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APPLICATION OF COLLOCATION

DETERMINATION OF A LOCAL APPROXIMATION TO THE ANOMALOUS POTENTIAL OF THE EARTH USING "EXACT" ASTRO-GRAVIMETRIC COLLOCATION

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1. INTRODUCTION

The set of functions harmonic in an open set and regular at infinity constitute a linear space. We will denote the open set by Ω and its boundary by ω. A subset of this space may become a Hilbert space by the introduction of a suitable inner product. We will denote the inner product of two functions f and g by (f,g) and the corresponding norm $\|f\|^2 = (f,f)$.

A Hilbert space of this kind will contain a numerable, complete orthonormal set $(V_i,\ i=0,1,\ldots,\infty)$, i.e. every function f may be expressed as a series

$$f = \sum_{i=0}^{\infty} a_i V_i \tag{1}$$

where $a_i \in R$ $\sum_{i=0}^{\infty} a_i^2 < \infty$ and $(V_i, V_j) = \delta_{ij}$.

The Hilbert space will have a reproducing kernel, i.e. the identity transformation (I) of the space into itself may be expressed by

$$I(f)(Q) = (K(\cdot,Q),f) = f(Q),$$

where Q is a point in Ω.

The reproducing kernel $K(P,Q)$ is, as it appears by the notation, a function of two variables. It is symmetric, and is, for one of the variables fixed, an element of the Hilbert space. We indicate, that one of the variables are fixed by writing a dot instead of the independent variable.

The reproducing kernel may be represented by

$$K(P,Q) = \sum_{i=0}^{\infty} V_i(P) \cdot V_i(Q). \tag{2}$$

Using the linearity of the inner product and the formulae (1) and (2), we easily see the reproducing property:

$$I(f)(Q) = (K(P,Q), f(P)) = \left(\sum_{i=0}^{\infty} V_i(P) V_i(Q), \sum_{j=0}^{\infty} a_j V_j(P) \right)$$

$$= \sum_{j=0}^{\infty} a_j \left(\sum_{i=0}^{\infty} V_i(Q) \cdot (V_i(P), V_j(P)) \right) = \sum_{j=0}^{\infty} a_j V_j(Q) = f(Q).$$

<u>Example 1.1.</u> The Dirichlet norm.
When Ω has a sufficiently smooth boundary, we may introduce the Dirichlet norm

$$\|f\|^2 = \int (\nabla f)^2 \, d\Omega ,$$

which correspond to the inner product

$$(f,g) = \int \nabla f \cdot \nabla g \, d\Omega .$$

($\nabla f = D_{x_1} f + D_{x_2} f + D_{x_3} f$, D_{x_i} the partial derivative. D_n will below denote the partial derivative with respect to the normal n to ω).

In this case, the reproducing kernel is the sum of the Green's and Neumann's functions for the set Ω,

$$K(P,Q) = G(P,Q) + N(P,Q).$$

Using Green's identity and the fact that $D_n\, G$ and N are zero on the boundary ω, we get

$$(f,K(\cdot,Q)) = \int \nabla f \cdot \nabla K(\cdot,Q)\, d\Omega$$

$$= -\int_{\omega} D_n K(\cdot,Q)\cdot f d\omega = -\int_{\omega} D_n N(\cdot,Q)\cdot f d\omega = f$$

or

$$= -\int_{\omega} K(\cdot,Q)\cdot D_n f d\omega = -\int_{\omega} G(\cdot,Q) D_n f d\omega = f.$$

In case Ω is the open set outside a sphere with centre at the origin, the usual solid spherical harmonics is a complete basis. The functions

$$\left(\frac{R}{r}\right)^{i+1} \cdot \left(\bar{S}_{ij}(\theta,\lambda) \text{ or } \bar{R}_{ij}(\theta,\lambda)\right) \quad \text{cf. [1] (1-73)}$$

are easily normalized because

$$\frac{1}{4\pi} \int_{\Omega} \nabla\left(\left(\frac{R}{r}\right)^{i+1} \bar{S}_{ij}(\theta,\lambda)\right)^2 d\Omega = R(i+1).$$

Hence the reproducing kernel is

$$K(P,Q) = \sum_{i=0}^{\infty} \frac{1}{R(i+1)} \left(\frac{R^2}{r\cdot r'}\right)^{i+1}$$

$$\sum_{j=0}^{i} \left(\bar{S}_{ij}(\theta,\lambda)\bar{S}_{ij}(\theta',\lambda')+\bar{R}_{ij}(\theta,\lambda)\cdot\bar{R}_{ij}(\theta',\lambda')\right),$$

where the spherical coordinateswithout mark refer to P and the others to Q.

Using the summation formula [1], (1-82) for the surface spherical harmonics, a more compact expression can be computed

$$K(P,Q) = \sum_{i=0}^{\infty} \frac{s^{i+1}}{R} \frac{2i+1}{i+1} P_i(t),$$

where $s = \frac{R^2}{r \cdot r'}$, $P_i(t)$ is the Legendre polynomial of degree i and t = cosine to the spherical distance (ψ) between P and Q.

The series representing the reproducing kernel can be reduced to a closed expression, cf. [3] p.26

$$K(P,Q) = \frac{s}{R}\left(\frac{2}{L} + \ln \frac{1-t}{L+s-t}\right),$$

where $L = (1-2st + s^2)^{\frac{1}{2}}$.

In a Hilbert space, the values of linear functionals may be expressed by

$$l'(f) = (f,l).$$

where l is a uniquely determined function. This relation between the functional l' and the function l defines an isometric isomorphy between the Hilbert space and its dual space. For a reproducing kernel Hilbert space, this mapping may be expressed using the kernel,

$$l'(f) = l'(K(\cdot,Q),f(Q)) = (l'(K(\cdot,Q)),f(Q)) = (l,f),$$

i.e. $l(P) = l'(K(\cdot,P))$.

Because this mapping is isometric, we can find the norm of l' as an element of the dual space:

$$\|l\|^2 = \|l'\|^2 = (l'K(P,Q), l'K(R,Q)) =$$

$$= l'(l'(K(P,Q), K(R,Q))) = l'l'K(P,R).$$

Hence we will find the square of the norm of a linear functional by applying it on the reproducing kernel one time with respect to each of its variables.

2. THE COLLOCATION PROBLEM

In a reproducing kernel Hilbert space a particular type of approximation problems may be solved in a unique and very simple manner.

We have measured some quantities m_i, $i=1,\ldots,n$ related to the anomalous potential of the Earth, T. Suppose these quantities may be expressed as the values of certain linear functionals l_i' operating on T:

$$l_i'(T) = m_i, \quad i=1,\ldots,n .$$

We then want to find an approximation to T, (noted $\tilde{T}$), which agrees exactly with these measurements,

$$\tilde{T} \in M = \left\{ T \mid l_i'(T) = m_i, \quad i=1,\ldots,n \right\}$$

and which has the least possible norm. The determination function $\tilde{T}$ is called the problem of ("exact") collocation.

M is a subset of our Hilbert space, and we want to find the element, which has the shortest distance from the harmonic function identically equal to zero. Now, note, that the subspace parallel to M (i.e. containing the zero-function) may be expressed by

$$M_o = \{ T \mid l_i'(T) = 0\} = \{ T \mid (l_i, T) = 0\}.$$

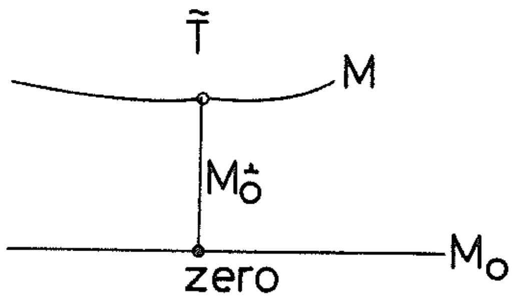


Fig. 1

The geometrical interpretation of
the collocation problem.

The function $\tilde{T}$, we are looking for, is the intersection between the set M and $M_o^\perp$ (the orthogonal complement to M_o in the Hilbert space).

The space M_o consist of all functions orthogonal on the functions l_i. This means, that $M_o^\perp$ must consist of all linear combinations of the l_i,

$$M_o^\perp = \left\{ T \mid T \sum_{i=1}^{n} a_i l_i, \quad a_i \in \dot{R} \right\} = \left\{ T \mid T \sum_{i=1}^{n} a_i l_i'(K), a_i \in \dot{R} \right\}.$$

The intersection between these two sets is determined by solving a set of linear equations having a_i, $i=1,\ldots,n$ as unknowns:

$$l_j'(\tilde{T}) = m_j \;, \qquad j=1,\ldots,n$$

(3)

$$l_j'\left(\sum_{i=1}^{n} a_i l_i'(K)\right) = \sum_{i=1}^{n} a_i l_i' l_j'(K) = m_j \;.$$

Theorem. The function

$$\tilde{T} = \sum_{i=1}^{n} a_i l_i' K$$

is the solution to approximation problem, which has the least norm.

Proof. Let us suppose, that we had two different solutions T,U. For $U_o = U-T$ we get

$$l_i' \; U_o = 0 \text{ for all } i.$$

Using the reproducing property of $K(P,Q)$ we get

$$(U_o,T) = \left(U_o, \sum_{i=1}^{n} a_i l_i' K\right)$$

$$= \sum_{i=1}^{n} a_i (U_o, l_i' K) = \sum_{i=1}^{n} a_i l_i' (U_o,K) = \sum_{i=1}^{n} a_i l_i' (U_o) = 0.$$

For the square norm we then get

$$(U,U) = (T+U_o,T+U_o) = (T,T)+(U_o,U_o)+2(U_o,T)$$

$$= (T,T)+(U_o,U_o) \geq (T,T),$$

i.e. U must have a norm greater than T.

Using the representation

(4)
$$\tilde{T}(P) = \sum_{i=1}^{n} a_i l_i'(K(\cdot,P))$$

we *define* the prediction of a quantity $p = l'(T)$, which is the value of a linear functional l' applied on T, to be

$$\tilde{p} = l'(\tilde{T}) = \sum_{i=1}^{n} a_i l'(l_i'K),$$

and the estimated error of prediction to be the square of the distance of the associated function l from its projection in the subspace M_o:

$$V(l')^2 = \|l\|^2 - \{l'l_i'K\}^T\{l_i'l_j'K\}^{-1}\{l'l_j'K\}$$

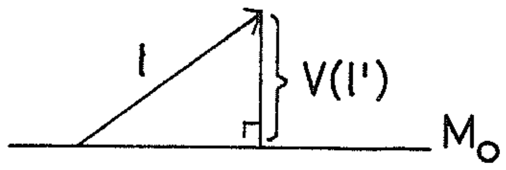


Fig. 2

The error of prediction

In this way we have a measure for the error of prediction, which is zero, when l' is a linear combination of the functionals associated with the measurements (l_i), and which gets its maximal value when l' is orthogonal on all l_i', i.e. when $l'l_i'K = 0$. Because of the well known isomorphybetween a reproducing kernel Hilbert space and a stochastic process, we will call the value of $l_i'l_j'K$ the covariance between the two functionals. Orthogonality is then equivalent to independence.

3. WHICH KIND OF HILBERT SPACE CAN WE USE?

From the relation (4) above, we see, that the solution to the collocation problem depends on the reproducing kernel and hence on the Hilbert space.

In case we should choose between different Hilbert space, we would naturally select the one which gave the "best" predictions. We could also ask for a Hilbert space, giving estimates for the error of prediction, which were near to the observed mean square prediction error.

But before we get the predictions, the reproducing kernel, the covariances $l_i' l_j' K$ etc. must be computed. Hence we will try to find a Hilbert space sufficiently simple, and see how good prediction results we can get.

We shall restrict our considerations to a few types of measurements, which correspond to the following (linearized) functionals:
Point gravity anomalies in a point P in Ω (P has spherical coordinates r, ϕ, λ and a reference gravity γ):

(5) $$l'_{\Delta g}(T) = -D_r T - \frac{2}{r} \cdot T(P).$$

Mean gravity anomalies $\overline{\Delta g}$ over an area a of total area A:

(6) $$l'_{\overline{\Delta g}}(T) = \frac{1}{A} \int_a l'_{\Delta g}(T)\, da.$$

The latitude component of the deflection of the vertical in P:

(7) $$l'_{\xi}(T) = -D_\phi T \cdot \frac{1}{\gamma}.$$

The longitude component of the deflection of the vertical in P:

(8) $$l'_\eta(T) = D_\lambda T \cdot \frac{1}{\gamma \cdot \cos\phi} .$$

The heightanomaly ρ:

(9) $$l'_\phi(T) = \frac{T}{\gamma} .$$

And finally the coefficients of T developed in the series (1)

$$l'_n(T) = (T, V_n) ,$$

which e.g. for the Hilbert space of example 1.1 is

(10) $$l'_{nm}(T) = \int \nabla T \cdot \nabla \left(\frac{R^{n+\frac{1}{2}}}{r^{n+1}} \frac{1}{\sqrt{n+1}} \bar{S}_{nm}(\theta,\lambda) \right) d\Omega .$$

Example 3.1. Collocation using observations of type (10). In this case the normal equations (3) become very simple. Let us suppose the coefficients c_1 to c_n have been observed:

$$\sum_{i=1}^{n} a_i l'_j l'_i K = c_j , \qquad j=1,\ldots,n.$$

Using the orthonormality we get

$$l'_j K = l'_j \left(\sum_{i=1}^{\infty} V_i(P) \cdot V_i(Q) \right) = V_j$$

or

$$\{a_1,\ldots,a_n\} \begin{Bmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{Bmatrix} = \{c_1,\ldots,c_n\} ,$$

i.e. $a_i = c_i$ for all i.

The solution to the collocation problem becomes (naturally) the finite series

$$\tilde{T}_0 = \sum_{i=1}^{n} c_i V_i \ .$$

<u>Example 3.2.</u> Collocation using observations of type (5) - (10). We have observations $c=\{c_1,\ldots,c_p\}^T$ of type (10) and $d=\{d_1,\ldots,d_q\}^T$ of type (5) - (9), $p + q = n$.

The normal equations (3) become (as a rough sketch):

$$\begin{Bmatrix} \left[l_c^2 K\right]_{p\times p} & \left[l_c' l_d' K\right]_{p\times q} \\ \\ \left[l_d' l_c' K\right]_{q\times p} & \left[l_d^2 K\right]_{q\times q} \end{Bmatrix} \begin{Bmatrix} [a] \\ \\ [b] \end{Bmatrix} = \begin{Bmatrix} [c] \\ \\ [d] \end{Bmatrix} \quad \text{or}$$

shortly

$$\begin{Bmatrix} A & B \\ B^T & C \end{Bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} c \\ d \end{Bmatrix}$$

which is

$$Aa + Bb = c$$
$$B^T a + Cb = d \ .$$

Again we have $A = I$, the identity matrix, and we get

$$Bb = c - a$$

and hence

$$B^T c - B^T Bb + Cb = d, \qquad \text{or}$$
$$(C - B^T B)b = d - B^T c.$$

We note that the elements of B are $l'_{d_i}(V_{c_j})$ and hence

$$B^T B = \left\{ l'_{d_i} l'_{d_j} \left(\sum_{k=1}^{p} V_k(P) \cdot V_k(Q) \right) \right\} \tag{11}$$

and

$$C - B^T B = \left\{ l'_{d_i} l'_{d_j} \left(\sum_{i=p+1}^{\infty} V_i(P) V_i(Q) \right) \right\}, \tag{12}$$

$$d - B^T c = d - \sum_{i=1}^{p} c_i l'_d V_i . \tag{13}$$

Hence we have reduced the collocation problem to a problem in a Hilbert space with the one dimensional subspaces corresponding to V_i, $i=1,\ldots,p$ removed and with a new reference potential, i.e. the approximation $\tilde{T}_0$ of example 3.1.

From the two examples, we may conclude, that it is an advantage to use the same orthonormal system in the Hilbert space as the one used e.g. in satellite geodesy.

The main problem in the practical application of collocation is the computation of the coefficients of the normal equations (3), the covariances $l'_i l'_j K$. In case the reproducing kernel may be represented by a closed expression as in example 1.1, the computation is much facilitated. The properties, which made a representation by a closed expression possible were (a) the set Ω was equal to the set outside a sphere, (b) this set is rotational invariant, (c) the inner product was rational invariant, (d) the solid spherical harmonics were orthogonal on each other and (e) the normalizing constants of the harmonics are determined by a simple rational function of the degree only $((2n + 1)/(n+1))$.

We will then in the following limit ourselves to Hilbert space of functions harmonic outside a sphere (radius R), and which have a rotational invariant inner product. In this case the reproducing kernel can always be expressed by

$$K(P,Q) = \sum_{n=0}^{\infty} \sigma_n^2 \, s^{n+1} \, P_n(t). \tag{14}$$

The constants σ_n^2 are called the degree-variances.

Example 3.3. Computation of the covariance between deflections. The computation of the coefficients of the equations (3) are simplified, when we use a rotational invariant norm:
We put

$$K(P,Q) = k(s,t),$$

indicating that K only depends on s and $t = \cos\psi$. For the deflections of the vertical we then get

$$l'_\xi(k) = -D_\phi t \cdot D_t k \cdot \frac{1}{\gamma}, \tag{15}$$

$$l'_\eta(k) = -D_\lambda t \cdot D_t k \cdot \frac{1}{\gamma \cdot \cos\phi}, \tag{16}$$

$$l'_\xi l'_{\xi'}(k) = -\left(D_\phi^2 t \cdot D_t k + D_\phi t \cdot D_t^2 k\right) \frac{1}{\gamma \gamma'}, \text{ etc.} \tag{17}$$

Thus, in this case we can use $D_t k$ and $D_t^2 k$ both in the computation of the covariance between deflections of the vertical and between deflections and height anomalies. For more details see [3].

Example 3.4. Computation of the covariance between gravity anomalies.

Using (5) we get

$$l'_{\Delta g}\left(s^{n+1}P_n(t)\right) = \frac{n-1}{R}\cdot s^{n+1}P_n(t),$$

because $s = \dfrac{R^2}{r\cdot r'}$.

Hence

$$(18)\qquad l'_{\Delta g}(k) = \sum_{n=0}^{\infty}\frac{n-1}{R}\,\sigma_n^2\cdot s^{n+1}\cdot P_n(t)$$

and

$$(19)\; l'_{\Delta g}(l'_{\Delta g},k) = \sum_{n=0}^{\infty}\frac{(n-1)^2}{R^2}\,\sigma_n^2\cdot s^{n+1}\cdot P_n(t).$$

As mentioned above, it is possible to obtain a closed expression for the reproducing kernel when for example the degree variances σ_n^2 are determined by a simple rational function of n. There is a relatively simple relation between a kind of scalar products and a kind of rational functions parameterizing σ_n^2. When

$$(f,g) = \int\Big(\sum_{\alpha} D^{\alpha}f\cdot D^{\alpha}g\Big)d\Omega\ ,$$

where α is a multiindex symbol of order m, the rational function parameterizing σ_n^2 is of order $2(1-m)$, (see [4]) and cf. example 1.1, where $\alpha\in\{1,2,3\}$ and $m=1$.

We have here decided to consider quantities, which are derivatives of order zero or one of T. Let us therefore consider norms, which minimalize the variation of these quantities, i.e. the second derivatives of $\tilde{T}$. α is hence of order 2 and σ_n^2 will vary like n^{-2}.

Using example 3.4 we note, that in case $\sigma_n^2 = \frac{1}{(n-1)^2}$, $n \geq 2$, we will get a very simple computation of the covariance between the gravity anomalies:

$$(20) \qquad l'_{\Delta g} l'_{\Delta g}, K = \frac{1}{R^2} \sum_{n=2}^{\infty} s^{n+1} P_n(t) = \frac{1}{L} - s^2 \cdot t \ .$$

Unfortunately the reproducing kernel corresponding to the series (20) can't be represented by a simple closed expression. But instead we can use

$$\sigma_n^2 = \frac{1}{n \cdot (n-1)} \text{ or } \frac{2}{(n-1)(n-2)} \ .$$

The inner products corresponding to these two parametrizations are

$$(21) \qquad \int_{\Omega} W_k(r) \cdot \sum_{\alpha} D^{\alpha} f \cdot D^{\alpha} g \, d\Omega = (f, g) \ ,$$

where α takes on the value of all pairs formed by the numbers 1,2 and 3. In the first case we have

$$W_1(r) = a_1 (12 \frac{R}{r} - 15 + 4 \frac{r}{R})$$

and in the second

$$W_2(r) = a_2 (24 \frac{R}{r} - 35 + 12 \frac{r}{R}) .$$

The corresponding closed expressions becomes

$$(22) \quad K_1(P,Q) = a_1 \cdot s (1 - L + (ts - 1) \ln \frac{2}{L_2})$$

and

$$(23) \quad K_2(P,Q) = a_2 \cdot s (\frac{3ts}{2} \cdot L_1 + s ((P_2(t) \cdot s - t) \cdot$$

$$\cdot \ln \frac{2}{L_2} + \frac{s}{4}(st^2 - 1))) ,$$

where $L_1 = 1-ts-L$ and $L_2 = 1-ts+L$ and a_1, a_2 are normalizing constants of dimension $(m/\sec)^4$.

The actual evaluation of the value of the functionals (5) - (9) applied on the kernel will in some cases be difficult. When both points of evaluation (the points where the measurement have taken place) are near to the boundary, the quantity s is nearly 1. We will, using e.g. the kernels (22) or (23), have to compute expressions, which consist of quantities, which are formed as differences between quantities of nearly equal magnitude. This is related to the fact, that the functionals (5) - (9) generally are not elements of the dual space, when the related points of evaluation are situated on the boundary.

This problem may be solved by a change in the Hilbert space. We may require the functions in the Hilbert space to the harmonic down to a Bjerhammar sphere, i.e. a sphere totally enclosed in the Earth. For a Hilbert space of this kind, the mentioned functionals will always be members of the dual space, i.e. having norm less than infinity.

(Note, that the use of Hilbert spaces of functions harmonic in an open set including the set outside the surface of the Earth, will not limit our possibilities of finding an arbitrary good approximation to T. This is due to the Runge theorem for harmonic functions, cf. [2], p.54).

The use of the method of collocation requires the formation of one equation per observation. We will then easily arrive at very big systems of equations. Elimination methods, used for the solution of the equations, requires the computation of many sums of products, thus producing rounding errors. Furthermore, the ratio between the smallest and biggest eigenvalue of the matrix of the equations must lie within certain numerical boundaries, which will depend on the properties of the actual computer. Hence, we must require, that the Hilbert space not cause a too strong correlation between the measurements.

Thus, the collocation process will not always work in a Hilbert space containing the low order harmonics, when the method is used to construct a local approximation. This problem can be partly solved by removing the "global" information as described in example 3.2, cf. the situation described in fig. 3.

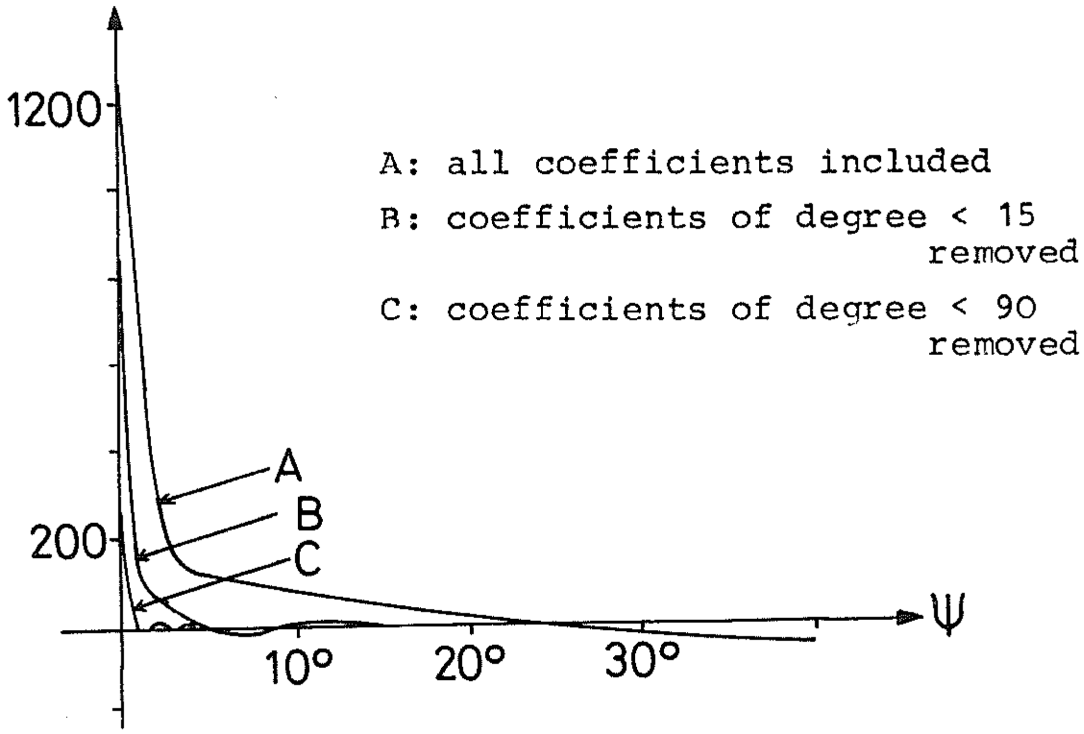


Fig. 3

Covariance function of gravity anomalies corresponding to different Hilbert spaces.

All the above mentioned criteria will still not be enough to determine a specific Hilbert space, which should be the best suited for determining the approximation $\tilde{T}$. In practice, the following procedure has given good prediction results. The idea behind is, that the above defined error of prediction should be a good estimate of the mean square prediction error. The mean square variation of the measurements are represented by the so called empirical covariance function. The Hilbert space was then determined by requiring that the covariance function derived from the reproducing kernel was the best possible approximation to the empirical covariance function, (see [3]).

Presumably, a better method is to compare the approximation computed in different spaces. One set of measurements should then be used in the construction of the approximation. Another set should be used as a test set to check the quality of the solution, the likelihood of the estimated errors of prediction, the stability of the solution of the equations etc. Some preliminary computational experiments in this direction indicates, that when a dense grid of measurements are used, different Hilbert spaces, will give nearly the same quality of the predictions.
Finally, theoretical methods should be used to check that the approximations will converge towards the solution to the appropriate boundary value problem, when the grid of measurements becomes more and more dense.

4. CONSTRUCTION OF A LOCAL APPROXIMATION

We will now consider the approximation of T in a subset included in Ω. The restriction of a harmonic function to an open set will be a harmonic function on this set. We could then regard Hilbert spaces of functions harmonic in this subset. Thus, this is generally not possible, because of the difficulties, which occur in the computation of an orthonormal system and thereby in the construction of the reproducing kernel.

We are then forced to consider Hilbert spaces of the type presented in section 3. And as mentioned, it is necessary to remove the "global" information e.g. by the procedure described in example 3.2. But even when the "global" information, corresponding to the about 400 coefficients, which is determined at present, is removed, measurements lying in a distance of 4° - 5° away, may influence a local approximation, cf. fig. 3. We must then find a method to improve our reference potential, so that it at least in the area of interest represents the influence of the harmonics up to a suitable order.

This has been tried with fairly good results using mean gravity anomalies and applying the collocation procedure.

Unfortunately it is very difficult to compute the value of the mean anomaly functionals applied on a reproducing kernel. A practical method is to represent the functional as

$$l'_{\Delta g}(V_i) = \frac{i-1}{r} \cdot s_o^{i+1} \cdot V_i \;, \tag{24}$$

where $s_o<1$, (i.e. regarding the mean anomalies as point gravity anomalies in a point lying in a certain height above the surface of the Earth). This representation reflects the damping effect of the mean anomalies on the harmonics of high degree and the samller mean square variation of the mean anomalies.

Of course, this procedure does not give us information of the number of harmonics, which can be regarded as removed. But as a pratical rule, we will require, that the reproducing kernel is changed so much, that the point gravity anomalies, lying in a distance equal to the diameter of the area a, cf. (6), becomes independent. For $1^\circ \times 1^\circ$ equal area anomalies the first zero point of the covariance function derived from the reproducing kernel shall then be found in a distance of 1°. This correspond to the removal of the harmonics up to degree 90 in a Hilbert space having a reproducing kernel of type (22). (We note, that this change corresponds very well to the observed change of the first zero point in the empirical covariance function of gravity anomalies, which refer to a locally improved reference potential).

Let us now suppose, that we in a local area want to represent the gravity anomalies with a mean square error 10-times better than the observed mean square variation. This variation is estimated to 1200 $mgal^2$, globally. The mean square variation of the $1^\circ \times 1^\circ$ mean gravity anomalies is estimated to 700 $mgal^2$. Thus even when an approximation is constructed, so that it agrees with the $1^\circ \times 1^\circ$ mean gravity anomalies, we must still take care of about 40% of the variation.

It may then, in same areas, be possible to select a reasonable number of gravity anomalies, so that a sufficient good approximation can be constructed. Otherwise, mean anomalies over e.g. 6' × 6' areas can be used first, and then point values. Gravity values are then predicted as the sum of the contributions from all the approximations:

$$\widetilde{\Delta g} = l'_{\Delta g}(\tilde{T})$$

where

(25) $$\tilde{T} = \tilde{T}_0 + \tilde{T}_1 + \dots + \tilde{T}_n .$$

Here $\tilde{T}_0$ is determined according to example 3.1, and $\tilde{T}_i$, $i>0$ according to example 3.2,

(25a) $$\tilde{T}_i = \sum_{j=1}^{M_i} a_j l'_j K_i \; ,$$

(25b) $$K_i = \sum_{j=J_i}^{\infty} \sigma_j^2 \; s^{j+1} \; P_j(t)$$

and $J_i > J_{i-1}$. In this way we have a method for going from the "big to the small" as required by Moritz.

Now, someone may propose, that the mean value of the local area could be computed first of all. Then the collocation procedure could be used in a Hilbert space, where the harmonics up to a certain order, corresponding to the magnitude of that area, was removed, and where the mean value was subtracted from the anomalies. Thus, such a procedure will not allow us to fit local solutions together, when we want to include e.g. height anomalies or deflections of the vertical in the computations.

The same procedure can be used when we want to use both gravity anomalies, deflections of the vertical and height anomalies. The final approximation $\tilde{T}_n$ of (25) is then computed from these measurements, using $\tilde{T}_0 + \cdots + \tilde{T}_{n-1}$ as a reference potential.

But one problem is left. Until now, we have considered "exact" collocation, i.e. we have regarded the measurements as having no errors. We will continue to regard the gravity anomalies as errorless, but the deflections of the vertical may contain both systematical and big random errors. The systematic error can originate from errors in star catalogues, corrections for polar motion, scale errors in the geodetic coordinates and errors in the transformation parameters, which determines the transformation from local geodetic coordinates to global.

It is possible to use the correlation between the gravity anomalies and the deflections for the determination of the systematic errors. We will represent this error as a change in the deflections (ξ_0,η_0) in a point in the middle of the area considered, $d_0 = (\delta\xi_0,\delta\eta_0)$. The change of the deflections (ξ_i,η_i) in a point p_i will then be

$$(26)\qquad \begin{Bmatrix} \delta\xi_i \\ \delta\eta_i \end{Bmatrix} = \begin{Bmatrix} a^i_{11} & a^i_{12} \\ a^i_{21} & a^i_{22} \end{Bmatrix} \begin{Bmatrix} \delta\xi_0 \\ \delta\eta_0 \end{Bmatrix} \text{ or } d_i = A_i\, d_0 .$$

The coefficients a^i_{jk} are determined by [1], (5-59).

The vector δ may be determined by requiring

$$(27)\qquad \|\tilde{T}\|^2 + (m-p-d)D^{-1}(m-p-d)$$

to be minimum as a function of the variables a_i in (3) and of d_o. m is the vector of measured values, p the vector of predicted values and D^{-1} a weight matrix. (The gravity anomalies have infinite weight according to the above mentioned). We will then get a set of normal equations like (3), but with two extra rows and columns and with the matrix D added to coefficients:

$$\left\{D_{ij}+K_{ij}\right\}\left\{a_j\right\} + A_i\, d_o = m_i$$

(28)

$$\left\{A_j^T\right\}\left\{a_j\right\}+\left\{A_j\right\}^T\left\{K_{ij}\right\}^{-1}\left\{A_i\right\}\cdot d_o=\left\{A_j\right\}^T\left\{K_{ij}\right\}^{-1}\left\{m_i\right\}.$$

A more simple method is to compute an approximation $\tilde{T}_n$ from the gravity anomalies alone, and then predict the deflections. d_o may then be determined by requiring

$$(m-p-u)\,D^{-1}\,(m-p-u) \tag{29}$$

to be minimum, or

$$A^T D^{-1} A\ d_o = A^T D^{-1}(m-p).$$

Now, finally, it is possible to compute local approximations to the anomalous potential using astro-gravimetric data.

Hence, an approximation to T covering an area A must be computed using

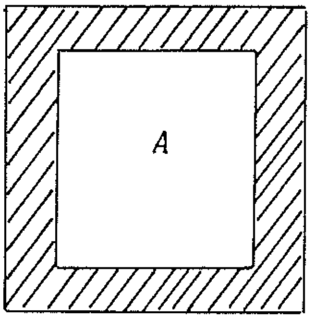
(a) the appropriate change of deflections and gravity anomalies to a global reference system,

(b) the best set of potential coefficients ($\tilde{T}_o$),

(c) mean gravity anomalies covering the area A and the surrounding area in a distance corresponding to the covariance function derived from the reproducing kernel K_1, $(\tilde{T}_1)$,

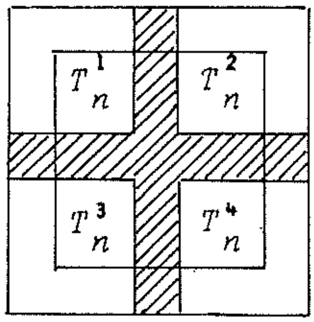
(d) if necessary, mean gravity anomalies corresponding to smaller areas than these used in (c), $(\tilde{T}_2,\ldots,\tilde{T}_{n-1})$,

(e) and finally the corrected deflections of the vertical and the point gravity anomalies.



Area covered by mean anomalies shaded.

Fig. 4a



Common area for the four approximations shaded.

Fig. 4b

When the number of observations necessary to obtain a specific accuracy is too great, the area can be divided in sub areas overlapping each other. We will then get solutions $\tilde{T}_n^i$ for sub-areas A_i of A. The described procedure (a) - (e) makes such a subdivision possible, because we simultaneously are changing the Hilbert space and hence the reproducing kernel, in such a way that the observations becomes more and more independent.

The procedure (a) - (e) has been tested in the Danish area in a $2^o \times 2^o$ square.

(a) The Potsdam correction was applied and the geodetic coordinates were changed from ED 1950 to a global reference system.

(b) The Rapp 14 × 14 solution from 1967 was used as $\tilde{T}_o$.

(c) $1^o \times 1^o$ mean gravity anomalies covering an area of 10^o diameter were used for the computation of $\tilde{T}_1$. The reproducing kernel (22) was used with the ratio between the radius of the Bjerhammar sphere and the mean radius of the Earth equal to 0.995 and J_1=15, cf. (25a).

(d) Was not used.

(e) In this step, the kernel (22) was used with J_2=110. Using 81 gravity anomalies spaced 15' apart, the change in the deflections d_o was determined according to (29), ($\delta\xi_o$ = 1" and $\delta\eta_o$ = -1"). In the final approximation $\tilde{T}_2$, 2 × 5 deflection components were used.

The approximation $\tilde{T} = \tilde{T}_o + \tilde{T}_1 + \tilde{T}_2$ was checked using a test set of 64 gravity anomalies (lying in the middle of the squares formed by the 81 observed anomalies) and 60 deflection components, cf. table 1.

The proposed method of fitting local solutions together has been tested in the Scandinavian area, using only deflections. But the results were not sufficient good, probably because we had not removed the global information. Overlap areas of width 1^o were used, but should have been broadend. The height anomalies vary in the middle of the overlap area with up to 1.5 m at certain points.

Table 1. Comparison of prediction and measurements

		Original value,ED	Difference observed-predicted
Δg	mean, mgal	15.0	0.0
	mean square variation, $mgal^2$	185.0	20.5
ξ	mean, arcsec	- 1.5	- 0.2
	mean square variation, $arcsec^2$	5.1	0.5
η	mean, arcsec	0.7	- 0.2
	mean square variation, $arcsec^2$	3.4	0.7

5. APPLICATION OF COLLOCATION

The original prediction method for gravity anomalies developed by Moritz and others was an application of the prediction theory for stochastic processes. The here described theory and prediction procedure is the functional analytic counterpart.

The statistical model has many weak points. It is difficult to give a clear description of the probability space underlying the stochastic process. And the estimation problems can't be solved before the anomalous potential is perfectly determined.

In the functional analytic model, the point of departure is a reproducing kernel Hilbert space. In this Hilbert space, we get a solution to a problem of astrogravimetric collocation following the described procedure. But we meet the same type of problems as in the statistical model: it is not easy to give good reasons for preferring one Hilbert space or another.

As far as the collocation procedure has been applied until now, the problem of finding a Hilbert space, which gave sufficient good predictions, has not been severe. When we used reproducing kernels, from which we could derive good approximations to the empirical covariance function, the prediction results were acceptable.

In another respect the Hilbert space problem is not so severe. Different Hilbert spaces will give nearly the same prediction results, when the measurements are situated sufficiently dense. - But there may exist functionals, for which the predictions are unstable, when the Hilbert space is changed.

Hence the collocation procedure should be applied with due caution. The approximations and corresponding predictions should be checked by comparing the results with the results of other methods. And a test set of measurements should be used for the control of the validity of the predictions.

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