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A Mass Density Covariance Function Consistent with the Covariance Function of the Anomalous Potential (*)

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Summary. — The disturbing mass densities, i.e. the differences between the actual mass densities and a mass density reference function are regarded as the values of a random function.

Steffen Lauritzen (Random Orthogonal Set Functions and Stochastic Models for the Gravity Potential of the Earth, Stoch. Proc. Appl., Vol 3, 1975, pp. 65-72) has derived a covariance function of the mass variations, which is consistent with (an estimate of) the covariance function of the anomalous potential. The disturbing mass covariance function indicates, that the mass distributions become more and more irregular as one approaches the center of the Earth.

In this paper other hypothetical density covariance functions are derived and explicitly given, which are consistent with the same covariance function of the anomalous potential as used by Lauritzen. These density covariance functions give maximal values for the density variation at the surface of the Earth.

1. — INTRODUCTION.

The disturbing masses, i. e. the differences between the actual mass densities and a certain mass density reference function (see e. g. Moritz (1973)) may be regarded as the values of a random function. Lauritzen (1975) describes the disturbing mass by a random *orthogonal* set function and shows, that the degree-variances of the covariance function of the anomalous potential are simply related to the moments of the covariance measure of the random function.

Using this relationship and an estimate of the covariance function of the anomalous potential (derived in Lauritzen (1973)), it is shown, as a consequence, that the mass distributions must become more and more irregular as one approaches the center of the Earth.

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Now describing the random disturbing masses by an orthogonal set function means, that disjoint clods of mass are regarded as independent. But physically it seems quite likely, that density anomalies lying in the same distance of the center of the Earth are more correlated than density anomalies situated in different distances from the center. We would also generally suppose, that the biggest mass anomaly variations occurred at the surface of the Earth.

In a statistical model having these properties, it would be possible to represent the "topographic" masses as (relatively) big mass variations in the region near to the surface of some regular body approximating the Earth.

We will in this paper give an example of such a statistical model, which furthermore has the advantage, that the covariance functions are easily computable through closed expressions.

Instead of prescribing some probabilistic property for the random function, we may as proposed by Tscherning (1974) regard mass density functions, which are solutions to a partial differential equation. Denoting the density anomaly function by d we will require, that

$$\Delta\,(r^n d) = 0, \tag{1}$$

where Δ is the Laplace operator (in three variables), r the distance from the center of the Earth and n is an integer. (See Krarup (1970) for an interpretation of the condition (1) in the form of a minimum condition for the mass distributions).

We will in section 2 give expressions (in the form of the sum of a Legendre series) for the density anomaly covariance functions $\mathrm{cov}(d_X, d_Y)$, which will produce the specific covariance function of the anomalous potential $\mathrm{cov}(T_P, T_Q)$ used by Lauritzen (1973, 1975).

For the actual numerical computation of the density anomaly covariance function it is necessary (and possible) to derive closed expressions for the Legendre series. This is done in section 3, and in section 4 we analyse the behaviour of the obtained covariance functions and discuss some other models. Finally, in section 5 we outline some subjects for future investigations.

2. — The series expression for the mass density covariance function.

The covariance function of the anomalous potential of the Earth, which has been developed by Lauritzen (1973), may be expressed as the sum of a Legendre series

$$\mathrm{cov}\,(T_P, T_Q) = \sum_{i=3}^{\infty} \frac{A\,R_B^2}{(i-1)\,(i-2)}\, u^{i+1} P_i\,(t)\,. \tag{2}$$

Here A is a constant in units of mGal², R_B is the radius of a (Bjerhammar-) sphere totally enclosed in the Earth, T_P and T_Q are the values of the anomalous potential in points P and Q (here outside the Earth), t is equal to the cosine of the spherical distance between the points, $P_i(t)$ the Legendre polynomial of degree i and

$$u = \frac{R_B^2}{r\,r'} ,$$

where r and r' are the distances of P and Q, respectively, from the origin.

Introducing the quantities

$$\sigma_i(T, T) = \frac{A\, R_B^2}{(i-1)\,(i-2)} , \tag{3}$$

which are denoted the (model) potential degree-variances, we have

$$\text{cov}\ (T_P, T_Q) = \sum_{i=3}^{\infty} \sigma_i(T, T)\, u^{i+1}\, P_i\ (t).$$

We may introduce a similar quantity:

$$\bar{\sigma}_i(T, T) = \frac{A R_B^2 p^{i+1}}{(i-1)\,(i-2)} , \tag{4}$$

where:

$$p = \left(\frac{R_B}{R_E}\right)^2,$$

and R_E is the mean radius of the Earth. Then

$$\text{cov}\ (T_P, T_Q) = \sum_{i=3}^{\infty} \bar{\sigma}_i(T, T) \left(\frac{R_E^2}{r.\,r'}\right)^{i+1} P_i(t) = \sum_{i=3}^{\infty} \bar{\sigma}_i(T, T)\, v^{i+1}\, P_i(t), \tag{5}$$

where $v = R_E^2\, /\, (r\, r')$.

It is possible to obtain the covariance function (2) from a covariance function of the density anomalies fulfilling (1), cf. Tscherning (1974). The density anomaly functions will have a covariance function

$$\text{cov}\ (d_P, d_Q) = \sum_{i=3}^{\infty} \bar{\sigma}_i(T, T)\ \frac{(2\,i+1)^2\,(2\,i+3-n)^2}{R_E^4\,(4\,\pi\,k)^2}\, w^{i-n}\, P_i(t) , \tag{6}$$

where n is the same "n" as in equation (1), $w = (r.r')/R_E^2$, k the gravitational constant and P, Q now both points inside the Earth. It may easily be verified, using (2) and (6), that

$$\mathrm{cov}\,(T_P, T_Q) = \int_{\text{Earth}} \left(\int_{\text{Earth}} \mathrm{cov}\,(d_X, d_Y) \frac{k}{\| P - X \|} \, dX \right) \frac{k}{\| Q - Y \|} \, dY ,$$

where P and Q are points outside and X, Y are points inside the Earth.

3. — Derivation of a closed expression.

The covariance function (2) may be represented by a closed expression, see e.g Tscherning and Rapp (1974, section 8). We have

$$\mathrm{cov}\,(T_P, T_Q) = A\, R_B^2 \left| u^3 \left(P_2(t) + \frac{1 - t^2}{4} \right) + \right. \tag{7}$$

$$\left. + \ln \frac{2}{N} \left(u^3 P_2(t) - u^2 t \right) + u\,(3\,t\,u - 1) \frac{M}{2} \right| ,$$

where for:

$$L = (1 - 2\,u\,t + u^2)^{1/2} \tag{8}$$

we have put

$$M = 1 - L - u\,t \quad \text{and} \tag{9}$$

$$N = 1 + L - u\,t\,. \tag{10}$$

(The quantities L, M, N will in the following be used with a variable s substituted for the variable u as well).

We will use the closed expression (7) for the derivation of a closed expression for (6). Now we note that (6) can be expressed as a function of a variable

$$s = r\,r' \frac{R_B^2}{R_E^4} :$$

$$\mathrm{cov}\,(d_P, d_Q) = \sum_{i=3}^{\infty} \frac{A\, R_B^2\, p^{n+1}}{(i-1)\,(i-2)} \cdot \frac{(2\,i+1)^2\,(2\,i+3-n)^2}{R_E^4\,(4\,\pi\,k)^2} \cdot \tag{10 a}$$

$$\left(\frac{R_B^2\, r\, r'}{R_E^2\, R_E^2} \right)^{1-n} P_i(t)\,.$$

Using (2) and (7) we see, that

$$G = \sum_{i=3}^{\infty} \frac{A\, R_B^2\, p}{(i-1)\,(i-2)}\, s^i\, P_i\,(t) = \tag{11}$$

$$= A\, R_B^2\, p \left[s^2 \left(P_2(t) + \frac{1-t^2}{4} \right) + \ln \frac{2}{N}\, (s^2 P_2(t) - s\,t) + \frac{(3\,t\,s - 1)\, M}{2} \right].$$

The idea is now to find an appropriate differential operator, which when applied at the function G defined by (11) will furnish us with a closed expression for (7).

For an arbitrary function f of s we have , by putting

$$R = \frac{R_E^2}{R_B}\,, \quad f' = \frac{\partial f}{\partial s}\,, \quad f'' = \frac{\partial^2 f}{\partial s^2}:$$

$$\frac{\partial^2 f}{\partial r\, \partial r'} = \frac{\partial}{\partial r} \left(\frac{\partial s}{\partial r'}\, f' \right) = \left(\frac{\partial}{\partial r}\, \frac{r}{R^2}\, f' \right) = \tag{12}$$

$$= \frac{1}{R^2}\, f' + \frac{\partial s}{\partial r}\, \frac{r}{R^2}\, f'' = \frac{1}{R^2}\, (f' + s f'')\,.$$

Then for:

$$h_i = 4\, s^i\, s^{1/2}$$

we have using (12)

$$\frac{\partial^2 h_i}{\partial r\, \partial r'} = \frac{1}{R^2} \left(4\,(i + 1/2)\, s^{i-1/2} + s\,.\,4\,(i + 1/2)\,(i - 1/2)\, s^{i-3/2} \right) = \tag{13}$$

$$= \frac{1}{R^2}\, s^{i-1/2}\, (2i + 1)^2\,.$$

Putting:

$$k_i = \frac{\partial^2 h_i}{\partial r\, \partial r'}\, (4 s^{2-n/2}) = \frac{4\,(2\,i + 1)^2}{R^2}\, s^{i-(n-3)/2}$$

we get, using (12) once more

$$\frac{\partial^2 k_i}{\partial r\, \partial r'} = \frac{(4\ 2i+1)^2}{R^4}\left(\left(i - \frac{n-3}{2}\right) s^{i-(n-1)/2} + \right. \tag{14}$$

$$\left. + \left(i - \frac{n-3}{2}\right)\left(i - \frac{n-1}{2}\right) s^{i-(n-1)/2}\right) =$$

$$= \frac{1}{R^4}\,(2i+1)^2\,(2i+3-n)^2\, s^{i-(n-1)/2}\,.$$

Hence, finally combining (11), (13) and (14), we have

$$\operatorname{cov}(d_P, d_Q) = \frac{p^n R^4}{(4\pi k R_E^2)^2}\, s^{-(n+1)/2}\left(\frac{\partial^2}{\partial r\, \partial r'}\left(4\, s^{2-n/2}\, \frac{\partial^2}{\partial r\, \partial r'}\,(s^{1/2}\,.\,4\,G)\right)\right). \tag{15}$$

The differential operator

$$D_n(G) = \frac{p^n,\ R^4\, s^{-(n+1)/2}}{4\,(\pi k R_E^2)^2}\left(\frac{\partial^2}{\partial r\, \partial r'}\left(4\, s^{2-n/2}\, \frac{\partial^2}{\partial r\, \partial r'}\,(s^{1/2}\,.\,4\,G)\right)\right)$$

may naturally be used to derive closed expressions for covariance functions, which correspond to other covariance models than the one discussed here.

We will then apply this operator on the closed expression (11).

Putting

$$G_1 = 4\, s^{1/2}\, G,$$

$$C = s^{5/2}\left(P_2(t) + \frac{1-t^2}{4}\right), \tag{16}$$

$$D = s^{5/2}\, P_2(t) - s^{3/2}\, t\,, \tag{17}$$

$$E = \ln \frac{2}{N}\,, \tag{18}$$

$$F = \frac{3\, t\, s^{3/2} - s^{1/2}}{2} \tag{19}$$

we have from (11):

$$G_1 = 4\, A\, R_B^2\, p\,[\,C + D\,E + F\,M\,]\,. \tag{20}$$

Using (12) we have (where the apostrophe means differentiation with respect to s):

(21)
$$\frac{\partial^2}{\partial r \partial r'} G_1 = \frac{1}{R^2} (G_1' + s \, . \, G_1'') =$$

$$= 4 \, . \, A \, . \, R_B^2 \, . \, p \, \frac{1}{R^2} [C' + D\,E' + D'\,E + F\,M' + F'\,M +$$

$$+ s\,(C'' + D\,E'' + 2\,D'\,E' + D''\,E + F\,M'' + 2\,F'\,M' + F''\,M)\,].$$

We must then compute these first and second derivatives. Let us first compute the auxiliar quantities L', L'' and N'. From (8) we have (with s substituted for u)

(22)
$$L' = (s - t) \,/\, L,$$

(23)
$$L'' = 1 \,/\, L + (s - t)\,(t - s) \,/\, L^3 =$$

$$= 1 \,/\, L - (s^2 - 2\,t\,s + t^2) \,/\, L^3 = (1 - t^2) \,/ L^3$$

and from (10)

(24)
$$N' = (s - t) \,/\, L - t.$$

We then have, using (9), (22) and (23)

(25a)
$$M' = (-s + t) \,/\, L - t,$$

(25b)
$$M'' = (t^2 - 1) \,/\, L^3,$$

from (16)

(26a)
$$C' = \frac{5}{2}\, s^{3/2}\, (P_2(t) + (1 - t^2) \,/\, 4)\,,$$

(26b)
$$C'' = 15\, s^{1/2}\, (P_2(t) + (1 - t^2) \,/\, 4) \,/\, 4\,,$$

from (17)

(27a)
$$D' = (5\, s^{3/2}\, P_2(t) - 3\, s^{1/2}\, t) \,/\, 2\,,$$

(27b)
$$D'' = (15\, s^{1/2}\, P_2(t) - 3\, s^{-1/2}\, t) \,/\, 4\,,$$

and from (18) and (22)

$$E' = \frac{N}{2} \cdot \frac{(-s + t + t\,L)\,2}{L\,N^2} = \frac{(t - s + t + L)}{N\,L}\,,$$

which may be further reduced to

(28a) $$E' = \frac{s\,(L - 1)\,(1 - t^2)}{s^2\,(t^2 - 1)\,L} = \frac{1 - L}{s\,L} = \frac{1}{s\,L} - \frac{1}{s}$$

and hence

(28b) $$E'' = \frac{1}{s^2} - \frac{1}{s^2\,L} + \frac{t - s}{L^3\,s}\,.$$

From (19) we finally get

(29a) $$F' = (\,9\,t\,s^{1/2} - s^{-1/2}\,)\,/\,4$$

(29b) $$F'' = (9\,t\,s^{-1/2} + s^{-3/2}\,)\,/\,8.$$

Substituting the expressions (25a,b) — (29a,b) in (21) we get after some reductions

(30) $$\frac{\partial^2}{\partial r\,\partial r'}\;G_1 = A\,p^3\,s^{-1/2}\;\Big|\;s^2(15 - 145\,t^2)\,/\,4 + E\,(P_2(t)\,25\,s^2 - 9\,t\,s)\;+$$

$$+\;30\,t\,s - 1/2 + L\,(1 - 75\,t\,s)\,/\,2 + (8\,t\,s - 4\,s^2)\,/\,L\;\Big|\,.$$

In order to facilitate the derivations, we will introduce new quantities H, I, J, K, Q, U, V and W and simultaneously compute the first and second derivatives with respect to s of the first five of these:

(31) $\mathrm{H} = s^2\,(15 - 145\,t^2)\,/\,4,\quad H' = s\,(15 - 145\,t^2)\,/\,2,\quad H'' = (15 - 145\,t^2)\,/\,2,$

(32) $I = P_2(t)\,25\,s^2 - 9\,t\,s,\quad I' = P_2(t)\,.\,50\,s - 9\,t,\quad I'' = 50\,P_2(t),$

(33) $J = 30\,t\,s - 1/2 \qquad J' = 30\,t, \qquad J'' = 0,$

(34) $K = (1 - 75\,t\,s)\,/\,2, \qquad K' = -\,75\,t\,/\,2, \qquad K'' = 0,$

$$(35)\quad \begin{cases} Q = (8\,t\,s - 4\,s^2)\,/\,L\,, \\ Q' = (8\,t\,s - 4\,s^2)\,(t - s)\,/\,L^3 + 8\,(t - s)\,/\,L = 4\,(t - s)\,(1\,/\,L^3 - 1\,/\,L) \\ Q'' = 4\,(-1\,/\,L^3 - 3\,(t - s)\,(s - t)\,/\,L^5 + (1 - t^2)\,/\,L^3) = \\ \quad = 4\,(\,(3 - t^2)\,/\,L^3 + 3\,(t^2 - 1)\,/\,L^5) \end{cases}$$

$$(36)\qquad U = s^{-(n-3)/2}$$

$$(37)\qquad V = s^{1/2}\,\frac{\partial^2}{\partial r\,\partial r'}\,G_1 = A\,p^3\,[\,H + E\,I + J + L\,K + Q\,]$$

and

$$(38)\qquad W = \frac{\partial^2}{\partial r\,\partial r'}\,(U\,V).$$

Using the equations (15) and (30) – (38) we see, that

$$(39)\qquad \operatorname{cov}\,(d_P\,,\,d_Q) = \frac{p^{n-2}\,s^{-(n+1)/2}}{4\,(\pi\,k)^2}\,W.$$

Putting $m = -(n - 3)\,/\,2$ and using (36) — (38) we get

$$(40)\quad \begin{cases} U' = m\,s^{m-1}, \qquad U'' = m\,(m - 1)\,s^{m-2} \\ V' = (H' + E\,I' + E'\,I + J' + L\,K' + L'\,K + Q')\,A\,p^3 \\ V'' = (H'' + I''\,E + 2I'\,E' + I\,E'' + L''\,K + 2\,L'\,K' + Q'')\,A\,p^3 \end{cases}$$

and by (12)

$$(41)\qquad W = \frac{1}{R^2}\,(V\,U' + V'\,U + s\,(V\,U'' + 2\,V'\,U' + V''\,U)\,) =$$

$$= s^{-(n-1)/2}\,\frac{1}{R^2}\,(m^2\,V + (2\,m + 1\,)\,s\,V' + s^2\,V'')\,.$$

Finally

$$(42)\qquad \operatorname{cov}\,(d_P\,,\,d_Q) = \frac{(p\,/\,s\,)^2}{4\,(\pi\,k\,R_B)^2}\,(m^2\,V + (2\,m + 1)\,s\,V' + s^2\,V'')\,,$$

and the computation of the covariances can be then easily be carried out for varying P and Q.

4. — THE BEHAVIOUR OF THE COVARIANCE FUNCTION FOR VARYING n, R_B, ψ AND s.

Values of the covariance function cov (d_P, d_Q) have been computed for $n = -2, -1, 0, 1, 2$ (equation (1)), for varying R_B (the radius of the Bjerhammar sphere), spherical distance ψ and distance of the points P and Q from the origin (as expressed through the quantity s). Figure 1 shows the graph of the covariance function computed for $0 \leqslant \psi \leqslant 3°$, $n = 0$ and $r = r' = 0.99 \,.\, R_E$, $0.999 \,.\, R_E$ and $1.000 \,.\, R_E$. The values given by Lauritzen (1973)

(43) $$A = 7.84888 \text{ mGal}^2$$

$$R_B \,/\, R_E = 0.9945$$

were used.

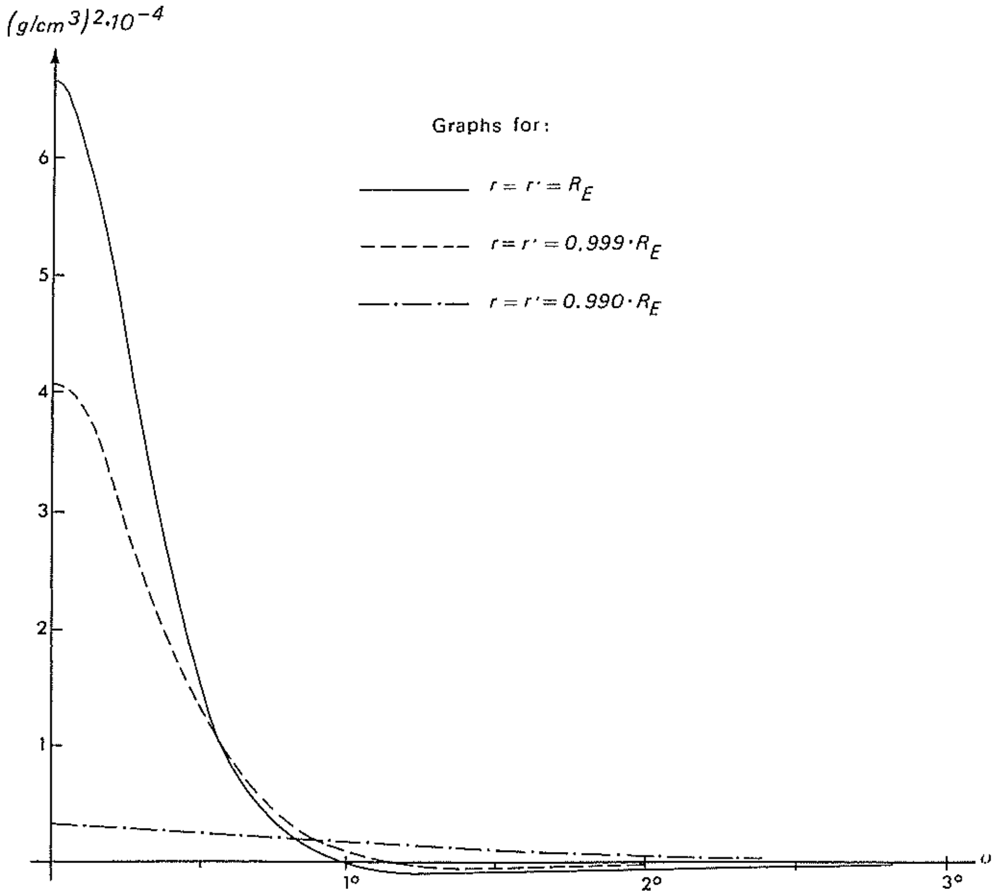

Fig. 1 – The covariance function of the mass density anomalies corresponding to a harmonic density function for varying spherical distance and depth.

The shape of the graph is very dependent on the radius of the Bjerhammar sphere. The general trend is an increase in the value for $\psi = 0°$ for $R_B \to R_E$ and a movement of the first zeropoint to the left. This means, that the mass density anomalies become more varying and independent for the radius of the Bjerhammar sphere increasing. Only a small variation was observed, when the covariance was computed for different n (= — 2, — 1, 0, 1, 2).

The variation of the density anomalies decreased in these cases very rapidly towards zero for r_P and $r_Q \to 0$. The variation was generally less than 1 % of the variation at the surface for a depth equal to around 150 km. (This trend will change for $n \geq 3$, because zero or negative powers of s will appear in equation (10a)). For $n > 3$ we are getting models, which give the same picture of the gravity variation as the Lauritzen (1975) model, i. e. the density anomaly variation increases as we approach the center of the Earth.

What will then be a realistic model for the density anomaly variation?

We know from the use of Bouguer-anomalies or terrain corrected free-air anomalies, that they are very much smoother than simple free-air gravity anomalies. This fact indicates, that very much information related to the local variation of the gravity anomalies is contained in the tpopography and the density variations in the upper crust.

The variation in the correlation between free-air anomalies and the topographic heights has even been used to estimate the depth of the density anomalies causing the gravity anomalies, see Vyscočil (1971, 1972) and Allan (1972).

Kaula (1967) has analysed the low order (long wavelength) variation of the topography and studied the correlation between the empirical degree-variances of the gravity and the degree-variances of the topography, transformed into a surface density layer. He found only a small correlation and concluded, that the long-wave variations did not have «an obvious direct connection to the visible topography» (cf. Kaula (1967), p. 778).

In the models discussed by Tscherning (1974) the variation of the correlation between density anomalies and (surface) gravity anomalies is determined by a factor $(r / R)^{i-n}$. For small i (and $i \geq n$) we see, that the correlation still may be significant at big depths. On the other hand, the correlation is only significant near the surface of the Earth for big i.

5. — Conclusions.

In the preceding sections we have analysed covariance functions of density anomalies all consistent with one specific estimate of the covariance function of the anomalous potential. Other estimates have been published (e.g. by Tscherning and Rapp (1974)) which could be analysed in a similar way. The analysis should not be confined to the behaviour of the density anomaly covariance function. The covariance functions should be used to predict gravity anomalies and deflections of the vertical e.g. from known height and density variations, potential coefficients and mean gravity anomalies applying the method of least squares collocation.

We will finally remark, that the study of density covariance function may be perceived as a first step toward the introduction of a general "reduction" or "correction" technique, applicable for gravity anomalies, deflections of the vertical and similar quantities. The reduction will not have the purpose of reducing any quantity up or down to for example an equipotential surface of the potential of the Earth, but it may have a similar effect. It will smooth out the observed data in a way, which hopefully should make the data more applicable for interpolation or extrapolation.

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