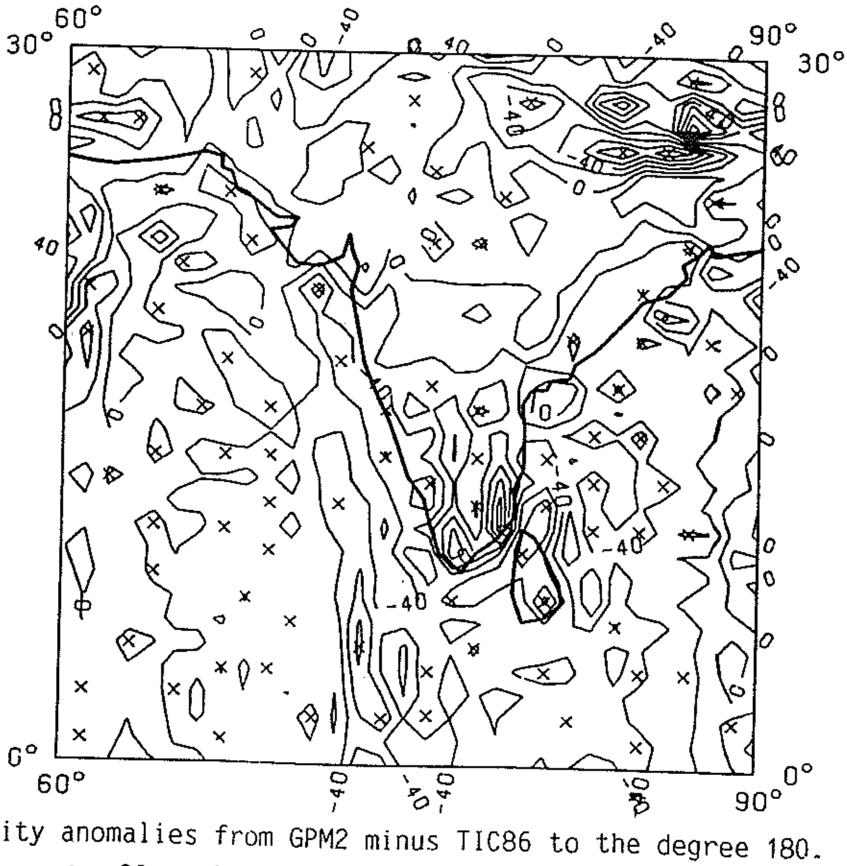

Fig.10 Gravity anomalies from GPM2 minus TIC86 to the degree 180. Contour interval 20 mgal.

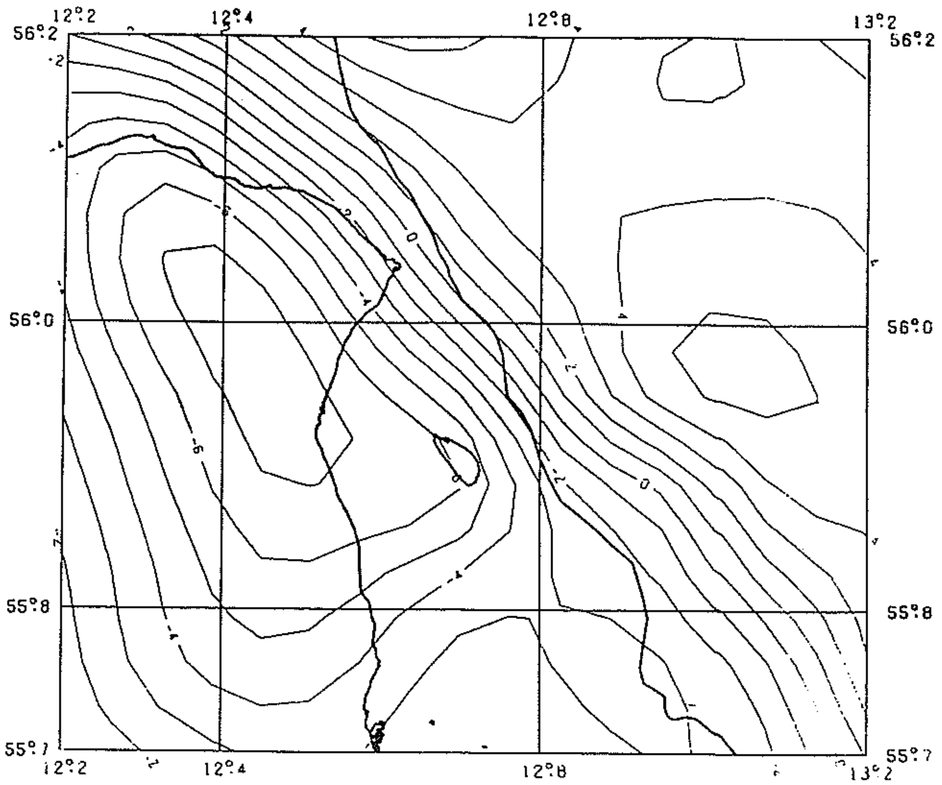

Fig. 11 Density anomalies from residual gravity anomalies at depth 0. Contour interval 10^{-2} g/cm³.

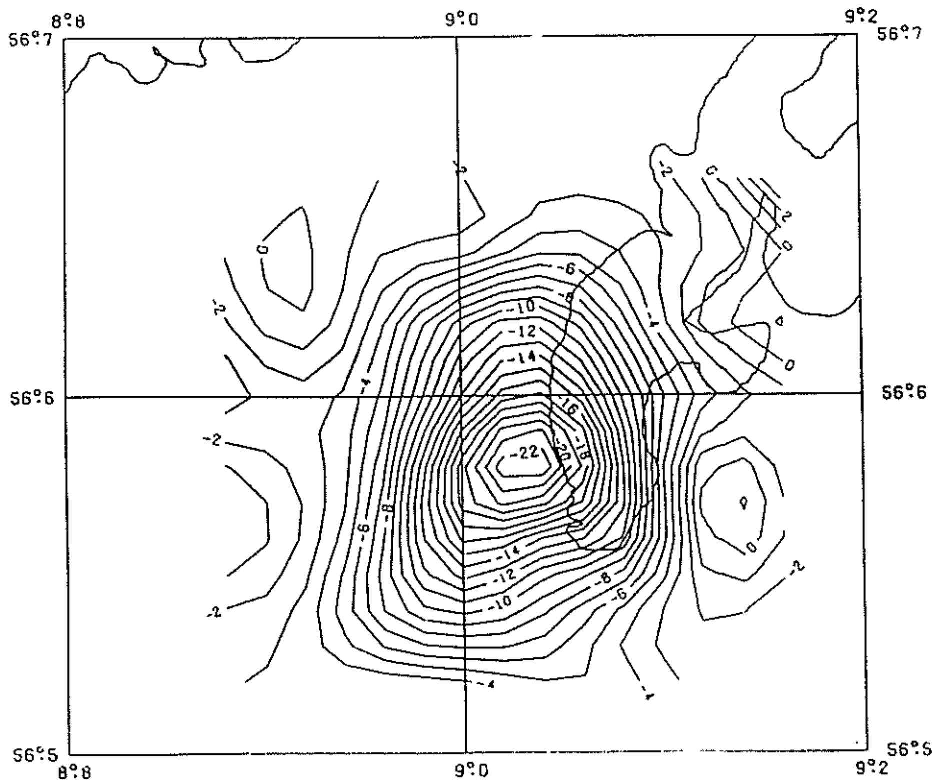

Fig. 12 Harmonic density anomalies from GPM2 minus TIC86 and local gravity data at depth 0. Contour interval 10^{-2} g/cm³.

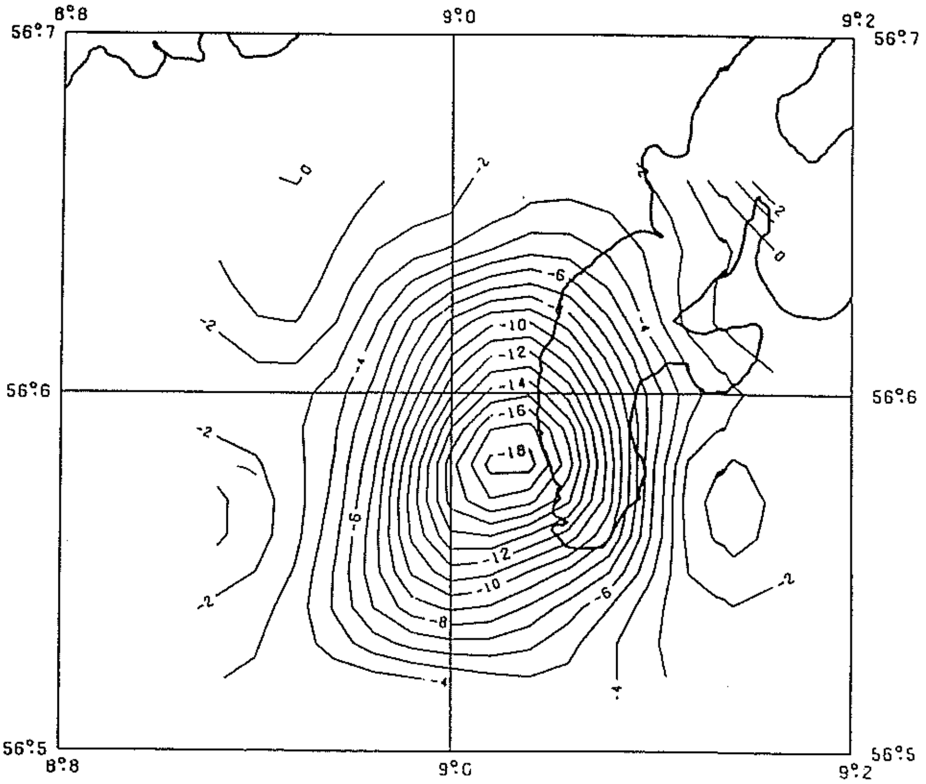

Fig. 13 Harmonic density anomalies from GPM2 minus TIC86 and local gravity data at depth 500 m. Contour interval 10^{-2} g/cm³.

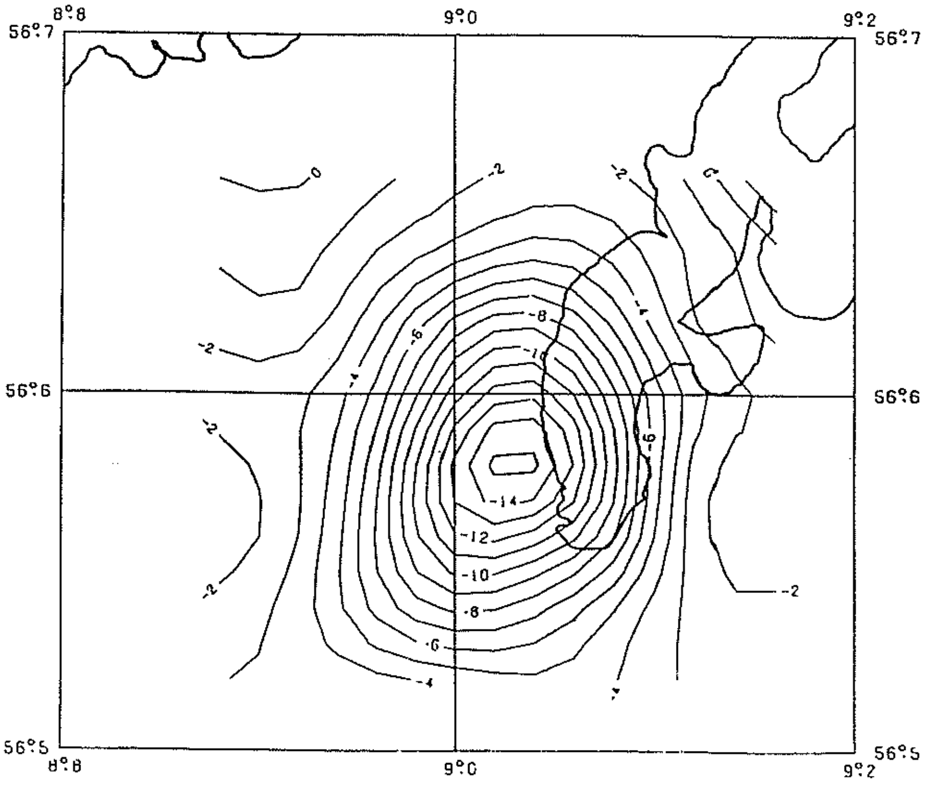

Fig. 14 Harmonic density anomalies from GPM2 minus TIC86 and local gravity data at depth 1000 m. Contour interval 10^{-2} g/cm³.

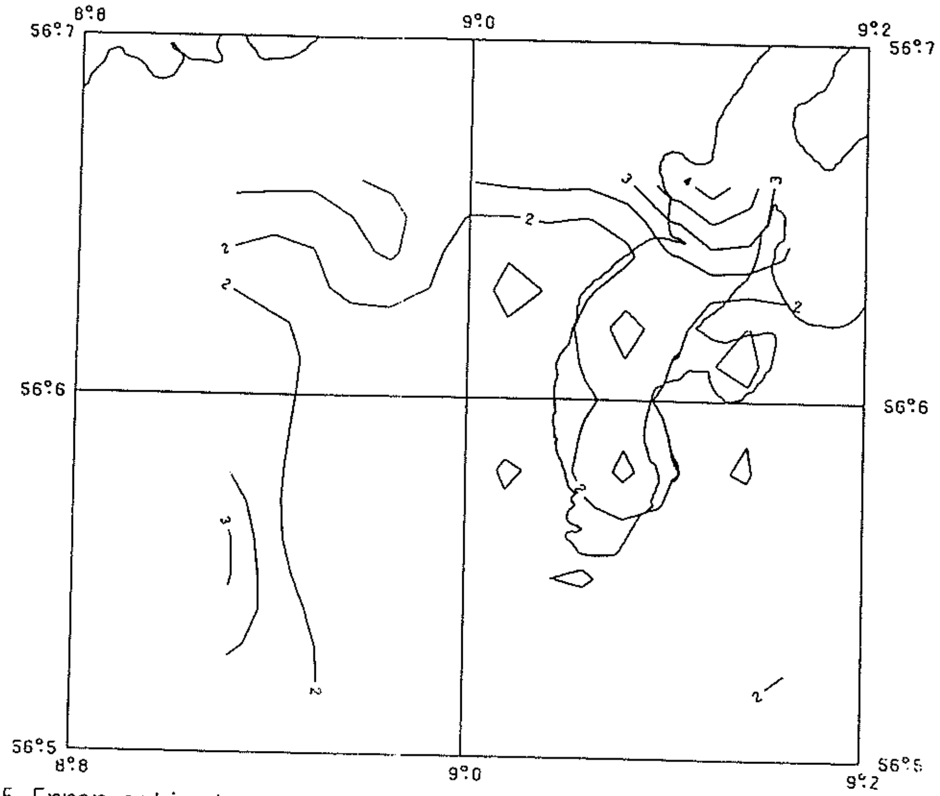

Fig. 15 Error estimates of harmonic density anomalies at depth 0. Contour 10^{-2} g/cm³.

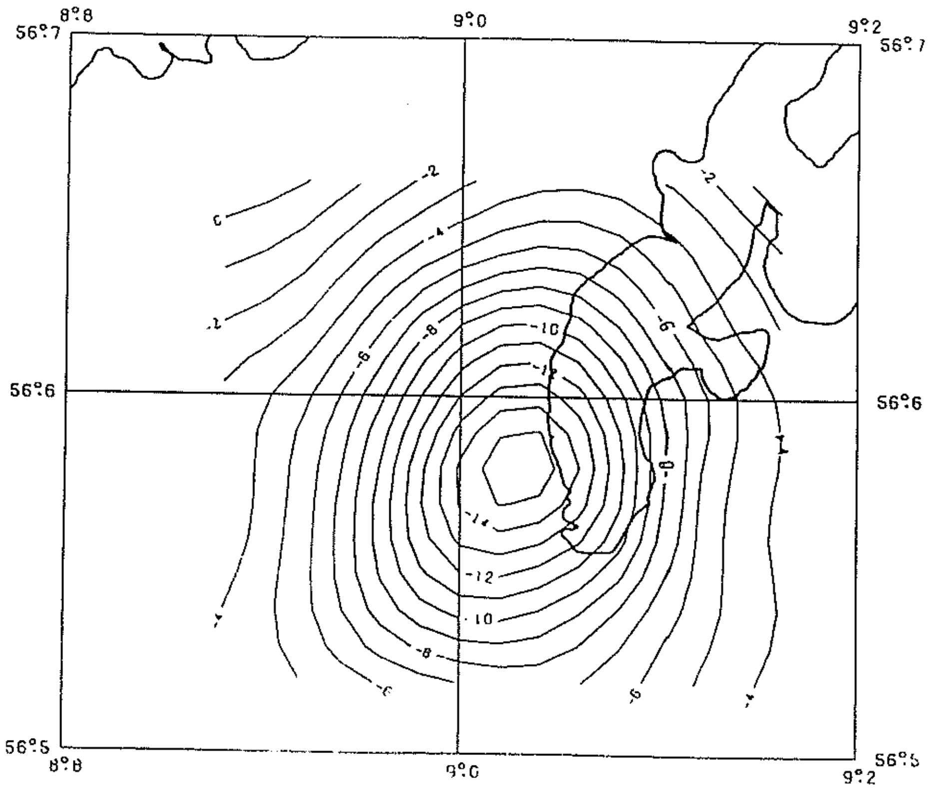

Fig 16 Gravity anomalies. Contour interval 1 mgal.