Models for the Auto- and Cross Covariances between Mass Density Anomalies and First and Second Order Derivatives of the Anomalous Potential of the Earth

by

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Summary

Using a combination of satellite determined potential coefficients and free-air gravity anomalies a global (isotropic) covariance function for the anomalous potential of the Earth was computed by TSCHERNING and RAPP in 1974.

The variation (as expressed for example through this covariance function) is uniquely determined by mass density anomalies, but the inverse relationship is not unique.

Uniqueness may be imposed by requiring for example the mass density anomaly function to be harmonic (and correspondingly the anomalous potential biharmonic inside the Earth). Under these and slightly different circumstances are cross covariances between density anomalies and derivatives of the anomalous potential derived. Corresponding to a free-air gravity anomaly variation of ± 42 mgal (at the surface of the Earth) the mass density variation becomes ± 0.80 g/cm^3 at the surface of the Earth and ± 0.04 g/cm^3 in 10 km's depth.

1. Introduction

The potential of the Earth (W), is the sum of the gravitational potential (V) and the rotational potential, where

$$(1) \quad V(P) = k \int_{\text{Earth}} \rho(Q) \, (\|P - Q\|)^{-1} \, dQ.$$

P and Q are points in R^3, k is the gravitational constant and ρ is the density distribution.

Let us suppose, that we have adopted a certain reference potential (U) (which includes the rotational part of W) and a corresponding density reference function. The difference $T = W - U$ is then denoted the anomalous potential. Denoting the difference between ρ and the reference density function by d we have

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$$(2) \quad T(P) = k \int_{\text{Earth}} d(Q)\,(\|P - Q\|)^{-1}\, dQ.$$

The anomalous potential will be harmonic outside the masses and may therefore be expanded as a series in (fully normalized) solid spherical harmonics (convergent outside a sphere enclosing all masses)

$$(3) \quad T(P) = \frac{kM}{r} \sum_{i=2}^{\infty} \left(\frac{a}{r}\right)^i \sum_{j=0}^{i} \bar{P}_{ij}\,(\sin\varphi)\,[\,\bar{C}_{ij} \cos(j\lambda) + \bar{S}_{ij} \sin(j\lambda)\,],$$

where (φ, λ, r) are the spherical coordinates of the point P, M is the mass of the Earth, a is the semi-major axis of the reference ellipsoid, $\bar{P}_{ij}(\sin\varphi)$ is the fully normalized associated LEGENDRE-polynomial of degree i and order j (cf. HEISKANEN and MORITZ 1967, eq. (1-77a) and (1-77b)) and $\bar{C}_{ij}$ and $\bar{S}_{ij}$ are the coefficients of the series.

It is well-known, that many different density distributions determine the same external gravity field. In spite of this a gravity survey is an indispensable matter in most geophysical explorations.

A gravity survey is useful, because it has been possible to impose such restrictions on d (or ρ), that numerical procedures for "inversion" of eq. (2) have given reliable results. The restrictions will typically require d to be an element of some finite dimensional function space or of the set of solutions to a certain partial differential equation (see e.g. VELIKOVICH and ZEL'DOVICH 1974 or TSCHERNING 1974 for examples of the last mentioned type of restrictions).

In some cases the restrictions are not explicitly stated, as for example when a certain probability distribution is prescribed for the anomalous densities. (See LAURITZEN 1975 and SCHWAHN 1975 for examples of such probability distributions).

A general characterization of the different kinds of restrictions on d which makes it possible to "invert" eq. (2) has not yet been worked out and requires sophisticated mathematical tools (see ANGER 1976a). Meanwhile we must, as pointed out e.g. in ANGER (1976b) evaluate the usefulness of the different proposed restrictions or "models".

We will in the following sections describe how the restriction

$$(4) \quad \Delta(r^n \cdot d) = 0,$$

where Δ is the LAPLACE-operator and n an integer, are used to construct the auto- and cross-covariance functions between density anomalies and the anomalous potential and its derivatives. From these covariance functions the mean square density variation in different depths have been computed based on the estimate of the covariance function of the anomalous potential computed by TSCHERNING and RAPP (1974). The computational procedure is outlined and finally the possibilities for further improvements of the models are briefly discussed.

2. The covariance function of the anomalous potential

When working with the anomalous potential and anomalous densities it is advantageous to work in spherical approximation. The Earth is approximated by a sphere having radius R = 6371.0 km. In the expansion of T in solid spherical harmonics (eq. (3)) the semi-major axis a is substituted by R and the distance from the origin (r) is put equal to R + the ellipsoidal height.

We will in the following frequently work with two points P and Q in $\mathbb{R}^3$ Q will have spherical coordinates distingushed from these of P by an apostrophe (φ', λ', r') and quantities evaluated in Q will also carry an apostrophe, e.g. $\Delta g'$, T'. We then have with this convention $T = T(P)$ and $T' = T(Q)$.

Following HEISKANEN and MORITZ (1967) we will define the covariance function of the anomalous potential at the surface of the Earth as (Ibid. 1967, (7-24):

$$(5)\quad \mathrm{cov}(T, T') = \frac{1}{8\pi^2} \int_{\lambda=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{\alpha=0}^{2\pi} T(R) \cdot T(S) \sin\theta \, d\theta \, d\lambda \, d\alpha,$$

where R has the polar distance θ and longitude λ, the azimuth from R to S is equal to α and the spherical distance between R and S is equal to the spherical distance (ψ) between P and Q. The covariance function will only depend on ψ and may be expanded in a LEGENDRE-series,

$$(6)\quad \mathrm{cov}(T, T') = \sum_{n=2}^{\infty} \sigma_n(T, T) \cdot P_n(\cos\psi),$$

where $P_n(\cos\psi)$ is the LEGENDRE-polynomium of degree n and $\sigma_n(T, T)$ are the so-called (potential) degree-variances. They are equal to the square-sum of the coefficients of T expanded in fully normalized spherical harmonics (cf. eq. (3)), i.e.

$$(7)\quad \sigma_i(T, T) = \sum_{j=0}^{i} (\bar{S}_{ij}^2 + \bar{C}_{ij}^2) \left(\frac{kM}{R}\right)^2.$$

The covariance function cov(T, T') will for either P or Q fixed be a harmonic function in space. The POISSON integral may therefore be applied (one time for P and one for Q), in order to obtain an expression for the covariance function valid at points outside the Earth's surface,

$$(8)\quad \mathrm{cov}(T, T') = \sum_{i=2}^{\infty} \sigma_i(T, T) \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\psi).$$

Covariance functions of other quantities may be defined in the same way as done by eq. (5). They may also, in case they are related to T through a linear functional, be derived by applying the functional directly on cov(T, T') as given by eq. (8). For the free-air gravity anomaly we have for example in spherical approximation

$$(9)\quad \Delta g(P) = \Delta g = -\frac{\partial T}{\partial r}\Big|_P - \frac{2}{r} T(P).$$

Hence

$$(10) \quad \mathrm{cov}(\Delta g, \Delta g') = -\frac{\partial}{\partial r}\left[-\frac{\partial \mathrm{cov}(T, T')}{\partial r'} - \frac{2}{r'}\mathrm{cov}(T, T')\right] + \\ + \frac{2}{r}\cdot\frac{\partial \mathrm{cov}(T, T')}{\partial r'} + \frac{4}{rr'}\mathrm{cov}(T, T') = \\ = \sum_{i=2}^{\infty} \sigma_i(T, T)\,\frac{(i-1)^2}{rr'}\left(\frac{R^2}{rr'}\right)^{i+1} P_i\ (\cos\Psi).$$

The quantity

$$(11) \quad \sigma_i\ (\Delta g, \Delta g\ ) = \sigma_i(T, T)\,\frac{(i-1)^2}{R^2}$$

is denoted the anomaly degree variance, and we have

$$(12) \quad \mathrm{cov}(\Delta g, \Delta g') = \sum_{i=2}^{\infty} \sigma_i(\Delta g, \Delta g)\left(\frac{R^2}{rr'}\right)^{i+2} P_i\ (\cos\Psi).$$

A global estimate of this function was first made by W.M. KAULA (1959). More recently TSCHERNING and RAPP (1974) estimated the covariance function by combining available potential coefficient information (as obtained from satellite pertubations and mean gravity anomalies) and point gravity anomaly data using eq. (5) and (7), (eq. (5) with Δg substituted for T).

In Ibid. (1974) is a representation, well suited for numerical evaluation, worked out. This representation was the one, which in between several models fitted the available data the best possible. The models differed by the expressions for the potential degree-variances

$$(13) \quad \sigma_i(T, T) = A \prod_{j=0}^{m} \frac{1}{(i+k_j)}\, s_o^{i+1}, \qquad i > i_o = \max_{j=0,\ldots,m} (-k_j)$$

where A is a constant in units of $(\mathrm{m/sec})^4$, m is a positive integer, k_j are integers and s_o is a constant < 1, equal to the square of the ratio between the so-called radius of the BJERHAMMAR-sphere and the mean radius of the Earth, R. These model potential degree-variances have the advantage, that we may convert the product to a sum

$$(14) \quad \prod_{j=0}^{m} \frac{1}{(i+k_j)} = \sum_{j=0}^{m} c_j/(i+k_j)$$

where

$$(15) \quad c_j = \prod_{i=0, m \neq j}^{m} 1/(k_i - k_j).$$

In case the degree-variances are multiplied with a polynomium in i (as for example when the anomaly degree-variances was computed) it is still possible to convert the product into a sum. There may in this case occur linear combinations of i^p, where $p \geq 0$.

The covariance function eq. (8) may be split accordingly,

$$(16) \quad \mathrm{cov}(T, T') = \sum_{j=0}^{m} \sum_{i=i_0+1}^{\infty} \frac{a\, c_j s_o^{i+1}}{(i+k_j)} \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\Psi) + $$

$$+ \sum_{i=2}^{i_0} \sigma_i(T, T) \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\Psi).$$

The functions

$$(17) \quad F_{k_j} = \sum_{i=i_0+1}^{\infty} \frac{s_0^{i+1}}{(i+k_j)} \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\Psi)$$

and

$$(18) \quad S_p = \sum_{i=i_0+1}^{\infty} i^p s_0^{i+1} \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\Psi)$$

(which may occur e.g. in $\mathrm{cov}(\Delta g, \Delta g')$) may all be represented by closed expressions (i.e. only having a finite number of terms), cf. TSCHERNING (1976a). The covariance functions as for example

$$(19) \quad \mathrm{cov}(T, T') = \sum_{j=0}^{m} A\, c_j F_{k_j} + \sum_{i=2}^{i_0} \sigma_i(T, T) \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\Psi)$$

are hence easily computable, while covariance functions of quantities involving differentiation with respect to φ or λ may involve the corresponding derivatives of F_{k_j} and S_p, see TSCHERNING (1976a) for details.

The model recommended by TSCHERNING and RAPP (1974) used $k_0 = -2$, $k_1 = -1$, $k_2 = 24$, $m = 2$ (and hence $i_o = 2$), $A = 425.28\ \mathrm{mgal}^2 * R^2$, $s_0 = 0.9967$ and $\sigma_2(T, T) = 7.5\ \mathrm{mgal}^2 * R^2$,

$$(20) \quad \mathrm{cov}(T, T') = 7.5 \cdot R^2 \left(\frac{R^2}{rr'}\right)^3 P_2(\cos\Psi) +$$

$$+ \sum_{i=3}^{\infty} \frac{425.28 \cdot R^2 s_o^{i+1}}{(i-2)(i-1)(i+24)} \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\Psi).$$

3. The density anomaly covariance function

As described in TSCHERNING (1974) the restriction to density functions fulfilling eq. (4) will imply a one-to-one correspondence between the external potential and the density functions. The density function d may be expanded (in spherical approximation) in internal solid spherical harmonics multiplied by r^{-n}, (see Ibid. 1974, eq. (37)),

$$(21)\quad d(P) = \sum_{i=2}^{\infty} \frac{1}{r^n} (\frac{r}{R})^i \sum_{j=0}^{i} \bar{P}_{ij} (\sin\varphi) [c_{ij} \cos (j\lambda) + s_{ij} \sin (j\lambda)].$$

In order that eq. (2) shall be fulfilled the ceofficients must be related in the following way

$$(22)\quad \left.\begin{matrix} c_{ij} \\ \\ s_{ij} \end{matrix}\right\} = \frac{(2i - n + 3)\ (2i + 1)}{R^{3-n}\ 4\pi\ k} \cdot \left\{\begin{matrix} \bar{C}_{ij} \\ \\ \bar{S}_{ij} \end{matrix}\right.$$

We may then first use eq. (5) with $d(P)$ substituted for $T(P)$ and subsequently with $d(Q)$ substituted for $T(Q)$ and get

$$(23)\quad cov(d,\ T') = \sum_{i=2}^{\infty} \sigma_i(T,\ T) \frac{(2i - n + 3)\ (2i + 1)}{r^n\ R^{3-n}\ 4\pi k} (\frac{r}{R})^i\ (\frac{R}{r'})^{i+1}\ P_i\ (\cos\psi),$$

$$(24)\quad cov(d,\ d') = \sum_{i=2}^{\infty} \sigma_i(T,\ T) \frac{(2i - n + 3)^2\ (2i + 1)^2}{(rr')^n\ (R^{3-n}\ 4\pi\ k)^2} (\frac{rr'}{R^2})^i\ P_i\ (\cos\psi).$$

The product of the model degree variances (eq. (13)) and the factors in eq. (22) can (as indicated for the covariance function of the anomalous potential above) be split into a sum of quantities depending on i^p or $1/(i + k_j)$. The covariance functions given by eq. (23) or (24) may hence be represented by a linear combination of the functions F_{k_j} and S_p given by eq. (17) and (18) when the model degree-variances eq. (13) are used.

A computer programme capable of evaluating the auto- and cross-covariance between density anomalies and the derivatives of T of up to second order has been described in TSCHERNING (1976). The programme has been used for the computation of the values given in Table 1 and 2.

Table 1. Values of cross- and auto-covariance functions for points having a varying spherical distance ψ and lying at the surface of the Earth. The covariance function given by eq. (20) has been used.

ψ	cov $(d,\ d')$ g^2/cm^6	cov (d, ζ') $g/cm^3 m$	cov $(d,\ g')$ $g/cm^3 mgal$	cov (d, ξ') g/cm^3 arcsec	cov $(d,\ T'_{zz})$ g/cm^3 E.U.
0°0	0.1615	0.1049	8.048	0.000	33.848
2°5	0.0160	0.0957	3.638	-0.485	3.355
5°0	0.0024	0.0865	1.910	-0.334	0.515
7°0	0.0007	0.0801	1.250	-0.245	0.145
10°0	0.0002	0.0753	0.912	-0.191	0.053
20°0	0.0000	0.0632	0.410	-0.098	0.000
30°0	-0.0001	0.0560	0.249	-0.065	-0.003

(ζ' = the height anomaly in Q, ξ' the latitude component of the deflection of the vertical in Q and T'_{zz} the second order normal derivative of T in Q)

Table 2. Values of the density anomaly covariance function for points having varying spherical distance ψ and varying height/depth. The covariance function given by eq. (20) has been used.

	Height/Depth (Meters)					
ψ	1000	500	-500	-1000	-5000	-10000
0°.0	4.9799	0.4647	0.0810	0.0485	0.0060	0.0018
2°.5	0.0037	0.0117	0.0169	0.0159	0.0050	0.0017
5°.0	0.0002	0.0014	0.0032	0.0037	0.0031	0.0014
7°.5	0.0	0.0003	0.0010	0.0012	0.0017	0.0011
10°.0	0.0	0.0001	0.0004	0.0005	0.0010	0.0008

4. Conclusion

We will initially point out, that the covariance function of the anomalous potential (eq. (20)) has been computed partly from free-air gravity anomalies, i.e. the effect of the topography was not removed. We should therefore expect much bigger values for the density variation at the surface of the Earth.

The small value (of ± 0.8 g/cm^3) we have obtained may be partly due to the damping factor $(s_0)^i$ occurring in eq. (13). In the model used here we have $s_0 = 0.9967$, while in the model discussed in TSCHERNING (1976b) $s_0 = 0.989$ with a corresponding density (root mean square) variation about ten times smaller.

The simple requirement, that the density anomaly function times the radial distance raised to a certain power is harmonic, has computational advantages. It should, by using other functions of the radial distance be possible to obtain a better agreement with the physical reality. Improved estimates of covariance function of the anomalous potential (especially of the variation of the degree-variances for increasing degree) would be very useful. The future possible availability of measurements of gravity gradients (which holds much information on the high frequency variation of the gravity field) should make it possible to obtain such new estimates.

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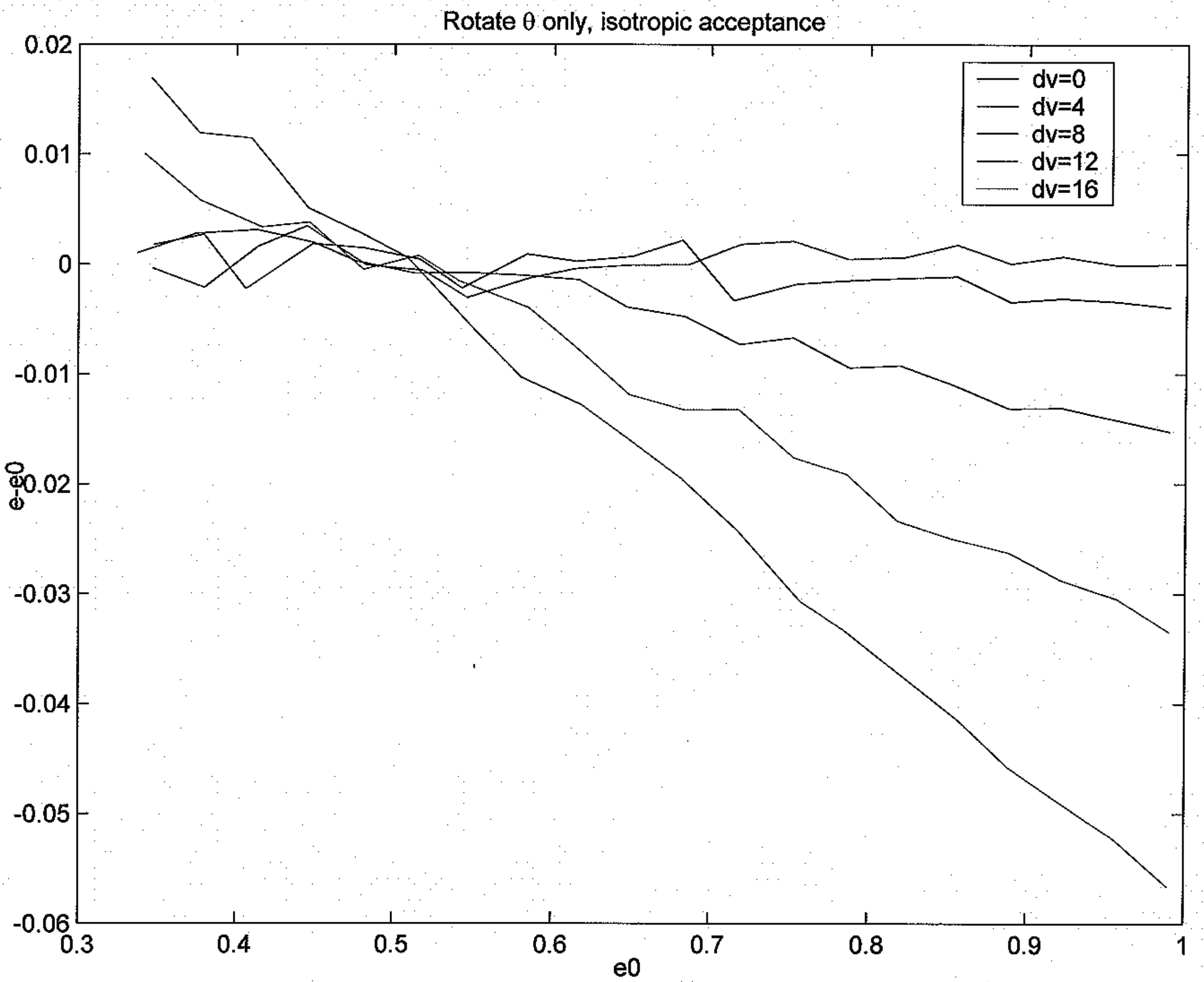

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