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Determination of Datum-Shift Parameters using Least Squares Collocation (*)

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Summary. — Datum-shift parameters (for example the components of the vector, which translates a local geodetic coordinate system into a geocentric coordinate system) may be determined by a least squares adjustment using the geodetic coordinates of points for which the geocentric coordinates are known. This procedure requires that the ellipsoidal height and hence the height anomalies of the points are known in the local geodetic coordinate system.

The paper describes how the method of least squares collocation may be applied for the simultaneous estimation of the height anomalies and the datum-shift parameters. A computational implementation is described and preliminary results of datum-shift determinations for Scandinavia and Greenland are reported.

1. — INTRODUCTION.

Results of several datum-shift determinations have been published in the last decade (see e.g. Veis (1968), Lambeck (1971), Marsh, Douglas, Klosko (1973), Anderle (1974), Seppelin (1974), Mueller (1974), and Schmid (1974)). All these determinations are based on the knowledge of the geodetic latitude, longitude and ellipsoidal height of certain points in which the geocentric coordinates have been determined using some satellite technique. The coordinate differences are treated as observations, and the datum-shift parameters are then determined by a least squares adjustment.

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Now, the geodetic coordinates contain errors. A part of the error is regarded as being systematic (e.g. scale and orientation errors) and may be determined simultaneously with the datum-shift parameters. Other errors are generally treated as being random. The error in the ellipsoidal height, for example, is regarded as random. Though, this error could be partly due to a systematic erroneous bending of an astro-geodetic geoid.

There exist now methods for detailed geoid determination, which may be superior to old-fashioned astro-geodetic (see e.g. Vincent and Marsh (1974)), using mean gravity anomaly and potential coefficient information. But these methods will determine geocentric geoidal heights and hence not improve the local geoid determinations.

On the other hand, the method of least squares collocation, may use deflections of the vertical given in a local geodetic coordinate system as well as mean (or point) gravity and potential coefficient information. This method has, when used for the determination of datum-shift parameters, a near connection to the earlier used methods for the establishment of a local geodetic datum. Here a weighted mean of the deflections of the vertical was required to attain a minimum. In least squares collocation deflections of the vertical and gravity anomalies are, for example, regarded as correlated and the quadratic form

$$Q = x^{T} C^{-1} x \tag{1}$$

is required to attain a minimum. In equation (1) x denotes a vector of observed quantities and C is their variance-covariance matrix. The superscript T denotes here and in the following transposition of a vector or a matrix.

2. — Determination of datum-shift parameters.

In least squares collocation we may, following Moritz (1972), take our departure from the following equation of observation

$$x = A\,X + s' + n\,, \tag{2}$$

where x as above is a r-vector of observed quantities, A a given $r \times q$ matrix, X a r-vector of parameters, s' a r-vector (denoted the signal in some relations) and n a r-vector (the « noise »).

A particular realization of the signal s' (which we also will denote by s), and the parameters X are estimated by

(3) $$\tilde{s} = C_s^T \bar{C}^{-1} (x - A \tilde{X}) \quad \text{and}$$

(4) $$\tilde{X} = (A^T \bar{C}^{-1} A)^{-1} A^T \bar{C}^{-1} x .$$

The matrix $\bar{C} = C + D$ is a $r \times r$ matrix equal to the sum of the given variance-covariance matrix C of the signal vector s' and the variance-covariance matrix D of the noise vector n. C_s is a r-vector of covariances between the specific signal value s and the vector of signal values s'.

We will restrict ourselves to regard only three datum-shift parameters, namely the components of the translation vector expressed through the change of the components of the deflections and the geoid height in a certain initial point, for example the point used as datum origin. Using a standard notation Heiskanen and Moritz (1967, section 5-9) we have

$$X_1 = - \delta \xi_0, \qquad X_2 = - \delta \eta_0 \text{ and } X_3 = - \delta \zeta_0 .$$

The geodetic coordinates of the initial point are denoted φ_0, λ_0.

We will regard the following kind of observations, (*a*)-(*d*), which have the corresponding equations of observation :

(*a*) point or mean free-air gravity anomalies

(5) $$\Delta g = \Delta g' + n (\Delta g)$$

(we disregard the change in reference gravity due to the change in latitude)

(*b*) latitude or longitude components of the deflections of the vertical in a point with geodetic latitude φ and longitude λ given in the local geodetic coordinate system, (a denotes the semi-major axis)

(6) $$\xi = (\cos\varphi \cos\varphi_0 + \sin\varphi \sin\varphi_0 \cos(\lambda - \lambda_0)) X_1 - \sin\varphi \sin(\lambda - \lambda_0) X_2 - \\ - \frac{\sin\varphi_0 \cos\varphi - \cos\varphi_0 \sin\varphi \cos(\lambda - \lambda_0)}{a} X_3 + \xi' + n(\xi) .$$

(7) $$\eta = \sin\varphi_0 \sin(\lambda - \lambda_0) X_1 + \cos(\lambda - \lambda_0) X_2 + \\ + \frac{\cos\varphi_0 \sin(\lambda - \lambda_0)}{a} X_3 + \eta' + n(\eta)$$

and in the geocentric coordinate system

$$\xi = \xi' + n(\xi) \; , \tag{8}$$

$$\eta = \eta' + n(\eta) \; , \tag{9}$$

(*c*) the height anomaly given in the local geodetic coordinate system

$$\begin{aligned} \zeta = & -(\cos\varphi_0 \sin\varphi - \sin\varphi_0 \cos\varphi \cos(\lambda - \lambda_0))\, a\, X_1 - \\ & - \cos\varphi \sin(\lambda - \lambda_0)\, a\, X_2 + \\ & + (\sin\varphi_0 \sin\varphi + \cos\varphi_0 \cos\varphi \cos(\lambda - \lambda_0))\, X_3 + \zeta' + n(\zeta), \end{aligned} \tag{10}$$

and in the geocentric system

$$\zeta = \zeta' + n(\zeta) \; , \tag{11}$$

d) the difference between the geocentric and the local geodetic latitude, longitude ($\times$ cos φ) and ellipsoidal height (h), (geocentric coordinates have a subscript c and refer to an ellipsoid having dimensions equal to these used in the local coordinate system)

$$\begin{aligned} \delta\varphi = \varphi_c - \varphi = & (\cos\varphi \cos\varphi_0 + \sin\varphi \sin\varphi_0 \cos(\lambda - \lambda_0))\, X_1 - \\ & - \sin\varphi \sin(\lambda - \lambda_0)\, X_2 - \\ & - \frac{\sin\varphi_0 \cos\varphi - \cos\varphi_0 \sin\varphi \cos(\lambda - \lambda_0)}{a} X_3 + n(\delta\varphi) \end{aligned} \tag{12}$$

$$\begin{aligned} \cos\varphi\, \delta\lambda = \cos\varphi\, (\lambda_c - \lambda) = & \sin\varphi_0 \sin(\lambda - \lambda_0)\, X_1 + \cos(\lambda - \lambda_0)\, X_2 + \\ & + \frac{\cos\varphi_0 \sin(\lambda - \lambda_0)}{a} X_3 + n(\cos\varphi\, \delta\lambda) \end{aligned} \tag{13}$$

$$\begin{aligned} -\delta h = h - h_c = & -(\cos\varphi_0 \sin\varphi - \sin\varphi_0 \cos\varphi \cos(\lambda - \lambda_0))\, a\, X_1 - \\ & - \cos\varphi \sin(\lambda - \lambda_0)\, a\, X_2 + \\ & + (\sin\varphi_0 \sin\varphi + \cos\varphi_0 \cos\varphi \cos(\lambda - \lambda_0))\, X_3 + n(-\delta h). \end{aligned} \tag{14}$$

The signal s, which we wish to determine, is a function of three variables $T = T(\varphi, \lambda, h)$, the anomalous potential of the Earth. The elements of the vector s' stemming from the observation of the kind (a) — (c) are values of certain linear functionals applied on the anomalous potential, cf. e.g. Tscherning (1974, p. 8). Generally, the values of linear functionals applied on T are the realizations of the signal s, which we may use as observations or estimate by equation (3) in least squares collocation.

The estimates of the mean square error of the estimates of s (denoted E_{ss}) and the estimate of the variance-covariance matrix of the parameters, E_{XX} becomes, cf. Moritz (1971, eq. (3-33) and (3-36)),

$$E_{ss} = C_{ss} - C_s^T \bar{C}^{-1} C_s + C_s^T \bar{C}^{-1} A\, E_{XX}\, A^T \bar{C}^{-1} C_s \tag{15}$$

and

$$E_{XX} = (A^T \bar{C}^{-1} A)^{-1}\,, \tag{16}$$

where C_{ss} is the variance of s.

Let us now suppose, that the covariance between the (totally p) observations of the kind (a) — (c) are uncorrelated with the $(r - p)$ observations of the kind (d) of differences between local geodetic coordinates and geocentric coordinates. Furthermore, let us suppose, that the quantities of the kind (d) are mutually uncorrelated. When the elements of the $r \times r$ matrix $\bar{C}$ are arranged with the first $r - p$ rows and columns belonging to these observations we may partition $\bar{C}$ in four submatrices V, W, $\emptyset$, $\emptyset^T$ correspondingly

$$\bar{C} = \begin{Bmatrix} V & \emptyset^T \\ \emptyset & W \end{Bmatrix}, \tag{17}$$

where V is a diagonal matrix of dimension $r - p$ with diagonal elements equal to the corresponding elements of D (the variance-covariance matrix of the vector n). $\emptyset$ is a matrix containing only zero elements. We may partition the A matrix and the vectors x and C_s correspondingly:

$$A = \begin{Bmatrix} Y \\ Z \end{Bmatrix}, \qquad x = \begin{Bmatrix} y \\ z \end{Bmatrix}, \qquad \text{and } C_s = \begin{Bmatrix} O \\ U \end{Bmatrix},$$

where O is a $r - p$ dimensional vector of zero elements.

Hence

$$\bar{C}^{-1} = \begin{Bmatrix} V^{-1} & \emptyset \\ \emptyset & W^{-1} \end{Bmatrix} \tag{18}$$

$$\tilde{s} = U^{\mathrm{T}} W^{-1} (z - Z^{\mathrm{T}} \tilde{X}) \,, \tag{19}$$

$$\tilde{X} = E_{\mathrm{XX}} (Y^{\mathrm{T}} V^{-1} y + Z^{\mathrm{T}} W^{-1} z) \,, \tag{20}$$

$$E_{ss} = C_{ss} - U^{\mathrm{T}} W^{-1} U + U^{\mathrm{T}} W^{-1} Z E_{\mathrm{XX}} Z^{\mathrm{T}} W^{-1} U \tag{21}$$

and

$$E_{\mathrm{XX}} = (Y^{\mathrm{T}} V^{-1} Y + Z^{\mathrm{T}} W^{-1} Z)^{-1} \,. \tag{22}$$

From these equations we see, that the estimation of the signal and the parameters is not much different with or without observations of the category (*d*).

3. — The organization of the computation of the parameters.

The complexity of the computational task is naturally in the general situation very dependent on the dimension of the x and X vectors, i.e. on the integers r, q (and p). In the situation we have chosen to regard, there will generally be less parameters than observed quantities, i.e. $r > p > q$ $(= 3)$.

Inspecting the equations (19)-(22), it may look like, that it will be necessary to invert two matrices, the $p \times p$ matrix V and the $q \times q$ matrix $G = (Y^{\mathrm{T}} V^{-1} Y + Z^{\mathrm{T}} W^{-1} Z)$.

It may be worthwhile to invert G even in cases where q is much bigger than in our case, because the correlation between the estimated parameters may be of interest. But the inversion of W can be avoided, because we only need the value of the quadratic form

$$Q = P^{\mathrm{T}} W^{-1} P \tag{23}$$

or the solution vector

$$v = W^{-1} u \,, \tag{24}$$

where P and u are p-vectors. (The vector P is in eq. (21) equal to U and in eq. (22) to the rows of the matrix Z).

The quantities Q and v are readily obtained using the Cholesky factorization of W:

$$W = L\,L^{T}\,, \tag{25}$$

where L is a lower triangular $p \times p$ matrix.

The computation of L, Q and v has been discussed e.g. in Poder and Tscherning (1973). The technique proposed there divides the upper triangular part of the matrix W in blocks of nearly equal size. They are then two by two simultaneously present in the core storage under the computation of L^{T}.

Q and v are then computed by

$$v = (L^{T})^{-1}\,(L^{-1}\,u)$$

and

$$Q = (L^{-1}\,P)^{T}\,L^{-1}\,P\,.$$

The method is easily programmed, when at least two p-vectors simultaneously can be stored in the core storage. Let us then suppose, that this is possible.

The first step will then be the computation of the elements of the A and the (upper triangular part of the) $\bar{C} = C + D$ matrices. Supposing that the x vector and the A and C matrices are available (contingently stored on a disc), we may execute the computations in the following sequence:

Compute:

(*a*) L	(a lower triangular $p \times p$ matrix, which may be stored in the same storage location, earlier occupied by the upper triangular part of W),
(*b*) $b = (L^{T})^{-1}\,(L^{-1}\,z)$	(a p-vector),
(*c*) $B = Z^{T}\,b$	(a q-vector),
(*d*) $F = (L^{T})^{-1}\,(L^{-1}\,Z)$	(a $p \times q$ matrix), (this matrix may now be stored on the disc, where Z was stored beforehand; the Z matrix is not needed any longer)

(e) $G = Z^{\mathrm{T}} (L^{\mathrm{T}})^{-1} L^{-1} Z + Y^{\mathrm{T}} V^{-1} Y$ (a $q \times q$ matrix; the elements of $Z^{\mathrm{T}} (L^{\mathrm{T}})^{-1} L^{-1} Z$ are computed as a part of step (d)),

(f) $E_{XX} = G^{-1}$ (a $q \times q$ matrix),

(g) $\widetilde{X} = E_{XX} (B + Y^{\mathrm{T}} V^{-1} y)$ (a q-vector),

(h) $c = b - F \widetilde{X}$ (p-vector),

(i) $\{ U^{\mathrm{T}}, C_{ss} \}$ ($p + 1$-vector),

(j) $\widetilde{s} = U^{\mathrm{T}} c$,

(k) $H = U^{\mathrm{T}} F$ (q-vector) and

(l) $E_{ss} = C_{ss} - U^{\mathrm{T}} (L^{\mathrm{T}})^{-1} L^{-1} U + H^{\mathrm{T}} E_{XX} H$.

An alternative way is to compute the $p \times p$ matrix $- W^{-1} + F^{\mathrm{T}} E_{XX} F$, but this will require the explicit calculation of W^{-1}, which as mentioned above, is not necessary when the other procedure is followed. The alternative may only be a proper procedure, when a big number of error estimates E_{ss} are to be computed, (number of error estimates $> > p$).

The above described computational procedure has been programmed in the algorithmic language FORTRAN IV for an IBM system 370 computer as a modification of the program published in Tscherning (1974).

4. — Application of the method for Scandinavia and Greenland.

The method has been tested in two geographical areas, where observations were available to the author, i.e. Scandinavia and Greenland.

In Scandinavia, the European Datum 1950 is used. The coordinates of points in this datum are not uniform, because the adjustment was performed in geographical independent blocks. No geoid height information was used in the adjustment.

This fact is reflected in the published datum-shift determinations, which show relatively big residuals between the observed geocentric coordinates and the transformed coordinates (see e.g. Lefebvre (1969)). Another fact causing errors is the non-uniform system of astronomic longitudes, and latitudes which has contributed errors into the geoid determination.

Scandinavia, on the other hand, was adjusted as one block, fixing the coordinates of some stations near to the Danish-German border. And the (astronomical) longitudes are all belonging to the same system (the Baltic longitude system). In this region, we should hence be able to determine a datum-shift, which should have a good internal consistency.

As observations was used (*A*) the potential coefficients of order less than or equal to 20 determined by Rapp (1973), (*B*) around 500 1° approximate equal area mean gravity anomalies covering Scandinavia and the area lying around 500 km's herefrom, (*C*) around 600 components of the deflections of the vertical from Denmark, Norway and Sweden and Germany (with latitude > 54°). (*D*) the geocentric height anomaly of three stations, Tromsø, Oslo and Copenhagen of which the geocentric coordinates were known and finally (*E*) the height anomaly in Landskrona, which is fixed to + 1.4 m in ED 1950, see Bomford (1971).

The datum-shift was determined as a residual with respect to (United States Department of Defence) World Geodetic Datum values $\Delta X = -84 \pm 10$ m, $\Delta Y = -103 \pm 10$ m and $\Delta Z = -127 \pm 10$ m, cf. Seppelin (1974, Figure 9). The residuals were, using Landskrona as datum origin :

$$\delta\xi_0 = -0.7'' \pm 0.1'', \quad \delta\eta_0 = 0.7'' \pm 0.2'', \quad \delta\zeta_0 = -7.4 \pm 0.5 \text{ m}$$

When observations of the geocentric latitude and longitude of the three stations (Tromsø, Oslo and Copenhagen) were added, the datum-shift residuals became

$$\delta\xi_0 = 0.2'' \pm 0.1'', \quad \delta\eta_0 = 0.0'' \pm 0.1'', \quad \delta\zeta_0 = 5.5 \pm 0.4 \text{ m.}$$

In Greenland, the geodetic coordinates refer to the Quornoq datum. The coordinates has most recently (1971) been determined in one simultaneous adjustment, which used measured distances in the southern part of the network. Due to very scarce deflection information, the distances were not reduced to the ellipsoid, taking the geoid heights into account. Anyway, the errors in the geodetic coordinates are not expected to exceed ± 10 meters.

One of the main satellite triangulation stations is fortunately situated in Greenland, and this station has then also been used by Anderle (1974) for the determination of datum-shift parameters for the Quornoq datum. Though the determination is based upon the geoid height being zero at Thule.

We have then made a datum-shift determination in the same way as explained for Scandinavia, using the potential coefficients (A), (*B*) around 250 1° mean gravity anomalies and (*C*) the 20 pairs of deflection components available together with (*D*) the absolute height anomaly in Thule and (*E*) the height anomaly fixed to 0 meters at the datum origin.

The datum-shift was determined as a residual with respect to the values given by Anderle, $\Delta X = 196$ m, $\Delta Y = 132$ m, $\Delta Z = -142$ m. The residuals were

$$\delta\xi_0 = -0.7'' \pm 1.1'', \quad \delta\eta_0 = +0.7'' \pm 1.2'', \quad \delta\zeta_0 = -33.6 \pm 1.7 \text{ m.}$$

When observations of the geocentric latitude and longitude of Thule was added, the datum-shift residuals changed to

$$\delta \xi_0 = 0.1'' \pm 0.3'', \qquad \delta \eta_0 = 0.0'' \pm 0.3'', \qquad \delta \zeta_0 = -33.6 \pm 1.7 \text{ m}.$$

A comparison of the datum-shift components obtained with and without observed geocentric latitudes and longitudes shows, that the two sets of values differ significantly. No obvious esplanation can be given for this fact. The reason could be either a too heigh weight put on the used data or - programming or data errors. The given values should therefore be regarded as preliminary.

5. — CONCLUSION.

The here described method for datum-shift determination is especially well-suited for areas with scarce astronomic information, i.e. for areas in which an astro-geodetic geoid determination is an impossible task. The method may directly use the observed deflections of the vertical and their standard deviations. Hence, it should give good estimates of the datum-shift parameters and their standard deviations, which otherwise would have to be based on e.g. astro-geodetic determined geoid heights and the standard deviations of these heights.

The method should be further developed and implemented as to become an integral part of a combined geometric and physical geodesy adjustment procedure like to one described by Eeg and Krarup (1973).

REFERENCES

R.J. ANDERLE, *Transformation of Terrestrial Survey Data to Doppler Satellite Datum.* J.G.R., Vol. 79, No. 35, 1974, p. 5319-5331.

G. BOMFORD, *The astro-geodetic geoid in Europe and connected areas* 1971. Appendix to Mrs. I. Fischer's report for SSG V-29, 1971.

J. EEG and T. KRARUP, *Integrated Geodesy.* The Danish Geodetic Institute, Internal Report No. 7, 1973, (also in « Methoden und Verfahren der Mathematischen Physik », Vol. 14. 1975).

W.A. HEISKANEN and H. MORITZ, *Physical Geodesy,* 1967.

K. LAMBECK, *The relation of some geodetic datums to a global geocentric reference system.* Bulletin Géodésique No. 99, 1971, p. 37-53.

M. LEFEBVRE, *Etude comparative des différents résultats obtenus en géodésie spatiale sur le réseau européen.* Bulletin Géodésique No. 94, 1969, p. 415-425.

J.G. MARSH, B.C. DOUGLAS and S.M. KLOSKO, *A global station coordinate solution based upon camera and laser data* – GSFC, 1973. – NASA/Goddard Space Flight Center, Greenbelt, Md., 1973.

H. MORITZ, *Advanced Least-Squares Methods.* Reports of the Department of Geodetic Science No. 175, 1972.

I.I. MUELLER, *Global Satellite Triangulation and Trilateration Results.* J.G.R., Vol. 79, No. 35, p. 5333-5348.

K. PODER and C.C. TSCHERNING, *Cholesky's Method on a Computer.* The Danish Geodetic Institute, Internal Report No. 8, 1973.

R.H. RAPP, *Numerical Results from Combination of Gravimetric and Satellite Data Using the principle of Least Squares Collocation.* Reports of the Department of Geodetic Science No. 200, 1973.

H.H. SCHMID, *Worldwide Geometric Satellite Triangulation.* J.G.R., Vol. 79, No. 35, p. 5349-5376.

T.O. SEPPELIN, *The Department of Defense World Geodetic System* 1972. DMA, Washington D.C., 1974.

C.C. TSCHERNING, *A FORTRAN IV Program for the Determination of the Anomalous Potential Using Stepwise Least Squares Collocation.* Reports of the Department of Geodetic Science, No. 212, 1974.

G. VEIS, *The Determination of the Radius of the Earth and other Geodetic Parameters as Derived from Optical Satellite Data.* Bulletin Géodésique No. 89, 1968, p. 253-275.

S. VINCENT and J.G. MARSH, *Global Detailled Geoid Computation and Model Analysis.* Proceedings of Symposium on The Earth's Gravitational Field and Secular Variations in Position, Sydney, Australia, 1973.