manuscripta geodaetica Vol. 1 (1976) 125-141

On the Chain-Rule Method for Computing Potential Derivatives

C. C. Tscherning

SUMMARY:

Among three methods for computing gravitational potential derivatives (the method of coefficient modification, the function method and the chain-rule method) the latter is according to L.J. Gulick (A Comparison of Methods for Computing Gravitational Potential Derivatives, ESSA Technical Report C&GS 40, 1970) the most favourable because of the fewest arithmetic operations and the least core storage used.

The chain-rule method takes advantage of the fact, that the gravitational potential may be expressed as a function of polar coordinates (latitude, longitude and distance from the origin). The coordinate system is singular at the poles, causing the algorithm developed by Gulick to fail at these points.

In the paper, algorithms valid at the poles are developed, thus using 7 % more arithmetic operations for the computation of the second order derivatives of the potential than used in the algorithm developed by Gulick. The new algorithms use, instead of polar coordinates, a (fixed) local orthogonal frame with origin in the point of evaluation, and coordinates x, y positive north and east, respectively, and z positive for increasing distance from the origin.

The different algorithms for the derivatives are strongly interrelated and may be regarded as one single algorithm giving e.g. all the first or all the second derivatives. This algorithm, which in the algorithmic language algol can be expressed as a procedure, is described and listed.

1. Introduction

The potential of the Earth, W, may be split into a gravitational part (V) and a rotational part Φ,

$$W(\varphi, \lambda, r) = V(\varphi, \lambda, r) + \Phi(\varphi, r), \tag{1}$$

where (φ, λ, r) are the usual polar coordinates (φ (geocentric) latitude, λ longitude and r distance from the origin). Let us suppose, that the gravitational potential (or an approximation to the potential) is represented in terms of solid spherical harmonics,

$$V(\varphi, \lambda, r) = \frac{GM}{r} \left[\sum_{n=o}^{N} \left(\frac{a}{r}\right)^n \sum_{m=o}^{n} P_n^m (\sin\varphi) \left(C_n^m \cos(m\lambda) + S_n^m \sin(m\lambda)\right) \right] \quad (2)$$

where GM is the product of the gravitational constant and the mass of the Earth, a is the semi-major axis of the reference ellipsoid, C_n^m and S_n^m are the so-called potential coefficients, P_n^m $(\sin\varphi)$ is the associated Legendre-polynomial of degree n and order m and N is the maximal degree of the available coefficients. The rotational potential is given by

$$\Phi(\varphi, r) = \frac{\omega^2}{2} (r \cdot \cos\varphi)^2 , \quad (3)$$

where ω is the speed of rotation (in units of sec-').

The first and second derivatives of W (or V) are needed for a variety of purposes and several procedures are available for their computation, see e.g. James (1969), Gulick (1970) or Hopkins (1974). The methods discussed differ from each other, both by their different methods of computation, their core-storage requirements and the number of arithmetic operations used.

A careful analysis of the methods, which directly evaluates the series eq. (2), has been published by Gulick (1970). He discussed three main types of methods, the method of coefficient modification, the chain-rule method and the function method, and concludes by recommending the chain-rule method.

Unfortunately the equations employed are not valid at the poles ($\varphi = 90^o$, $\varphi = -90^o$). This is due to the singularity of the polar coordinate system.

In Tscherning (1976) relations between the first and second order partial derivatives with respect to polar coordinates an3 with respect to a local orthogonal (x, y, z) coordinate system are developed. This coordinate system has its origin in the point of evaluation, z is positive in the direction opposite to the center of the Earth (= the "center" of the reference ellipsoid), x is positive north and y is positive east. Let the point of evaluation have rectangular coordinates (X_o, Y_o, Z_o) in a coordinate system with origin at the "center" of the reference ellipsoid, Z-axis coinciding with the semi-minor axis of the reference ellipsoid, X-axis equal to the line of intersection between the equatorial plane and the Greenwich meridian plane and Y-axis lying in the equatorial plane, orthogonal on the X-axis, so that the X, Y and Z axis form a right-handed coordinate system. Let the point have carresponding polar coordinates (φ_o, λ_o, r_o) and let furthermore an arbitrary point have coordinates (X, Y, Z) and (φ, λ, r) in each of the two global coordinate systems. We then have

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{Bmatrix} X_o \\ Y_o \\ Z_o \end{Bmatrix} + \underline{\underline{C}}^T \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \qquad (4)$$

where

$$\underline{\underline{C}}^T = \begin{Bmatrix} -\sin\varphi_o\cos\lambda_o & -\sin\lambda_o & \cos\varphi_o\cos\lambda_o \\ -\sin\varphi_o\sin\lambda_o & \cos\lambda_o & \cos\varphi_o\sin\lambda_o \\ \cos\varphi_o & 0 & \sin\varphi_o \end{Bmatrix} \qquad (5)$$

cf. Tscherning (1976, eq. (8) and (9)). The derivatives with respect to x, y or z evaluated in (x, y, z) = (0,0,0) are simply related to the derivatives with respect to polar coordinates.

According to Ibid. (1976, eq. (51) - (57)) we have

$$\frac{\partial}{\partial x} = \frac{1}{r}\frac{\partial}{\partial\varphi} \qquad (6)$$

$$\frac{\partial}{\partial y} = \frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda} \qquad (7)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r} \qquad (8)$$

$$\frac{\partial^2}{\partial x^2} = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} \qquad (9)$$

$$\frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r} - \frac{\tan\varphi}{r^2}\frac{\partial}{\partial\varphi} + \frac{1}{r^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} \qquad (10)$$

$$\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} \qquad (11)$$

$$\frac{\partial^2}{\partial x \partial y} = \frac{1}{r^2} \frac{\partial}{\partial \varphi} \left(\frac{1}{\cos\varphi} \frac{\partial}{\partial \lambda}\right) \tag{12}$$

$$\frac{\partial^2}{\partial x \partial z} = \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial \varphi}\right) \tag{13}$$

$$\frac{\partial^2}{\partial y \partial z} = \frac{\partial}{\partial r} \left(\frac{1}{r\cos\varphi} \frac{\partial}{\partial \lambda}\right) = \frac{1}{r\cos\varphi} \left(\frac{\partial^2}{\partial \lambda \partial r} - \frac{1}{r} \frac{\partial}{\partial \lambda}\right) \tag{14}$$

From the vector of first order derivatives

$$\left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z}\right)^T$$

or the 3 x 3 matrix of second order derivatives

$$\begin{bmatrix} \frac{\partial^2 W}{\partial x^2} & \frac{\partial^2 W}{\partial x \partial y} & \frac{\partial^2 W}{\partial x \partial z} \\ \frac{\partial^2 W}{\partial x \partial y} & \frac{\partial^2 W}{\partial y^2} & \frac{\partial^2 W}{\partial y \partial z} \\ \frac{\partial^2 W}{\partial x \partial z} & \frac{\partial^2 W}{\partial y \partial z} & \frac{\partial^2 W}{\partial z^2} \end{bmatrix}$$

we may easily obtain the partial derivatives in the (X, Y, Z) coordinate system or in other (orthogonal) coordinate systems by linear transformations, cf. Ibid (1976, eq. (14) and (15)).

Hence, we must apply the equations (6) - (14) to the function W in order to derive the necessary computational formulae. This will be done in section 2 for Φ and in section 3 for V. Finally in section 4, we discuss the algorithmic implementation of the formulae in the form of an algol procedure. The algol procedure is listed in an appendix.

2. Derivatives of the rotational potential

By straight forward differentiation we get, applying the equations (6) - (14) with $t = \sin\varphi$ and $u = \cos\varphi$:

$$\left.\begin{aligned}
\frac{\partial\Phi}{\partial x} &= -\omega^2\, r\cos\varphi\sin\varphi = -\omega^2 utr,\\
\frac{\partial\Phi}{\partial y} &= \frac{\partial^2\Phi}{\partial x\partial y} = \frac{\partial^2\Phi}{\partial z\partial y} = 0,\\
\frac{\partial\Phi}{\partial z} &= r\cos^2\varphi\,\omega^2 = \omega^2 ru^2,\\
\frac{\partial^2\Phi}{\partial x^2} &= \cos^2\varphi\,\omega^2 + (\sin^2\varphi - \cos^2\varphi)\,\omega^2 = (\omega t)^2\\
\frac{\partial^2\Phi}{\partial y^2} &= \cos^2\varphi\,\omega^2 + \sin^2\varphi\,\omega^2 = \omega^2,\\
\frac{\partial\Phi}{\partial z^2} &= \cos^2\varphi\,\omega^2 = (\omega u)^2,\\
\frac{\partial^2\Phi}{\partial x\partial z} &= -\omega^2\cos\varphi\sin\varphi = -\omega^2 tu.
\end{aligned}\right\} \qquad (15)$$

3. Computation of $V(\varphi, h, v)$ and of its first and second derivatives

Using eq. (2), we see, that V and its derivatives can be evaluated when we are able to compute the quantities

$$\frac{1}{r}\left(\frac{a}{r}\right)^n P_n^m(\sin\varphi)\cdot\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}$$

and the derivatives of these quantities.

We will in the following write down recursion formulae for the evaluation of these quantities. For the most easy formulation of these formulae, it is advantageous to extend the definition of the associated Legendre-polynomials so that $P_n^m(\sin\varphi)$ is identically zero for $n < 0$ or $m > n$. With these definitions we have with the initial value $P_o(t) = 1.0$, the basic recursion algorithms:

$$P_n(t) = ((2n-1)t P_{n-1}(t) - (n-1) P_{n-2}(t))/n \tag{16}$$

for $n > 0$ and $m = 0$ and

$$P_n^m(t) = (2n-1)u\, P_{n-1}^{m-1}(t) + P_{n-2}^m(t) \tag{17}$$

for $0 < m \leq n$. (Here we have again used $t = \sin\varphi$ and $u = \cos\varphi$).

Using

$$\frac{\partial}{\partial\varphi} P_n(t) = P_n^1(t), \tag{18}$$

cf. e.g. Gulick (1970, eq. (8b)), and that $P_0^1(t) = 0$ we have from eq. (17)

$$\frac{\partial}{\partial\varphi} P_n(t) = P_n^1(t) = (2n-1)u\, P_{n-1}(t) + P_{n-2}^1(t), \tag{19}$$

for $n > 0$ and $m = 0$. Using eq. (19) and (18) we have

$$\frac{\partial^2}{\partial\varphi^2} P_n(t) = (2n-1)\left[u\, P_{n-1}^1(t) - t\, P_{n-1}(t)\right] + \frac{\partial}{\partial\varphi} P_{n-2}^1(t). \tag{20}$$

Hence, when computing $\frac{\partial^2}{\partial v^2} P_n(t)$, the associated Legendre-polynomials of degree n-l and n-2, order 1 must already have been computed. (A somewhat simpler equation involving $P_n^2(t)$ may also be applied, but this quantity is not available at the moment when it (from an algorithmic standpoint) seems natural to compute $\frac{\partial^2}{\partial\varphi^2} P_n(t)$.

For the associated Legendre-polynomials we have, using eq. (17), for $0 < m \leq n$

$$\frac{\partial}{\partial\varphi} P_n^m(t) = (2n-1)\left[u \frac{\partial}{\partial\varphi} P_{n-1}^{m-1}(t) - t\, P_{n-1}^{m-1}(t)\right] + \frac{\partial}{\partial\varphi} P_{n-2}^m(t), \tag{21}$$

and

$$\frac{\partial^2}{\partial\varphi^2} P_n^m(t) = (2n-1)\left[u \frac{\partial^2}{\partial\varphi^2} P_{n-1}^{m-1}(t) - 2t \frac{\partial}{\partial\varphi} P_{n-1}^{m-1}(t) - u P_{n-1}^{m-1}(t)\right] + \frac{\partial^2}{\partial\varphi^2} P_{n-2}^m(t). \tag{22}$$

(Note, that the corresponding equations used by Gulick (1970) are not valid for $u = 0$).

Example 1

(This, and the following examples are included in order to facilitate the check of the algorithms in some simple cases),

$$\frac{\partial}{\partial\varphi} P_1(t) = \frac{\partial \sin\varphi}{\partial\varphi} = \cos\varphi = u,$$

$$\frac{\partial}{\partial\varphi} P_2(t) = \frac{\partial}{\partial\varphi} \left(\frac{3}{2} t^2 - \frac{1}{2}\right) = 3\ tu,$$

$$\frac{\partial^2}{\partial\varphi^2} P_1(t) = -t$$

$$\frac{\partial^2}{\partial\varphi^2} P_2(t) = 3(u^2 - t^2)$$

$$\frac{\partial}{\partial\varphi} P_1^1\ (t) = \frac{\partial \cos\varphi}{\partial\varphi} = -t,\ \frac{\partial^2}{\partial\varphi^2} P_1^1\ (t) = -u.$$

For the derivatives with respect to y we have

$$\frac{\partial}{\partial y} \left[\frac{1}{r} \left(\frac{a}{r}\right)^n P_n^m\ (\sin\varphi) \cdot \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \right]$$

$$= \frac{1}{r^2} \left(\frac{a}{r}\right)^n \frac{1}{\cos\varphi} P_n^m\ (\sin\varphi) \begin{Bmatrix} -m \sin eh \\ m \cos m\lambda \end{Bmatrix} \Bigg] \quad . \qquad (23)$$

Here we have to be careful for $\cos\varphi = u = 0$. For $m = 0$, we see that the derivative is zero. And for $m > 0$ we have a recursion formulae (cf. eq. (17))

$$\frac{1}{u} P_n^m\ (t) = (2n-1)\ P_{n-1}^{m-1}\ (t) + \frac{1}{u} P_{n-2}^m\ (t), \qquad (24)$$

valid for all values of u.

Example 2

$$\frac{1}{u} P_1^1(t) = 1, \quad \frac{1}{u} P_2^1(t) = 3t$$

For the second order derivatives we get from eq. (10)

$$\frac{\partial^2}{\partial y^2}\left[\frac{1}{r}\left(\frac{a}{r}\right)^n P_n^m(t) \begin{Bmatrix}\cos m\lambda \\ \sin m\lambda\end{Bmatrix}\right]$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{t}{ur^2}\frac{\partial}{\partial\varphi} + \frac{1}{u^2r^2}\frac{\partial^2}{\partial\lambda^2}\right) \circ \left[\frac{1}{r}\left(\frac{a}{r}\right)^n P_n^m(t) \begin{Bmatrix}\cos m\lambda \\ \sin m\lambda\end{Bmatrix}\right] . \tag{25}$$

Executing the second order differentiation with respect to h and dropping the differentiation with respect to r, we have first for m = 0 from eq. (19)

$$-\frac{t}{u}\frac{\partial}{\partial\varphi} P_n(t) = -(2n-1)t\, P_{n-1}(t) - \frac{t}{u} P_{n-2}^1(t) \tag{26a}$$

and for m > 0

$$\left[\frac{t}{u}\frac{\partial}{\partial\varphi} P_n^m(t) - \frac{m^2}{u^2} P_n^m(t)\right] \frac{1}{r^3}\left(\frac{a}{r}\right)^n \begin{Bmatrix}\cos m\lambda \\ \sin m\lambda\end{Bmatrix}$$

$$= \left[\frac{2n-1}{u}\left(-tu\frac{\partial}{\partial\varphi} P_{n-1}^{m-1}(t) + t^2 P_{n-1}^{m-1}(t) - m^2 P_{n-1}^{m-1}(t)\right)\right.$$

$$\left. - \frac{t}{u}\frac{\partial}{\partial\varphi} P_{n-2}^m(t) - \frac{m^2}{u^2} P_{n-2}^m(t)\right] \frac{1}{r^3}\left(\frac{a}{r}\right)^n \begin{Bmatrix}\cos m\lambda \\ \sin m\lambda\end{Bmatrix}$$

$$= \left[(2n-1)\left(-t\frac{\partial}{\partial\varphi} P_{n-1}^{m-1}(t) + (t^2 - m^2)\frac{1}{u} P_{n-1}^{m-1}(t)\right)\right.$$

$$\left. - \frac{t}{u}\frac{\partial}{\partial\varphi} P_{n-2}^m(t) - \frac{m^2}{u^2} P_{n-2}^m(t)\right] \frac{1}{r^3}\left(\frac{a}{r}\right)^n \begin{Bmatrix}\cos m\lambda \\ \sin m\lambda\end{Bmatrix} . \tag{26}$$

Note that this equation is valid for u = 0 because the critical quantity $(t^2 - m^2)/u$ is equal to $(t^2 - 1)/u = -u$ for m = 1.

Example 3

$$-\frac{t}{u}\frac{a}{\partial\varphi}P_1(t)-\frac{0}{u2}\ {}_1P(t)=-t\ ,$$

$$-\frac{t}{u}\frac{\partial}{\partial\varphi}P_1^1(t)-\frac{1}{u^2}P_1^1(t)=\frac{1}{u}(t^2-1)=-u\ ,$$

$$-\frac{t}{u}\frac{\partial}{\partial\varphi}P_2^1(t)-\frac{1}{u^2}P_2^1(t)=\frac{3}{u}\left[-2u^2t\right]=-6\ ut\ .$$

For $\frac{\partial^2}{\partial x\partial y}$ we get using eq. (12)

$$\frac{\partial^2}{\partial x\partial y}\left[P_n^m(t)\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}\right]=\frac{1}{r^2}\frac{\partial}{\partial\varphi}\left(\frac{1}{u}P_n^m(t)\right)\frac{\partial}{\partial\lambda}\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}\ . \qquad (27)$$

For $m>0$ we have (dropping the terms $\frac{\partial}{\partial\lambda}\cos m\lambda=-m\sin m\lambda$ and $\frac{\partial}{\partial\lambda}\sin m\lambda=m\cos m\lambda$)

$$\frac{\partial}{\partial\varphi}\left(\frac{1}{u}P_n^m(t)\right)=(2n-1)\frac{\partial}{\partial\varphi}P_{n-1}^{m-1}(t)+\frac{\partial}{\partial\varphi}\left(\frac{1}{u}P_{n-2}^m(t)\right)\ , \qquad (28)$$

which may be used in connection with the algorithm given by eq. (21).

Example 4

$$\frac{\partial}{\partial\varphi}\left(\frac{1}{u}P_1^1(t)\right)=0,\ \frac{\partial}{\partial\varphi}\left(\frac{1}{u}P_2^1(t)\right)=3u\ ,$$

$$\frac{\partial}{\partial\varphi}\left(\frac{1}{u}P_2^2(t)\right)=\frac{\partial}{\partial\varphi}(3u)=-3t\ .$$

In the same way we get for $\frac{\partial^2}{\partial x\partial z}$ using eq. (13),

$$\frac{\partial^2}{\partial x\partial z}\left[\frac{1}{r}\left(\frac{a}{r}\right)^n P_n^m(t)\right]=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\varphi}\left[\frac{1}{r}\left(\frac{a}{r}\right)^n P_n^m(t)\right]\right)$$

$$=\frac{-(n+2)}{r^3}\left(\frac{a}{r}\right)^n\frac{\partial}{\partial\varphi}P_n^m(t)\ , \qquad (29)$$

which may be evaluated using eq. (21).

Using eq. (14) we have for $\frac{\partial^2}{\partial y \partial z}$,

$$\frac{\partial^2}{\partial y \partial z}\left[\frac{1}{r}\left(\frac{a}{r}\right)^n \begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}\right]$$

$$= \frac{\partial}{\partial r}\left(\frac{1}{ru}\frac{\partial}{\partial \lambda}\left[\frac{1}{r}\left(\frac{a}{r}\right)^n P_n^m(t)\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}\right]\right)$$

$$= -\frac{(n+2)}{r^3}\left(\frac{a}{r}\right)^n \frac{1}{u} P_n^m(t)\begin{Bmatrix}-m \sin m\lambda\\ m \cos m\lambda\end{Bmatrix} \qquad (30)$$

which may be evaluated using eq. (24).

The first and second order derivatives with respect to z (r) may be evaluated using eq. (16), (17) and

$$\frac{\partial}{\partial z}\left[\frac{1}{r}\left(\frac{a}{r}\right)^n P_n^m(t)\right] = \frac{-(n+1)}{r^2}\left(\frac{a}{r}\right)^n P_n^m(t), \qquad (31)$$

$$\frac{\partial^2}{\partial z^2}\left[\frac{1}{r}\left(\frac{a}{r}\right)^n P_n^m(t)\right] = \frac{(n+1)(n+2)}{r^3}\left(\frac{a}{r}\right)^n P_n^m(t) \qquad (32)$$

4. The implementation of the recursion formulae in an algorithmic language

The recursion formulae are strongly interrelated (see Table 1 and 2) and it is advantageous to assemble the formulae in one algorithm capable of evaluating the first derivatives or the first and second derivatives.

A precise, explicit description of an algorithm may be given in the algorithmic language algol, which furthermore has the advantage that most electronic computers may interpret the algorithm.

In the appendix is listed an algol procedure, which (when adapted to an actual computer) is capable of evaluating W and its first and second derivatives for given values of t, u, r, cos(λ), sin(λ), GM, ω, a, N and the potential coefficients C_n^m, S_n^m of order and degree up to N, cf. eq. (2).

The central and most time consuming part of the computations is the evaluation of the recursion formulae eq. (16) - (32). In the equations it is generally so, that for the computation of a quantity of degree n and order m the same quantities of degree and crder (n-1, m-l) and (n-2, m) are needed (contingently together with one or more other quantities). The relationship between the different quantities is pictured in Table 1 for m = 0 and in Table 2 for m > 0.

Table 1. Interrelation between the left-hand side and the quantities occurring at the right-hand side in the recursion algorithms, m = 0. (The "x" indicate the use of the quantity occurring in the head of the column in the recursion formula used for the evaluation of the quantity occurring in the column denoted "Result".)

Result	Quantities on right-hand side						
	order	$P_n^m(t)$ 0	$P_n^m(t)$ 1	$\frac{\partial}{\partial\varphi} P_n^m(t)$ 0	$\frac{\partial}{\partial\varphi} P_n^m(t)$ 1	$\frac{1}{u} P_n^m(t)$ 0	$\frac{1}{u} P_n^m(t)$ 1
	degree						
$P_n(t)$	n - 2	x					
	n - 1	x					
$\frac{\partial}{\partial\varphi} P_n(t)$	n - 2		x				
	n - 1	x					
$\frac{\partial^2}{\partial\varphi^2} P_n(t)$	n - 2				x		
	n - 1	x	x				
$\frac{-t}{u} \frac{\partial}{\partial\varphi} P_n(t)$	n - 2						x
	n - 1	x					

Table 2. Interrelation between the left-hand side and the quantities occurring on the right-hand side in the recursion algorithm, $m > 0$. (The "x" indicate the use of the quantity occurring in the head of the column in the recursion formula used for the evaluation of the quantity occurring in the column denoted "Result".)

Result	Quantities on right-hand side										
		P_n^m		$\frac{\partial}{\partial\varphi} P_n^m$		$\frac{\partial^2}{\partial\varphi^2} P_n^m$		$\frac{1}{u} P_n^m$		$-\frac{t}{u}\frac{\partial}{\partial\varphi} P_n^m - \frac{m^2}{u} P_n^m$	$\frac{\partial}{\partial\varphi}(\frac{1}{u} P_n^m)$
	order	m-1	m	m-1	m	m-1	m	m-1	m	m-1 m	m
	degree										
P_n^m	n-2		x								
	n-1	x									
$\frac{\partial}{\partial\varphi} P_n^m$	n-2				x						
	n-1	x		x							
$\frac{\partial^2}{\partial\varphi^2} P_n^m$	n-2						x				
	n-1	x		x		x					
$\frac{1}{u} P_n^m$	n-2								x		
	n-1	x									
$-\frac{t}{u}\frac{\partial}{\partial\varphi} P_n^m - \frac{m^2}{u^2} P_n^m$	n-2									x	
	n-1			x				x			
$\frac{\partial}{\partial\varphi}(\frac{1}{u} P_n^m)$	n-2										x
	n-1			x							
$\frac{\partial}{\partial r}\left[\frac{1}{r}(\frac{a}{r})^n P_n^m\right]$	n		x								
$\frac{\partial^2}{\partial r^2}\left[\frac{1}{r}(\frac{a}{r})^n P_n^m\right]$	n		x								
$\frac{\partial}{\partial r}\left[\frac{1}{r \cdot u}\frac{\partial}{\partial\lambda}(\frac{1}{r}(\frac{a}{r})^n P_n^m)\right]$	n								x		
$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial\varphi}(\frac{1}{r}(\frac{a}{r})^n P_n^m)\right]$	n				x						

Note in the tables, that as soon as a quantity of degree n-2, order m is used once, it is not used anymore. This fact makes it possible to limit the storage requirements for the algorithm to (five) double subscripted arrays with bounds for example 0,1 and 0,N (cf. the declarations of the arrays P, DX, DY, DXY and DYY in the algol procedure listed in the appendix). The total core storage requirements will therefore increase like N^2 (necessary for the potential coefficients) and not like 2 N^2 as used in the algorithms developed by Gulick (1970).

Counting all the arithmetic operations (additions, subtractions and multiplications) shows that the number of operations are significantly bigger than the number of operations used in the algorithms developed by Gulick (1970). But we may take advantage of the kind of coordinate system employed in which the Laplace equation has its usual form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \qquad (33)$$

i.e. we may from two of these derivatives compute the third. In the algol procedure we have used

$$\frac{\partial^2 V}{\partial x^2} = - \frac{\partial^2 V}{\partial y^2} - \frac{\partial^2 V}{\partial z^2} . \qquad (34)$$

The number of operations occurring in the algol procedure becomes thereby only 7% bigger than the number of operations used by Gulick (1970, eq. (65)).

This difference seems very reasonable, considering the numerical advantages which have been achieved.

REFERENCES:

Gulick, L.J.: A comparison of Methods for Computing Gravitational Potential Derivatives, ESSA Technical Report, C & GS 40, 1970.

Hopkins, J.: Mathematical Models of Geopotential Gradients, In Mather R.S. and P.V. Angus-Leppan (Ed.): Proceedings of Symposium on Earth's Gravitational Field and Secular Variations in Position, pp. 93-105, June 1974.

Jarnes, R.W. : Transformation of Spherical Harmonics Under Change of Reference Frame, Geophys. J.R. astr. Soc. (1969) 17, pp. 305-316.

Tscherning, C.C.: Computation of the Second-Order Derivatives of the Normal Potential Based on the Representation by a Legendre Series, in print "Manuscripta Geodaetica", 1976.

Appendix.

```
real procedure g_pot_c(pos_i, C, N, all, G);
value N, all; boolean all: integer N; array pos_i, C, G;

comment Reference:
Tscherning, C.C.: On the Chain-rule method for Computing potential
-                 derivatives, 1976.

The procedure computer the value, the first derivative and if
wanted the second derivatives of the potential of the Earth
(W) or of its corresponding anomalous potential (T). Infor-
mation defining the position of the point of evaluation must
be stored in the array pos_i, see below.

The potentials ere represented by a series in solid spherical
harmonic), C must hold the (un-normalized) coefficientr, C(0) =
C(0, 0) = 1.0 for W and 0.0 for T, C(1) = C(1, 0), C(2) = C(1, 1),
C(3) = S(1, 1) etc. up to S(N, N), where N is the maximal
degree and order, (cf. ref., eq.(2)).

Parameters:
(a) Call values;
pos_i  holds position information end must be declared with
-      bounds (1:6). pos_i(1) = the radial distance (r),
-      pos_i(2) = p, the distance from the rotation (Z) - axis,
-      pos_i(3) = cosine of the geocentric polar distance,
-      pos_i(4) = sine   of -                 -    -   ,
-      pos_i(5), pos_i(6) = sine and cosine of the longitude,
C      must be declared with bounds (-3: (N+1)**2-1). It has
-      C(-3) = gm (the product of the mass and the gravitatio-
-      nal constant), C(-2) = the semi-major axis of the refe-
-      rence ellipsoid, C(-1) = the rotation speed of the Earth
-      (sec**-1), and C(0) - C((N+1)**2-1) = the coefficients
-      of the series.
N      the order of the series.
all    true when the second order derivatives are computed and
-      false when only the first order derivatives and W or T
-      are evaluated.
```

(b) Return values;

G The result is stored in G as follows:

- G(0, 1) = dW/dx, G(0, 2) = dW/dy, G(0, 3) = dW/dz and
- when all is true:
- G(1, 1) = ddW/ddx, G(1, 2) = G(2, 1) = ddW/dxdy, G(1, 3) =
- G(3, 1) = ddW/dxdz, G(2, 2) = ddW/ddy, G(2, 3) = G(3, 2) =
  ddW/dydz and G(3, 3) = ddW/ddz,
- where W may be interchanged by T and the variables x, y
- end z are the coordinates of a locel (fixed) frame, x is
- positive North, y positive East and z positive In the
- direction opposite to the center of the Earth, (cf. ref.
- eq. (4) and (5)).

The values of W or T will be returned by g-pot-c;

```
begin
  integer i, i1, m, ic, ia, ib;
  real t, u, r, p, s, si, si1, si2, si3, sl, cl, t2, u2,
  p0, dx, dy, dyy, dxy, p1, dx1, dy2, xi1, xi2, x2i, om2,
  V, Vx, Vy, Vz, Vyy, Vxy, Vzz, Vxz, Vyz, vx, vy, vz,
  vyy, vxy, cs0, csi, cic, b, c0, x2it, x2iu;
  array sml, cml, xm2(0:N), P, DX, DY, DYY, DXY(0:1, 0:N);

  r:= pos-i(1);  p:= pos-i(2);  t:= pos-i(3);  u:= pos-i(4);
  sl:= pos-i(5);  cl:= pos-i(6);  u2:= u*u;  t2:= t*t;

  comment Initialisations in order to assure that the associated
  Legendre-polynomials are zero for order > degree;
  for m:= 0 step 1 until N do
  for i:= 0, 1 do
  P(i, m):= DX(i, m):= DY(i, m):= DYY(i, m):= DXY(i, m):= 0.0;


  comment P(0, 0) Is the zero-degree Legendre-polynomial at this
  stage, s is a/r, si is s**0, cml(0) Is cos(0) and xi1 is 0+1;
  s:= C(-2)/r;  si:= cml(0):= xi1:= P(0, 0):= 1.0;
  sml(0):= xm2(0):= 0.0;

  comment ic points on the next potential coefficient to be
  used (in the array C), ia is the subscript of the row hol-
  ding recursion quantities of degree 1 and i-2 and ib is the
  subscript of the row holding quantities of degree i-1;
  ic:= 1;  ia:= 0;  ib:= 1;  x2i:= -1.0;  xi2:= 2.0;

  comment The computations are now performed in two loops, The
  i-loop increasing the degree and the m-loop the order, The
  contribution to the different derivatives comming from each
  degree 1 ( > 0) are first accumulated In the variables vx,
  vy, vx etc, Theese quantities ere then again accumulated
  in the variables Vx, Vy, Vz etc, (Note, that the variables
  Vyy first at the very end of the procedure will conteln the
  quantity Vz/r, cf, ref, eq. (10)), This kind of accumulation
  is used in order to avoid the dlseppearance of the possible
  significant contribution from a number of #small¹ quantities.
  Of the same reason will the contribution from the zero degree
  term first be added at the end of the procedure;

  V:= Vx:= Vy:= Vz:= Vyy:= Vzz:= Vyz:= Vxz:= Vxy:= 0.0;
```

```
for i:= 1 step 1 until N do
begin
  comment xi1 is equal to i+1, xi2 to i+2 and x2i to 2*i+1;
  xi1:= xi2;  xi2:= xi2+1;  x2i:= x2i+2;  i1:= i-1;
  x2it:= x2i*t;  x2iu:= x2i*u;
  cic:= C(ic);  ic:= ic+1;
  si:= si*s;  si1:= xi1*si;  si2:= xi2*si;  si3:= xi2*si1;

  comment Interchange of the two rows in the arrays P - DXY
  by means of the following three statements;
  m:= ia;  ia:= ib;  ib:= m;

  comment sml(i):= sin(i*long.), cml(i):= cos(i*long.);

  sml(i):= sml(i1)*cl+cml(i1)*sl;
  cml(i):= cml(i1)*cl-sml(i1)*sl;
  xm2(i):= if i = 1 then -u else t2-i*i;

  comment computation of the contribution from the zonal*
  harmonic coefficients and of the derivatives of the i'th
  degree zonal-harmonic (a Legendre-polynomial), cf. ref. eq.
  (16), (18) and (19);

  p1:= P(ib, 0);  p0:= P(ia, 0):= (x2it*p1-i1*P(ia, 0))/i;
  dx1:= DX(ib, 0);  dx:= DX(ia, 0):= x2iu*p1+P(ia, 1);
  vx:= cic*dx;  vy:= 0.0;  vz:= cic*p0;

  14 ell then
  begin
    comment cf. ref. eq.(26a). The variable dy2 is used when
    evaluating ref. eq.(26) for m = 1;
    dyy:= DYY(ia, 0):= -x2it*p1-t*DY(ia, 1);
    vyy:= dyy*cic;  vxy:= 0.0;   dy2:= p1;
  end second order deriv.;

  for m:= 1 step 1 until i do
  begin
    comment cf. ref. eq. (17), (21), (23) and (24);
    cs0:= C(ic)*cml(m)+C(ic+1)*sml(m);
    csi:= (-C(ic)*sml(m)+C(ic+1)*cml(m))*m;  ic:= ic+2;
    p0:=  P(ia, m):= x2iu*p1          + P(ia, m);
    dx:= DX(ia, m):=-x2it*p1+x2iu*dx1+DX(ia, m);
    dy:= DY(ia, m):= x2i* p1          +DY(ia, m);

    vx:= vx+cs0*dx;  vy:= vy+csi*dy;  vz:= vz+cs0*p0;

    if all then
    begin
      comment cf. ref. eq. (26), (27) and (28);
      dxy:= DXY(ia, m):= x2i*dx1+DXY(ia, m);
      dyy:= DYY(ia, m):= -x2it*dx1+xm2(m)*x2i*dy2+DYY(ia, m):
      vyy:= vyy+cs0*dyy;
      vxy:= vxy+csi*dxy;  dy2:= DY(ib, m);
    end second order deriv.;

    p1:= P(ib, m);  dx1:= DX(ib, m);
  end m-loop;
```

```
    V:= V+si*vz;  Vx:= Vx+si*vx;  Vy:= Vy+si*vy;  Vz:= Vz-si1*vz;

    if all then
    begin
      comment cf. ref. eq. (25), (32), (27), (29) and (30);
      Vyy:= Vyy+si*vyy;
      Vzz:= Vzz+si3*vz;  Vxy:= Vxy+si*vxy;
      Vxz:= Vxz-si2*vx;  Vyz:= Vyz-si2*vy;
    end second order deriv. accumm.;
  end i-loop;

  comment adding zero-order term and centrifugal force contri-
  bution, cf. ref. eq. (1), (3) and (15);

  b:= C(-3)/r;  c0:= C(0);  om2:= C(-1)**2;

  gpot_c:= b*(c0+V)+om2*p**2/2;

  b:= b/r;  Vz:= b*(Vz-c0);
  G(0, 1):= b*Vx-t*p*om2;
  G(0, 2):= b*Vy;
  G(0, 3):= Vz+u2*r*om2;

  if all then
  begin
    b:= b/r;  Vyy:= b*Vyy+Vz/r;  Vzz:= b*(2*c0+Vzz);
    G(1, 1):= -(Vyy+Vzz)+om2*t2;  comment cf. ref. eq.(34);
    G(1, 2):= G(2, 1):= b*Vxy;
    G(1, 3):= G(3, 1):= b*Vxz-u*t*om2;
    G(2, 2):= Vyy+om2;
    G(2, 3):= G(3, 2):= b*Vyz;
    G(3, 3):= Vzz+u2*om2;
  end second order deriv.;
end proc. body;
```

date received: 1976-07-19