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Computation of the Second-Order Derivatives of the Normal Potential Based on the Representation by a Legendre Series

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SUMMARY:

The normal potential of the Earth $U(t,r)$, (t = cosine of the polar distance, r = the radial distance) may be represented by the sum of a Legendre Series (with coefficients depending on r) and a rotational term. Expressions for the (first and) second derivatives with respect to usual Euclidian rectangular coordinates, with respect to the local frame having z-axis opposite to the radius vector, x, y positive north and east, respectively and with respect to the local frame having z'-axis coinciding with the negative gradient vector of U, x', y' positive north and east, respectively, are derived as functions of the first and second derivatives with respect to t and r.

Recursion formulae for the computation of these derivatives of the Legendre-series are developed and programmed in Algol 60.

It is described how the correctness of the algorithms has been tested and the magnitude of the introduced round-off errors has been estimated by applying the well-known physical properties of U.

1. Introduction

A geodetic reference system consists of the values for four Earth-parameters

- a the semi-major axis of the reference ellipsoid,
- GM the product of the gravitational constant and the mass of the Earth,
- J_2 the coefficient of the second order zonal harmonic,
- ω the rotation speed of the Earth.

We also associate with reference system a rectangular X, Y, Z-coordinate system with origin at the "center" of the reference ellipsoid, the Z-axis coinciding with the semi-minor axis of the reference ellipsoid, the X-axis lying in principle arbitrary in the equatorial plane and the Y-axis orthogonal on the X- and Z-axes, so that the three axes form a right-handed orthogonal coordinate system, cf. Figure 1.

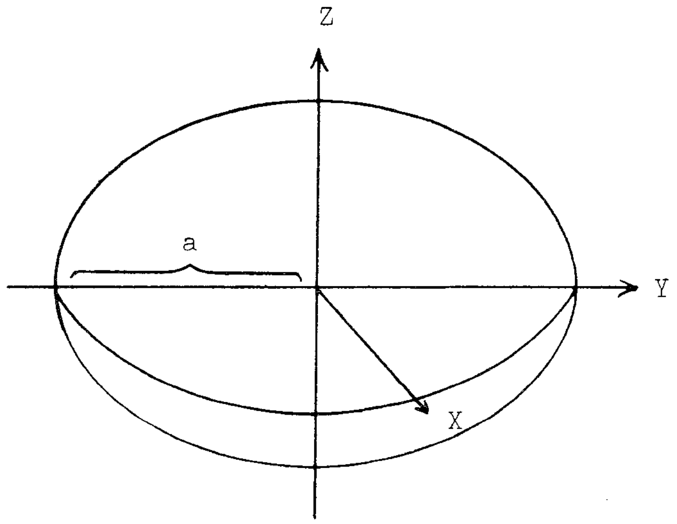

Figure 1. The reference ellipsoid and the X,Y,Z-axes.

Note, that the definition of the reference system and the related X,Y,Z-coordinate system is purely "mathematical". The difficult problems of placing this coordinate system properly with respect to the Earth and taking into account the time variation of the position of points are not relevant for the subject of this paper.

The four parameters a, GM, J_2 and ω determine, besides the excentricity of the reference ellipsoid, a so-called normal potential U for which the reference ellipsoid is an equipotential surface, cf. e.g. Aeiskanen and Moritz (1967, section 2-7). (This book will in the following be denoted "PG").

Following PG (eq. (6-14) - (6-16)) we will express U as the sum of the gravitational potential V and the rotational potential R where

$$U(\varphi,r) = V(\varphi,r) + R(\varphi,r), \tag{1}$$

$$V(\varphi,r) = \frac{GM}{r}\left[1 - \sum_{n=1}^{\infty} J_{2n} \left(\frac{a}{r}\right)^{2n} P_{2n}(\sin\varphi)\right], \tag{2}$$

$$R(\varphi,r) = \frac{\omega^2}{2}(\cos\varphi\ r)^2 \tag{3}$$

Here $\varphi = \tan^{-1}(Z/(X^2+Y^2)^{\frac{1}{2}})$ is the (geocentric) latitude, $r = (X^2+Y^2+Z^2)^{\frac{1}{2}}$, the distance from the origin of the point of evaluation and $P_{2n}(\sin\varphi)$ are the Legendre polynomials of degree 2n. Putting $t = \sin\varphi$,

$$\left.\begin{aligned} A_o &= 1 \\ A_{2n} &= -J_{2n}, \qquad n=1, \ldots, \infty \quad \text{and} \\ A_{2n+1} &= 0 \qquad n=0, \ldots, \infty \end{aligned}\right\} \tag{4}$$

we have

$$V(\varphi, r) = \frac{GM}{r} \sum_{i=o}^{\infty} A_i \left(\frac{a}{r}\right)^i P_i(t) \tag{5}$$

i.e. expressed as the sum of a Legendre series with coefficients depending on r.

The computation of the first derivatives of U are discussed e.g. in PG (section 6-3). Now, higher order derivatives of U are needed for example as reference values for torsion balance or gradiometer measurements. The derivatives may either be wanted with respect to the coordinates (X,Y,Z) or with respect to the coordinates (x',y',z') of a local orthogonal coordinate system. This coordinate system will have its origin in the point of evaluation (X_o, Y_o, Z_o), z' will be positive in the direction opposite to the gradient vector of U, x' will be positive north and y' positive east. The relation between the "global" X,Y,Z-coordinates and the "local" x',y',z'-coordinates are given in eq. (70).

For computational purposes it is convenient first to compute the derivatives in another local coordinate system (coordinates denoted x,y,z) which is closely related to the spherical coordinates φ, h, r (λ = tan-'(Y/X)). The origin will again be in the point of evaluation (X_o, Y_o, Z_o), z will be positive in the direction opposite to the origin of the X,Y,Z-coordinate system, x will be positive north and y positive east. The relations between the two coordinate systems are given in eq. (7).

In the following we will when convenient denote the different sets of coordinates by $\underline{X} = (X,Y,Z) = (X_1, X_2, X_3)$, $\underline{x}' = (x',y',z') = (x_1', x_2', x_3')$ and $\underline{x} = (x,y,z) = (x_1, x_2, x_3)$.

Let the points (X_o, Y_o, Z_o) and (X,Y,Z) have spherical coordinates $(\varphi_o, \lambda_o, r_o)$, (φ,h,r), respectively. We then have

$$\left.\begin{aligned} X_o &= r_o \cos\varphi_o \cos\lambda_o \\ Y_o &= r_o \cos q_o \sin\lambda_o \\ Z_o &= r_o \sin\varphi_o \end{aligned}\right\} \tag{6}$$

and

$$\left.\begin{aligned} X &= X_o + x(-\sin\varphi_o \cosh_o) - y\sin\lambda_o + z\cos\varphi_o \cosh_o \\ Y &= Y_o + x(-\sin\varphi_o \sinh_o) + y\cos\lambda_o + z\cos\varphi_o \sinh_o \\ Z &= Z_o + x\cos\varphi_o + z\sin\varphi_o \end{aligned}\right\} \qquad (7)$$

or on matrix form

$$\underline{X} = \underline{X}_o + \underline{\underline{C}}^T \cdot \underline{x} \,, \qquad (8)$$

where

$$\underline{C}^T = \begin{bmatrix} -\sin\varphi_o \cos\lambda_o & -\sin\lambda_o & \cos\varphi_o \cosh \\ -\sin\varphi_o \sinh_o & \cos\lambda_o & \cos\varphi_o \sinh_o \\ \cos\varphi_o & 0 & \sin\varphi_o \end{bmatrix} \qquad (9)$$

(cf. e.g. Moritz (1971, eq. (1-22)).

(We note, that the matrix C is not well-defined for points on the Z-axis. This difficulty can be overcomed by assigning to these points the longitude 0 (zero) and the latitude 90^o for $Z \geq 0$ and -90^o for $Z < 0$).

We may rewrite eq. (7) in the following manner

$$\left.\begin{aligned} X &= x(-\sin\varphi_o \cosh_o) - y\sinh_o + (r_o+z)\cos\varphi_o \cos\lambda_o \\ Y &= x(-\sin\varphi_o \sinh_o) + y\cosh_o + (r_o+z)\cos\varphi_o \sin\lambda_o \\ Z &= x\cos\varphi_o + (r_o+z)\sin\varphi_o \end{aligned}\right| \qquad (10)$$

2. The derivatives in an orthogonal coordinate system expressed by means of the derivatives with respect to the spherical coordinates.

In order to obtain the second order derivatives with respect to for example the local coordinate x_i it, is necessary to compute

$$\frac{\partial r}{\partial x_i}, \frac{\partial^2 r}{\partial x_i \partial x_j}, \frac{\partial t}{\partial x_i}, \frac{\partial^2 t}{\partial x_i \partial x_j}, \frac{\partial \lambda}{\partial x_i}, \frac{\partial^2 \lambda}{\partial x_i \partial x_j} \,.$$

Using the chain-rule for differentiation we have

$$\frac{\partial}{\partial x_i} = \frac{\partial r}{\partial x_i}\frac{\partial}{\partial r} + \frac{\partial t}{\partial x_i}\frac{\partial}{\partial t} + \frac{\partial \lambda}{\partial x_i}\frac{\partial}{\partial \lambda} \qquad (11)$$

and

$$\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2 r}{\partial x_i \partial x_j}\frac{\partial}{\partial r} + \frac{\partial r}{\partial x_i}\left[\frac{\partial r}{\partial x_j}\frac{\partial^2}{\partial r^2} + \frac{\partial t}{\partial x_j}\frac{\partial^2}{\partial t \partial r} + \frac{\partial \lambda}{\partial x_j}\frac{\partial^2}{\partial r \partial \lambda}\right]$$

$$+ \frac{\partial^2 t}{\partial x_i \partial x_j}\frac{\partial}{\partial t} + \frac{\partial t}{\partial x_i}\left[\frac{\partial r}{\partial x_j}\frac{\partial^2}{\partial r \partial t} + \frac{\partial t}{\partial x_j}\frac{\partial^2}{\partial t^2} + \frac{\partial \lambda}{\partial x_j}\frac{\partial^2}{\partial \lambda \partial t}\right]$$

$$+ \frac{\partial^2 \lambda}{\partial x_i \partial x_j}\frac{\partial}{\partial \lambda} + \frac{\partial \lambda}{\partial x_i}\left[\frac{\partial r}{\partial x_j}\frac{\partial^2}{\partial r \partial \lambda} + \frac{\partial t}{\partial x_j}\frac{\partial^2}{\partial \lambda \partial t} + \frac{\partial \lambda}{\partial x_j}\frac{\partial^2}{\partial \lambda^2}\right]. \tag{12}$$

Using eq. (8) and the fact that C^T is an orthogonal matrix we have

$$\left\{\frac{\partial X_i}{\partial x_j}\right\} = \underline{\underline{C}}^T \tag{13}$$

and thereby

$$\left\{\frac{\partial}{\partial X_i}\right\} = \underline{\underline{C}}^T \left\{\frac{\partial}{\partial x_i}\right\} \tag{14}$$

and

$$\left\{\frac{\partial^2}{\partial X_i \partial X_j}\right\} = \underline{\underline{C}}^T \left\{\frac{\partial^2}{\partial x_i \partial x_j}\right\} \underline{\underline{C}} \tag{15}$$

This express the well-known fact, that when we know all the derivatives of a specific order in one orthogonal coordinate system, we are able in a straight forward manner to compute the derivatives (of the same order) in any other orthogonal coordinate system. (Cf. Hotine (1969), Ch. 2 and 3) for a description of the same matter using tensors).

We can then either differentiate with respect to the (X,Y,Z) coordinates or with respect to the local (x,y,z) coordinates. The latter leads to the most simple equations, and will be carried through here.

Introducing

$$\begin{aligned} p^2 &= X^2 + Y^2 = r^2\cos^2\varphi \\ &= ((r_o+z)\cos\varphi_o - x\sin\varphi_o)^2 + y^2 \end{aligned} \tag{16}$$

we have

$$r^2 = p^2 + Z^2 = (r_o+z)^2 + x^2+y^2 \tag{17}$$

$$t = Z/r \tag{18}$$

and

$$\tan h = Y/X. \tag{19}$$

Applying straight forward differentiation we get

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{r_o+z}{r} \tag{20}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{3x^2}{r^3}, \quad \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{3y^2}{r^3}, \quad \frac{\partial^2 r}{\partial z^2} = -\frac{(r_o+z)^2}{r^3} + \frac{1}{r} \tag{21}$$

$$\frac{\partial^2 r}{\partial x \partial y} = -\frac{xy}{r^3}, \quad \frac{\partial^2 r}{\partial x \partial z} = -\frac{x(r_o+z)}{r^3}, \quad \frac{\partial^2 r}{\partial y \partial z} = -\frac{y(r_o+z)}{r^3}, \tag{22}$$

which when evaluated for (x,y,z) = (0,0,0) gives

$$\left.\begin{aligned} &\frac{\partial r}{\partial x} = \frac{\partial r}{\partial y} = \frac{\partial^2 r}{\partial x \partial y} = \frac{\partial^2 r}{\partial x \partial z} = \frac{\partial^2 r}{\partial y \partial z} = \frac{\partial^2 r}{\partial z^2} = 0, \\ &\frac{\partial r}{\partial z} = 1, \quad \frac{\partial^2 r}{\partial x^2} = \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}. \end{aligned}\right\} \tag{23}$$

Because

$$\frac{1}{\cos^2 h}\frac{\partial \lambda}{\partial x_i} = \frac{X \cdot \frac{\partial Y}{\partial x_i} - Y \frac{\partial X}{\partial x_i}}{X^2} \tag{24}$$

we get

$$\frac{\partial \lambda}{\partial x} = \cos^2\lambda \left[X(-\sin\varphi_o \sinh_o) - Y(-\sin\varphi_o \cosh_o)\right] / X^2,$$

which may be reduced to

$$\frac{\partial \lambda}{\partial x} = \frac{y \sin\varphi_o}{r^2 \cos^2\varphi}. \tag{25}$$

Correspondingly

$$\frac{\partial\lambda}{\partial y} = \cos^2\lambda \left[X\cos\lambda_o - Y(-\sin\lambda_o) \right] / X^2$$

$$= \frac{1}{p^2} \left[(r_o+z)\cos\varphi_o + x(-\sin\varphi_o) \right] , \qquad (26)$$

$$\frac{\partial A}{\partial z} = \cos^2\lambda \left[X\cos\varphi_o \sinh_o - Y \cos\varphi_o \cosh_o \right] / X^2$$

$$= \frac{1}{p^2} (-\cos\varphi_o y), \qquad (27)$$

$$\frac{\partial^2\lambda}{\partial x^2} = y\sin\varphi_o \cdot \frac{\partial}{\partial x}\left[\frac{1}{p^2} \right], \qquad (28)$$

$$\frac{\partial^2\lambda}{\partial z^2} = -y\cos\varphi_o \cdot \frac{\partial}{\partial z}\left[\frac{1}{p^2} \right], \qquad (29)$$

$$\frac{\partial^2\lambda}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{1}{p^2} \right] ((r_o+z)\cos\varphi_o - x \sin\varphi_o)$$

$$= \frac{2y}{p^3} \left[(r_o+z)\cos\varphi_o - x\sin\varphi_o \right] , \qquad (30)$$

$$\frac{\partial^2\lambda}{\partial x\partial y} - \frac{\sin\varphi_{\blacktriangledown}}{r^2\cos^2\varphi} \;\&\; \frac{2}{p^3}\sin\varphi_o , \qquad (31)$$

$$\frac{\partial^2\lambda}{\partial x\partial z} = -2y\sin\varphi_o \left[(r_o+z)\cos^2\varphi_o - x\cos\varphi_o - x\cos\varphi_o\sin\varphi_o \right] / p^3, \qquad (32)$$

$$\frac{\partial^2\lambda}{\partial y\partial z} = \frac{\cos\varphi_o}{p^2} - 2((r_o+z)\cos^2\varphi_o - x\cos\varphi_o\sin\varphi_o) \left[(r_o+z)\cos\varphi_o \right.$$

$$\left. - x\sin\varphi_o) \right] / p^4. \qquad (33)$$

Evaluated in $(x,y,z) = (0,0,0)$ we get

$$\frac{\partial\lambda}{\partial x} = \frac{\partial\lambda}{\partial z} = \frac{\partial^2\lambda}{\partial x^2} = \frac{\partial^2\lambda}{\partial z^2} = \frac{\partial^2\lambda}{\partial y^2} = \frac{\partial^2\lambda}{\partial x\partial z} = 0, \qquad (34)$$

$$\frac{\partial\lambda}{\partial y} = \frac{1}{\cos\varphi_o \cdot r} \tag{35}$$

$$\frac{\partial^2\lambda}{\partial x\partial y} = \frac{\sin\varphi_o}{r_o^2\cos^2\varphi_o} = \frac{\tan\varphi_o}{r_o^2 \cdot \cos\varphi_o} \tag{36}$$

$$\frac{\partial^2\lambda}{\partial y\partial z} = -\frac{1}{r_o^2\cos\varphi_o} \tag{37}$$

Finally for $t = \cos(\theta)$ we have

$$t = Z/r = \frac{(r_o+z)\sin\varphi_o + x\cos\varphi_o}{((r_o+z)^2+x^2+y^2)^{\frac{1}{2}}} \tag{38}$$

and hence

$$\frac{\partial t}{\partial x} = \frac{\cos\varphi_o \cdot r - \left[(r_o+z)\sin\varphi_o + x\cos\varphi_o\right] x/r}{r^2}$$

$$= \frac{\cos\varphi_o}{r} - \frac{x \cdot Z}{r^3}, \tag{39}$$

$$\frac{\partial t}{\partial y} = \frac{-yZ}{r^3}, \tag{40}$$

$$\frac{\partial t}{\partial z} = \frac{\sin\varphi_o r - Z(r_o+z)/r}{r^2} = \frac{\sin\varphi_o}{r} - \frac{Z(r_o+z)}{r^3}, \tag{41}$$

$$\frac{\partial^2 t}{\partial x^2} = \cos\varphi_o\left[-\frac{x}{r^3}\right] - \frac{Z}{r^3} - x \cdot \frac{\partial}{\partial x}\left[\frac{Z}{r^3}\right], \tag{42}$$

$$\frac{\partial^2 t}{\partial y^2} = -\frac{Z}{r^3} - \frac{\partial}{\partial y}\left[\frac{Z}{r^3}\right], \tag{43}$$

$$\frac{\partial^2 t}{\partial z^2} = \frac{\sin\varphi_o}{r^2}\left[-\frac{r_o+z}{r}\right] - \frac{Z}{r^3} - (r_o+Z)\left[\frac{\sin\varphi_o r^3 - 3Z(r_o+z)}{r^6}\right] \tag{44}$$

$$\frac{\partial^2 t}{\partial x \partial y} = -y\,\frac{\partial}{\partial x}\left[\frac{Z}{r^3}\right], \qquad \frac{\partial^2 t}{\partial z \partial y} = -y\,\frac{\partial}{\partial z}\left(\frac{Z}{r^3}\right), \tag{45}$$

$$\frac{\partial^2 t}{\partial z \partial x} = \cos\varphi_o\left[-\frac{r_o+z}{r^3}\right] - x\,\frac{\partial}{\partial z}\left[\frac{Z}{r^3}\right], \tag{46}$$

which, when evaluated in (0,0,0), give

$$\frac{\partial t}{\partial x} = \frac{\cos y_o}{r} \tag{47}$$

$$\frac{\partial^2 t}{\partial x^2} = \frac{\partial^2 t}{\partial y^2} = -\frac{t}{r^2} \tag{48}$$

$$\frac{\partial^2 t}{\partial x \partial z} = -\frac{\cos y_o}{r^2} \tag{49}$$

$$\frac{\partial t}{\partial y} = \frac{\partial t}{\partial z} = \frac{\partial^2 t}{\partial x \partial y} = \frac{\partial^2 t}{\partial z \partial y} = \frac{\partial^2 t}{\partial z^2} = 0 \quad . \tag{50}$$

Using eq. (11) and (12) we get

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{\cos\varphi}{r}\frac{\partial}{\partial t} = \frac{1}{r}\frac{\partial}{\partial\varphi} \\ \frac{\partial}{\partial y} &= \frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda}, \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial r}, \end{aligned}\right\} \tag{51}$$

$$\frac{\partial^2}{\partial x^2} = \frac{1}{r}\frac{\partial}{\partial r} - \frac{t}{r^2}\frac{\partial}{\partial t} + \frac{\cos^2\varphi}{r^2}\frac{\partial^2}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}, \tag{52}$$

$$\frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r} - \frac{t}{r^2}\frac{\partial}{\partial t} + \frac{1}{\cos\varphi^2 r^2}\frac{\partial^2}{\partial\lambda^2} \quad \frac{1}{r}\frac{\partial}{\partial r} \quad \frac{\tan\varphi}{r^2}\frac{\partial}{\partial\varphi} + \frac{1}{\cos^2\varphi r^2}\frac{\partial^2}{\partial\lambda^2}, \tag{53}$$

$$\frac{\partial}{\partial z^2} = \frac{\partial^2}{\partial r^2}\,, \tag{54}$$

$$\frac{\partial}{\partial x\partial y} = \frac{\cos\varphi}{r}\left[\frac{1}{\cos\varphi r}\frac{\partial^2}{\partial\lambda\partial t}\right] + \frac{\sin\varphi}{r^2\cos^2\varphi}\frac{\partial}{\partial\lambda}$$

$$= \frac{1}{r^2}\left[\frac{\partial^2}{\partial\lambda\partial t} + \frac{\tan\varphi}{\cos\varphi}\frac{\partial}{\partial\lambda}\right] = \frac{1}{r^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\lambda}\right]\,, \tag{55}$$

$$\frac{\partial^2}{\partial x\partial z} = -\frac{\cos\varphi}{r^2}\frac{\partial}{\partial t} + \frac{\cos\varphi}{r}\frac{\partial^2}{\partial t\partial r} = \frac{\cos\varphi}{r}\left[-\frac{1}{r}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t\partial r}\right] = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial\varphi}\right]\,, \tag{56}$$

$$\frac{\partial^2}{\partial y\partial z} = -\frac{1}{r^2\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{1}{\cos\varphi r}\frac{\partial^2}{\partial\lambda\partial r}$$

$$= \frac{1}{r\cos\varphi}\left[-\frac{1}{r}\frac{\partial}{\partial\lambda} + \frac{\partial^2}{\partial\lambda\partial r}\right] = \frac{\partial}{\partial r}\left[\frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda}\right]. \tag{57}$$

The matrix of second order derivatives of eq. (15) becomes

$$\left\{\frac{\partial^2}{\partial x_i\partial x_j}\right\} =$$

$$\left\{\begin{array}{c|c|c}
\frac{1}{r}\frac{\partial}{\partial r} - \frac{t}{r^2}\frac{\partial}{\partial t} + \frac{\cos^2\varphi}{r^2}\frac{\partial^2}{\partial t^2} & - & - \\
\hline
\frac{1}{r^2}\left[\frac{\partial^2}{\partial t\partial\lambda} + \frac{\tan\varphi}{\cos\varphi}\frac{\partial}{\partial\lambda}\right] & \frac{1}{r}\frac{\partial}{\partial r} - \frac{t}{r^2}\frac{\partial}{\partial t} + \frac{1}{\cos^2\varphi r^2}\frac{\partial^2}{\partial\lambda^2} & - \\
\hline
\frac{\cos\varphi}{r}\left[\frac{\partial^2}{\partial t\partial r} - \frac{1}{r}\frac{\partial}{\partial t}\right] & \frac{1}{r\cos\varphi}\left[\frac{\partial^2}{\partial\lambda\partial r} - \frac{1}{r}\frac{\partial}{\partial\lambda}\right] & \frac{\partial}{\partial r^2}
\end{array}\right\} \tag{58}$$

where the upper triangular part appear by applying the symmetry of the matrix. (Cf. Moritz (1971, eq. (1-25)) for another formulation of eq. (58)).

The trace of the matrix is equal to the Laplace operator,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - \frac{2t}{r^2}\frac{\partial}{\partial t} + \frac{1}{\cos^2\varphi \cdot r^2}\frac{\partial^2}{\partial \lambda^2} + \frac{\cos^2\varphi}{r^2}\frac{\partial^2}{\partial t^2} \tag{59}$$

cf. P.G. (eq. (1-41)).

Using eq. (58) and (15) we are able to write down the second order derivatives with respect to the X,Y and Z coordinates. They become after some rearrangement

$$r^2\frac{\partial^2}{\partial X^2} = \cos^2\varphi\cos^2\lambda\left(r^2\frac{\partial^2}{\partial r^2} + \sin^2\varphi\frac{\partial^2}{\partial t^2} - 2r\sin\varphi\frac{\partial^2}{\partial t\partial r}\right)$$
$$+ \frac{\sin^2\lambda}{\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} + r(1-\cos^2\lambda\cos^2\varphi)\frac{\partial}{\partial r} + \sin\varphi(3\cos^2\varphi\cos^2\lambda - 1)\frac{\partial}{\partial t}$$
$$+ 2\cos\lambda\sin\lambda\,\text{[redacted]}\frac{\partial}{\partial\lambda} + \sin\varphi\frac{\partial^2}{\partial\lambda\partial t} - r\frac{\partial^2}{\partial\lambda\partial r}) , \tag{60}$$

$$r^2\frac{\partial^2}{\partial Y^2} - \cos^2\varphi\sin^2\lambda\left(r^2\frac{\partial^2}{\partial r^2} + \sin^2\varphi\frac{\partial^2}{\partial t^2} - 2r\sin\varphi\frac{\partial^2}{\partial t\partial r}\right)$$
$$+ \frac{\cos^2\lambda}{\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} + r(1-\sin^2 h\cos^2\varphi)\frac{\partial}{\partial r} + \sin\varphi(3\cos^2\varphi\sin^2\lambda - 1)\frac{\partial}{\partial t}$$
$$-2\cos h\sin h\,\text{[redacted]}\frac{\partial}{\partial\lambda} + \sin\varphi\frac{\partial^2}{\partial\lambda\partial t} - r\frac{\partial^2}{\partial\lambda\partial r}) , \tag{61}$$

$$r^2\frac{\partial^2}{\partial Z^2} = r^2\sin^2\varphi\frac{\partial^2}{\partial r^2} + \cos^2\varphi(\cos^2\varphi\frac{\partial^2}{\partial t^2} + 2r\sin\varphi\frac{\partial^2}{\partial t\partial r}$$
$$+ r\frac{\partial}{\partial r} - 3\sin\varphi\frac{\partial}{\partial t}), \tag{62}$$

$$r^2 \frac{\partial^2}{\partial X \partial Y} = \Big[\cos^2\varphi(\sin^2\varphi \frac{\partial^2}{\partial t^2} + r^2 \frac{\partial^2}{\partial r^2} - r \frac{\partial}{\partial r} + \sin\varphi\,(3\frac{\partial}{\partial t} - 2r \frac{\partial^2}{\partial t \partial r}))$$

$$- \frac{1}{\cos^2\varphi} \frac{\partial^2}{\partial \lambda^2} \Big] \sin\lambda\cos\lambda$$

$$+ (\sin^2 A - \cos^2\lambda) \Big[\sin\varphi \frac{\partial^2}{\partial t \partial \lambda} + \frac{1}{\cos^2\varphi} \frac{\partial}{\partial \lambda} - r \frac{}{\partial\lambda\partial r}\Big], \qquad (63)$$

$$r^2 \frac{\partial^2}{\partial X \partial Z} = \cos\lambda \Big[\cos\varphi\sin\varphi(-\cos 2\varphi \frac{\partial^2}{\partial t^2} + r2 \frac{\partial^2}{\partial r^2} - r \frac{\partial}{\partial r})$$

$$+ \cos\varphi((3\sin^2 v - I) \frac{\partial}{\partial t} + (\cos^2\varphi - \sin^2\varphi) r \frac{\partial^2}{\partial t \partial r})\Big]$$

$$- \sin\lambda \Big[\cos\varphi \frac{\partial^2}{\partial\lambda\partial t} + r \frac{\sin\varphi}{\cos\varphi} \frac{\partial^2}{\partial\lambda\partial r}\Big], \qquad (64)$$

$$r^2 \frac{\partial^2}{\partial Y \partial Z} = \sin\lambda \Big[\cos\varphi\sin\varphi(-\cos^2\varphi \frac{\partial^2}{\partial t^2} + r^2 \frac{\partial^2}{\partial r^2} - r \frac{\partial}{\partial r})$$

$$+ \cos\varphi((3\sin^2\varphi - 1)\frac{\partial}{\partial t} + (\cos^2\varphi - \sin^2\varphi) r \frac{\partial^2}{\partial t \partial r})\Big]$$

$$+ \cosh \Big[\cos\varphi \frac{\partial^2}{\partial\lambda\partial t} + r \frac{\sin\varphi}{\cos\varphi} \frac{\partial^2}{\partial\lambda\partial r}\Big] . \qquad (65)$$

Expressions for the first derivatives can be found e.g. in PG (eq. (6-18)).

For the normal potential, which is independent of λ, we have

$$\left.\begin{aligned} \frac{\partial U}{\partial X} &= \cos\varphi\cos\lambda \frac{\partial U}{\partial r} - \sin\varphi\cos\lambda \frac{\cos\varphi}{r} \frac{\partial U}{\partial t} \\ \frac{\partial U}{\partial Y} &= \cos\varphi\sin\lambda \frac{\partial U}{\partial r} - \sin\varphi\sin\lambda \frac{\cos\varphi}{r} \frac{\partial U}{\partial t} \\ \frac{\partial U}{\partial Z} &= \sin\varphi \frac{\partial U}{\partial r} + \frac{\cos^2\varphi}{r} \frac{\partial U}{\partial t} \end{aligned}\right\} \qquad (66)$$

and hence

$$(\frac{\partial^2 U}{\partial X^2})^2 + (\frac{\partial^2 U}{\partial Y^2})^2 = (\cos\varphi \frac{\partial U}{\partial r} - \frac{\sin\varphi\cos\varphi}{r} \frac{\partial U}{\partial t})^2, \qquad (67)$$

the normal gravity

$$\gamma = \left[(\frac{\partial U}{\partial r})^2 + (\frac{\cos\varphi}{r} \frac{\partial U}{\partial t})^2 \right]^{\frac{1}{2}} \tag{68}$$

and the geographical latitude (cf. PG (2-29)),

$$\Phi = \tan^{-1}(- \frac{\partial U}{\partial Z} / ((\frac{\partial U}{\partial X})^2 + (\frac{\partial U}{\partial Y})^2))). \tag{69}$$

Observations carried out on the surface of the Earth (for example using a torsion balance instrument) will as a local approximate reference frame use the frame having the gradient vector of U as its z'-axis, the x'-axis orthogonal hereon, pointing north and the g'-axis orthogonal on the two other, forming a left-handed system.

The transformation between this local frame (with coordinates (x', y', z')) and the (X,Y,Z)-coordinate system can be expressed in the same wag as in eq. (8),

$$\underline{X} = \underline{X}_o + \underline{\underline{B}}^T \underline{x}' , \tag{70}$$

where

$$\underline{\underline{B}}^T = \begin{Bmatrix} -\sin\Phi \cos\Lambda & -\sin\Lambda & \cos\Phi \cos\Lambda \\ -\sin\Phi \sin\Lambda & \cos\Lambda & \cos\Phi\sin \\ \cos\Phi & 0 & \sin\Phi \end{Bmatrix} \tag{71}$$

where $\Lambda = h$ in this case.

We thereby have a direct transformation between the two local frames

$$\underline{x}' = \underline{\underline{B}}\, \underline{\underline{C}}^T \underline{x} \tag{72}$$

and between the second order derivatives

$$\left\{ \frac{\partial^2}{\partial x_i' \partial x_j'} \right\} = \underline{\underline{B}}\, \underline{\underline{C}}^T \left\{ \frac{\partial^2}{\partial x_i \partial x_j} \right\} \underline{\underline{C}}\, \underline{\underline{B}}^T . \tag{73}$$

We may naturally express the elements of the matrix $B\,C^T$ directly by the angle

$$\delta = \Phi - \varphi \,, \quad \text{i.e.} \tag{74}$$

$$\underline{\underline{B}}\;\underline{\underline{C}}^T = \begin{Bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{Bmatrix} \tag{75}$$

(where we are rotating between left-handed coordinate systems).

The matrix eq. (73) applied on the normal potential give us the so-called Marussi tensor (cf. Hotine (1969, eq. (2-162)), and it is well-known (cf. PG eq. (2-17), (2-18), (2-22a), 2-22b)) that the elements express for example the curvature of the plumb-line and of the equipotential surfaces.

On the reference ellipsoid we have from PG (eq.(2-17), (2-18) and (2-210)),

$$\frac{\partial^2 U}{\partial (x')^2} = -\gamma \cdot \frac{1}{M} = -\gamma \frac{b}{a^2}(1+(e')^2\cos^2\Phi)^{\frac{3}{2}} \tag{76}$$

$$\frac{\partial^2 U}{\partial (y')^2} = -\gamma \cdot \frac{1}{N} = -\gamma \frac{b}{a^2}(1+(e')^2\cos^2\Phi)^{\frac{1}{2}} \tag{77}$$

where b is the semi-minor axis and e' the second excentricity, and obviously for a99 points

$$\frac{\partial U}{\partial x'} = \frac{\partial U}{\partial y'} = \frac{\partial^2 U}{\partial x'\,\partial y'} = \frac{\partial^2 U}{\partial z'\,\partial y'} = 0 \tag{78}$$

$$\sum_{i=1}^{3} \frac{\partial^2 U}{\partial x_i'^2} - 2\omega^2 = 0 \tag{79}$$

cf. PG eq. (2-20), and finally

$$\left| \frac{\partial U}{\partial z'} \right| = \gamma \,.$$

3. Recursion algorithms for the derivatives of the sum of a Legendre series

Based on Clenshaw (1955) recursion formulae for the sun of a Legendre series have been developed e.g. in Tscherning and Rapp (1974, section 9). Here the algorithms were applied for the numerical evaluation of different covariance functions related to the anomalous potential of the Earth. The same algorithms has been used as a part of a FORTRAN-subroutine named GRAVC published in Tscherning (1974) and used for tne computation of the normal gravity in altitudes above 25 km (traditional series were used below this height).

The algorithm is very well-suited for the computation of derivatives

$$\frac{\partial^m V}{\partial t^i \, \partial r^j}, \qquad i + j = m ,$$

when the Legendre series (eq. (5)) have been truncated at a certain degree n.

Let us now take V equal to the sum of this truncated series. We then have for the derivatives of this series with respect to r,

$$\frac{\partial^j V}{\partial r^j} = \frac{GM}{r^{j+1}} \sum_{k=o}^{n} A_k \cdot \prod_{q=1}^{j} (-k-q)\left(\frac{a}{r}\right)^k P_k(t), \tag{80}$$

which for

$$s = a/r$$

and

$$A_k^j = \prod_{q=1}^{j} (-k-q) A_k \tag{81}$$

becomes

$$\frac{\partial^j V}{\partial r^j} = \frac{GM}{r^{j+1}} \sum_{k=o}^{n} A_k^j \, s^k \, P_k(t) . \tag{82}$$

Clenshaw regards the sum of a series

$$S_n = \sum_{k=o}^{n} a_k p_k(t), \tag{83}$$

for which there exist a three-term recursion formula

$$p_{k+1}(t) + e_k p_k(t) + f_k p_{k-1}(t) = 0 \tag{84}$$

where the coefficients e_i and f_i may depend on t as well as on i.

It is proved, that the recursion algorithm

$$\left.\begin{aligned} & b_{n+2} = b_{n+1} = 0 \\ & b_k = - e_k b_{k+1} - f_{k+1} b_{k+2} + a_k, \\ & i = n, \ldots, 1, 0 \end{aligned}\right\} \tag{85}$$

will furnish us with the sum eq. (83):

$$S_n = b_o p_o(t) + b_1 (p_1(t) + e_o p_o(t)) \tag{86}$$

For the Legendre polynomials we have the well-known recursion formula

$$P_{k+1}(t) - \frac{2k+1}{k+1} t \cdot P_k(t) + \frac{k}{k+1} P_{k-1}(t) = 0 \tag{87}$$

But, what is more important in this connection, is that we by multiplication with s^{k+1} obtain

$$s^{k+1} P_{k+1}(t) - \left[\frac{2k+1}{k+1} t \cdot s\right] s^k P_k(t) + \left[\frac{k \cdot s^2}{k+1}\right] s^{k-1} P_{k-1}(t) = 0 \tag{88}$$

For

$$e_k = - \frac{2k+1}{k+1} t \, s \,, \quad f_k = \frac{k \cdot s^2}{k+1} \quad \text{and} \quad p_k(t) = s^k P_k(t)$$

we have using eq. (84) - (86) the following recursion algorithm for [illegible] derivative of V with respect to r

$$\left.\begin{aligned}
&b^j_{n+2} = b^j_{n+1} = 0 \;, \\
&b^j_k = \frac{2k+1}{k+1}\, t\cdot s\; b^j_{k+1} - \frac{(k+1)}{k+2}\, s^2 b^j_{k+2} + A^j_k \\
&S^j_n = b^j_o + b^j_1(st-st) = b^j_o \\
&\frac{\partial^j V}{\partial r^j} = \frac{GM}{r^{j+1}}\, S^j_n \quad .
\end{aligned}\right\} \qquad (89)$$

The i'th derivative with respect to t is easily obtained by differentiating the equations i-times with respect to t.

$$\left.\begin{aligned}
&b^{j,i}_{n+2} = b^{j,i}_{n+1} = 0 \\
&b^{j,i}_k = \frac{2k+1}{k+1}\, s \left[i\cdot b^{j,i-1}_{k+1} + t\; b^{j,i}_{k+1} \right] - \frac{k+1}{k+2}\, s^2 b^{j,i}_{k+2} \\
&S^{j,i}_n = b^{j,i}_o \\
&\frac{\partial^{i+j} V}{\partial r^j \partial t^i} = \frac{GM}{r^{j+1}}\, S^{j,i}_n \quad .
\end{aligned}\right\} \qquad (90)$$

Note, that in order to carry through the k'th step in the algorithm for the J, I th derivative, the k'th step must have been carried out for all J, i derivatives, $i < I$.

The algorithm eq. (90) has been programmed in Algol for the RC 4000 computer of the Danish Geodetic Institute in the form of a procedure, see the appendix. The algorithm may naturally very easily be programmed in any other algorithmic computer language.

For a discussion of the favorable numerical properties of a recursion algorithm of this type see Clenshaw (1955).

4. Test of the algorithms and the algol-procedure

The listed algol-procedure is one of the modules in a system of applicative programs for physical geodesy being developed at the Danish Geodetic Institute.

One module will for example for given values of GM, a, J_2 and w (or other consistent set of Earth-parameters) compute the excentricity of the reference ellipsoid and the coefficients A_i of eq. (4). Another module will for given geodetic latitude, longitude and ellipsoidal height evaluate U and its first and second derivatives in one of the reference-frames discussed in section 2.

It is important to have possibilities for straightforward checks of the computations, including the round-off errors. Fortunately we have several possibilities which easily are applied:

(1) Use the known physical properties of U

(a) On the reference-ellipsoid:

U is constant,

Φ (eq. (69)) is equal to the geodetic latitude,

$-\frac{1}{\gamma}\frac{\partial^2 U}{\partial x'^2}$ and $\frac{1}{\gamma}\frac{\partial^2 U}{\partial y'^2}$ are equal to two curvatures cf. eq. (76) and (77),

(b) In all points

$$\frac{\partial U}{\partial x'} = \frac{\partial U}{\partial y'} = \frac{\partial^2 U}{\partial x'\partial y'} = \frac{\partial^2 U}{\partial z'\partial y'} = 0 ,$$

$$\Delta U = 2\omega^2 ,$$

(c) Use that U is symmetric with respect to the Equatorial plane and independent of λ.

(2) Use other algorithms available,

(a) The closed expressions for the normal gravity cf. PG(section 6-2)

(b) Algorithms intended for the evaluation of sums and derivatives of series in solid spherical harmonics (cf. e.g. Gulick (1970)).

(c) Use numerical differentiation methods.

(3) Use available computer soft-ware for numerical investigations as for example the possibility of carrying through the computations using another number of significant digits.

All these types of test has been carried out, some in the form af class-work done while the author was teaching a course in the field of physical geodesy at the University of Copenhagen. The subject is very much recommended for others teaching in this field.

The tests gave as a result that for n = 8 in eq. (80), the total effect of the round-off and other errors never exceeded 2^3 times the precision with which the computer evaluated cosine or sine.

5. Conclusion

We should point out, that several of the here developed equations may be applied in connection with the evaluation of other types of second order derivatives than these of the normal potential.

The equations (15), (58) and (73) may for example be useful when computing the second order derivatives of the satellite potential, which generally is given by its expansion as a series in solid spherical harmonics. The matrix $\underline{\underline{B}}\,\underline{\underline{C}}^T$ of eq. (75) would in this case also contain terms related to the difference $A-h$, where

$$A = \tan^{-1}\left(\frac{\partial W}{\partial y} \Big/ \frac{\partial W}{\partial x}\right)$$

and W here denotes the satellite potential.

In Tscherning (1976) covariance expressions for the second order derivatives of the anomalous potential were derived and given as derivatives with respect to φ and h. These covariance quantities can be related to the actually measured quantities in either spherical or ellipsoidal approximation using the in this paper derived equations.

Also the recursion algorithms may be useful (in the proper modified form) for the evaluation of the sum of other types of series and their derivatives. A simple example is the well-known Horner's scheme for the evaluation of a usual polynomial and its derivatives.

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Appendix

```
procedure dnpot(gm, a, aj, n, m, t, r, sum);
value gm , a, n, m, r, t;  real gm, a, r, t;  integer n, m;
array aj, sum:

comment The procedure computes (1) the value of the gravitational
Part (V) of a normal potential and (2) the derivatives of V of
order up to m with respect to the radial distance (r) and the
cosine of the co-latitude (t). All units S.I. .

Variables:
gm    (call value) product of the mass of the Earth and the gra-
                   vitational constant in units of m**3/sec**2,
a     ( -      - ) the semi major axis of the reference ellipsoid
                   in units of meters,
aj    ( -      - ) the coefficients of degree 0 - n of the Legen-
                   dre-series representing the normal potential,
                   cf. ref., eq.(4).
n     ( -      - ) the maximal degree of the coefficients,
m     ( -      - ) the maximal order of the derivatives,
r     ( -      - ) the radial distance to the point of evalu-
                   ation.
t     ( -      - ) cosine to the co-latitude (the polar dis-
                   tance) of the point of evaluation.

sum (return values) the element having subscript ((2*m-i+3)*i)//2
                    +j is equal to the derivative of order i
                    with respect to t and order j with respect
                    to r. The array must be declared with sub-
                    script limits (0:((m+1)*(m+2))//2-1).

Reference: Tscherning, C.C.: Computation of the Second-Order
           Derivatives of the Normal Potential based on the
           representation by a Legendre Series. 1976;

begin
  integer i, j, k, ij, mm, k1, k2;
  real ek, fk1, s, s2, bk2:
  array c(0:m+1), bk1(-m-2:((m+1)*(m+2))//2-1);

  s:= a/r; s2:= s*s;
  mm:= ((m+1)*(m+2))//2-1;
  comment mm is equal to the total number of derivatives which
  will be computed:

  for k:= -m-2 step 1 until mm do bk1(k):= 0.0:
  for k:= 0 step 1 until mm do sum(k):= 0.0:
  k1:= n+1;  k2:= n+2;
```

```
  comment evaluation of the value of the Legendre-series and its
  derivatives, cf. ref., section 3:
  for k:= n step -1 until 0 do
  begin
    ek:= (2*k+1)*s/k1; fk1:= -k1*s2/k2;
    k2:= k1; k1:= k; ij:= 0;

    comment computation of the coefficients A(j,k), cf. ref.
    eq.(81);
    c(0):= aj(k);
    for j:= 1 step 1 until m do c(j):= -c(j-1)*(k+j);

    for i:= 0 step 1 until m do
    for j:= 0 step 1 until m- ido
    begin
      comment cf. ref. eq.(90);
      bk2:= bk1(ij); bk1(ij):= sum(ij);
      sum(ij):= ek*(bk1(ij)*t+i*bk1(ij-m+i-2))+fk1*bk2+c(j);
      ij:= ij+1;  c(j):= 0.00:
    end i, j loop (differentiation with respect to t, r);
  end k-loop;

  ij:= 0;  for i:= 0 step 1 until m do
  begin real b;
    b:= gm:
    for j:= 0 step 1 until m- ido
    begin
      b:= b/r;  sum(ij):= sum(ij)*b;  ij:= ij+1;
    end;
  end;
end of procedure body:
```

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