Reports of the Department of Geodetic Science
Report No. 213

SOME SIMPLE METHODS FOR THE UNIQUE ASSIGNMENT OF A DENSITY DISTRIBUTION TO A HARMONIC FUNCTION

by
C. C. Tscherning

The Ohio State University
Department of Geodetic Science
1958 Neil Avenue
Columbus, Ohio 43210

August, 1974

Reports of the Department of Geodetic Science

Report No. 213

SOME SIMPLE METHODS FOR THE UNIQUE ASSIGNMENT OF A DENSITY DISTRIBUTION TO A HARMONIC FUNCTION

by

C.C. Tscherning

The Ohio State University
Department of Geodetic Science
1958 Neil Avenue

August, 1974

Foreword

This report was prepared by C.C. Tscherning, Research Associate, Department of Geodetic Science, and the Danish Geodetic Institute. This work was sponsored, in part, by the Danish Geodetic Institute, and in part by the Air Force Cambridge Research Laboratories, Bedford, Massachusetts, under Air Force Contract No. F19628-72-C-0120, The Ohio State University Research Foundation Project No. 3368B1, Project Supervisor, Richard H. Rapp. The Air Force Contract Monitor is Mr. George Hadgigeorge and Dr. Donald H. Eckhardt is Project Scientist.

A travel grant was supplied by the NATO Senior Fellowship Scheme.

The reproduction and distribution of this report was carried out through funds supplied by the Department of Geodetic Science. This report was also distributed by the Air Force Cambridge Research Laboratories as document AFCRL-TR-74-0506 (Scientific Report No. 19 under contract no. F19628-72-C-0120).

Abstract

One to one relationships are established between functions harmonic outside a sphere and density functions (1) with support equal to the sphere and (2) having the property that the density functions ρ, multiplied by a positive function of the distance from the center of the sphere, are harmonic, i.e.,

$$(*) \quad \Delta(f(r) \cdot \rho) = 0, \; f(r) > 0 \text{ for } 0 \leq r \leq R.$$

The relationship is established by specifying the relation for each external solid spherical harmonic V^{*}_{nm} of degree n and order m. The Poisson equation is first used to obtain a density function, equal to a distribution with support in the center of the sphere. This density is then spread out inside the whole sphere. As spreading operators are used the identity operators on Hilbert spaces of density functions fulfilling (*).

The derived relations may be used to assign a density distribution to the harmonic part of the potential of the Earth, and a covariance function of a density anomaly distribution to the covariance function of the anomalous potential of the Earth.

1. Introduction

Unfortunately, there is no unique mass-distribution which will produce the external potential of the Earth. But restrictions on the mass distribution may be prescribed in such a way, that a unique relationship is obtained.

We will illustrate this fact, by regarding a mass distribution ρ, which has support equal to a sphere, i.e. it is zero outside the sphere and non-zero in any open set inside the sphere. It would, for the study of mass distributions inside the Earth, have been more realistic to regard functions with support inside an ellipsoid (cf. e.g. Moritz (1973)). But this would result in quite complicated derivations and not serve the purpose of this presentation.

In this paper we will describe some simple algorithms that will assign a unique volume distribution to a given potential which is harmonic outside a sphere. In the main, this new approach uses the reproducing kernels of Hilbert spaces of density functions. This has been suggested by Dr. W. J. Davis of the Department of Mathematics, The Ohio State University.

We will regard the potential of a volume distribution ρ, which has support equal to a sphere Ω with radius R and center at the origin. The related total mass is denoted M.

The potential V, is then

$$(1)\quad V(P) = k \int_{\Omega} \frac{\rho(Q)}{\|P-Q\|}\, d\Omega_Q$$

where Q is a point in Ω, and k the gravitational constant.

The harmonic part of V can be expanded as a series in solid spherical harmonics (degree ℓ and order m)

$$(2)\quad V^{\circ}_{\ell m}(P) = \left(\frac{R^{\ell}}{r^{\ell+1}}\right) \cdot \bar{S}_{\ell m}(\theta, \lambda), \quad \ell = 0, \ldots, \infty, \; -\ell \le m \le \ell,$$

where r, θ, λ are the spherical coordinates of P and

$$(3)\quad \bar{S}_{\ell m}(\theta,\lambda) = \begin{cases} \sqrt{2\ell+1}\; P_{\ell}(\cos\theta) & \text{for } m = 0 \\ \sqrt{2(2\ell+1)\dfrac{(\ell-|m|)!}{(\ell+|m|)!}} \cdot \begin{cases} P_{\ell|m|}(\cos\theta)\cos m\lambda, & m < 0 \\ P_{\ell m}(\cos\theta)\sin m\lambda, & m > 0. \end{cases} \end{cases}$$

$P_{\ell m}(\cos\theta)$ are the associated Legendre polynomials. The overbar is used to signify the normalization, i.e. that

$$(4)\quad \frac{1}{4\pi}\int_{\sigma}(\bar{S}_{\ell m}(\theta,\lambda))^2\,d\sigma = 1$$

where σ is the surface of the unit sphere.

Hence, for P outside the sphere:

$$(5)\quad V(P) = kM\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}\, V^{\circ}_{\ell m}(P)$$

$$= \frac{kM}{r}\left(1 + \sum_{\ell=1}^{\infty}\left(\frac{R}{r}\right)^{\ell}\sum_{m=-\ell}^{\ell} a_{\ell m}\,\bar{S}_{\ell m}(\theta,\lambda)\right)$$

For the normalized surface harmonics (3), there is a very useful summation formulae, which will be applied several times (Heiskanen and Moritz, (1962, eq. (1-82′)):

$$(6)\quad P_{\ell}(\cos\psi) = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell}\bar{S}_{\ell m}(\theta,\lambda)\cdot\bar{S}_{\ell m}(\theta',\lambda'),$$

where θ and λ are the colatitude and longitude of a point P, θ' and λ' the colatitude and longitude of a point Q and ψ the spherical distance between the points, i.e,

$$(7)\quad \cos\psi = \cos\theta\cdot\cos\theta' + \sin\theta\cdot\sin\theta'\cos(\lambda'-\lambda)$$

(In the following developments, some of the quantities will not carry the correct units, but all the equations are consistent with respect to the units, and the final result, equation (47) will have proper units of mass divided by $(\text{length})^3$.)

2. The relations between density functions and potentials.

The basic relationship is given in equation (1). The inverse relationship is given by the Poisson equation:

$$(8)\quad \Delta(V(P)) = -4\pi \cdot k \cdot \rho,$$

where Δ is the Laplace operator.

For the potential V, we will now consider that our knowledge is limited to the harmonic part of the potential. This means that we can not use equation (8) to obtain formation about ρ.

As mentioned above, we will regard density function ρ with support equal to a sphere Ω with radius R. The set of harmonicity is then the set outside Ω.

Furthermore, we will restrict ourselves to considering the potentials, which are equal to the harmonic functions, e.q. (2). These functions are defined in the whole three-dimensional space $\mathbb{R}^3$. Could we use Poisson's equation on these and obtain a set of density functions $\rho_{\ell m}$? Unfortunately we have

$$(9)\quad (-4\pi k)^{-1}\Delta(V^{\circ}_{\ell m}(P)) = \begin{cases} 0 \text{ in } \mathbb{R}^3 \setminus \{0\} \\ \text{a singular quantity at } \{0\} \end{cases}$$

($\{0\}$ is the origin).

The idea is now to spread out this singular density function (a so called "distribution", see e.g. Yosida, (1971, p. 47)) to the whole sphere Ω, so that this density function will create the external harmonic $V^{\circ}_{\ell m}$ by application of equation (1).

3. Spreading out a singular density function.

We are looking for an operator which can handle the singular densities (9). But for those density functions, which are all ready spread out,we will require that they are not changed by the operator, i.e., the operator is the identity operator for at least some class of bounded density functions.

Let us regard the functions which are harmonic inside the sphere and for which the norm

$$(10)\quad \| \rho \|^2 = \frac{1}{4\pi} \int_{\Omega} \rho^2 d\Omega$$

is finite. This set of functions forms a Hilbert space with a reproducing kernel, K(P,Q), i.e.

$$(11)\quad \frac{1}{4\pi} \int_{\Omega} \rho(Q) \cdot K(P,Q)\, d\Omega = \rho(P)$$

where P and Q both are points in Ω, K(P,Q) is symmetric and for either P or Q fixed, K(P,Q) is a density function, which is harmonic.

An orthogonal set of functions spanning this Hilbert space is the set of internal solid spherical harmonics:

$$(12)\quad V^{i}_{\ell m}(P) = \left(\frac{r}{R}\right)^{\ell} \bar{S}_{\ell m}(\theta, \lambda), \quad m = -\ell, \ldots, \ell$$

The reproducing kernel is the sum of the products of the normalized solid spherical harmonics regarded as functions in P and Q respectively.

$$(13)\quad \| V^{i}_{\ell m} \|^2 = \frac{1}{4\pi} \int_{\Omega} \left(\left(\frac{r}{R}\right)^{\ell} \bar{S}_{\ell m}(\theta, \lambda)\right)^2 r^2 dr d\sigma$$

$$= \int_0^R \frac{r^{2\ell+2} dr}{R^{2\ell dr}} = \frac{R^3}{2\ell+3}$$

The reproducing kernel is then:

$$(14)\quad K(P,Q) = \sum_{\ell=0}^{\infty} \frac{2\ell+3}{R^3} \left(\frac{r \cdot r'}{R^2}\right)^{\ell} \sum_{m=-\ell}^{\ell} \bar{S}_{\ell m}(\theta,\lambda) \cdot \bar{S}_{\ell m}(\theta',\lambda')$$

and, using (6) the kernel becomes:

$$(15)\quad K(P,Q) = \sum_{\ell=0}^{\infty} \frac{(2\ell+3)(2\ell+1)}{R^3} \left(\frac{r \cdot r'}{R^2}\right)^{\ell} P_{\ell}(\cos\psi).$$

The identity operator (I) in this space is then:

$$(16)\quad I(f)(P) = \frac{1}{4\pi} \int_{\Omega} K(P,Q) \cdot f(Q)\, d\Omega = f(P)$$

where f(P) is a harmonic density function.

We can now apply this operator to the singular density function (distribution)(9):

$$(17)\quad \rho_{nm} = I\left(\frac{-1}{4\pi \cdot k} \Delta V^{\circ}_{nm}\right)(P) = \frac{1}{4\pi} \int_{\Omega} K(P,Q) \frac{-1}{4\pi \cdot k} \cdot \Delta V^{\circ}_{nm}(Q)\, d\Omega .$$

From Green's identity and the fact that K(P, Q) is harmonic in each variable the density function ρ_{nm}, eq. (17) becomes (with ω being the surface of Ω):

$$(18)\quad \rho_{nm}(P) = \frac{1}{4\pi} \iint_{\omega} \frac{1}{4\pi \cdot k} \left(V^{\circ}_{nm}(Q) \frac{\partial}{\partial n_Q} K(P,Q) - K(P,Q) \frac{\partial}{\partial n_Q} V^{\circ}_{nm}(Q)\right) d\omega$$

$$= \frac{1}{(4\pi)^2 k} \iint_{\omega} \left(V^{\circ}_{nm}(Q) \left(\sum_{\ell=0}^{\infty} \frac{(2\ell+3)(2\ell+1)}{R^3} \frac{\ell}{R} \left(\frac{r \cdot R}{R^2}\right)^{\ell} P_{\ell}(\cos\psi)\right)\right.$$

$$\left. - \sum_{\ell=0}^{\infty} \left(\frac{(2\ell+3)(2\ell+1)}{R^3} \left(\frac{r \cdot R}{R^2}\right)^{\ell} P_{\ell}(\cos\psi)\right) \frac{(-n-1)}{R} \cdot \frac{R^n}{R^{n+1}} \cdot \bar{S}_{nm}(\theta',\lambda')\right) d\omega$$

$$= \frac{1}{(4\pi)^2 k} \iint_{\omega} \sum_{\ell=0}^{\infty} \frac{(2\ell+3)(2\ell+1)}{R^3} \left(\frac{r}{R}\right)^{\ell} P_{\ell}(\cos\psi) \left(\frac{\ell}{R^2} \bar{S}_{nm}(\theta', \lambda')\right.$$

$$\left. + \frac{(n-1)}{R^2} \bar{S}_{nm}(\theta', \lambda')\right) d\omega$$

Using the expansion formula (6) and the orthogonality of the surface harmonics, $\bar{S}_{\ell m}(\theta, \lambda)$:

$$\rho_{nm}(P) = \frac{1}{4\pi k} \frac{(2n+1)(2n+3)}{R^3} \left(\frac{r}{R}\right)^n S_{nm}(\theta, \lambda) \text{ and}$$

(19) $$\rho_{nm}(P) = \frac{1}{4\pi R^3 k} (2n+3)(2n+1) V^1_{nm}(P)$$

<u>Example:</u>

$$kM \cdot V^{\circ}_{00}(P) = \frac{kM}{r} \text{ is mapped into}$$

(20) $$I\left(\frac{-1}{4\pi k} \Delta \left(\frac{kM}{r}\right)\right) = \frac{kM \cdot 3}{4\pi R^3 k} = \frac{M}{\frac{4}{3}\pi R^3}$$

i.e. the mass divided by the volume of the sphere.

Let us now show that the density function $\rho_{nm}(P)$ (19) really will produce the potential function $V^{\circ}_{nm}(Q)$ outside the sphere. We will denote the potential as computed by using ρ_{nm} in equation (1) by $W_{nm}(Q)$, and we must then show that $V^{\circ}_{nm} = W_{nm}$.

Denoting $||P - Q||$ in (1) by L we have

(21) $$L = ||P-Q|| = \sqrt{r^2 + (r')^2 - 2r \cdot r' \cdot \cos\psi}$$

and then

(22) $$\frac{1}{L} = \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{(r')^{\ell+1}} P_{\ell}(\cos\psi)$$

Substituting ρ_{nm} in equation (1) and using (22) we get:

(23) $$W_{nm}(Q) = \frac{k}{4\pi k \cdot R^3}(2n+1)(2n+3)\int_{\Omega} \frac{1}{L} V^i_{nm}(P)\, d\Omega_P$$

$$= \frac{(2n+3)}{4\pi R^3} \iint_{\sigma} \int_0^R \sum_{\ell=0}^{\infty} \left(\frac{r^{\ell}}{(r')^{\ell+1}} P_{\ell}(\cos\psi)\right)(2n+1)\left(\frac{r}{R}\right)^n \bar{S}_{nm}(\theta,\lambda)\, d\sigma$$

$$\cdot r^2 dr$$

and again using the summation formula (6) and the orthogonality property of the surface harmonics:

(24) $$W_{nm}(Q) = \frac{2n+3}{R^3}\int_0^R \frac{r^{2n} \cdot r^2}{(r')^{n+1} R^n} \bar{S}_{nm}(\theta', \lambda')\, dr$$

$$= \frac{2n+3}{R^3} \frac{R^{2n+3}}{(2n+3)(r')^{n+1} \cdot R^n} \bar{S}_{nm}(\theta', \lambda')$$

$$= \frac{R^n}{(r')^{n+1}} \bar{S}_{nm}(\theta', \lambda') = V^e_{nm}(Q),$$

i.e. we reproduce the external harmonic.

We will now regard an example of a Hilbert space where the identity operator performs the required smoothing, but in fact, does it too well. Let us regard a space of harmonic density functions with the norm:

(25) $$||\rho||^2 = \frac{1}{4\pi}\int_{\Omega} r \cdot \rho^2 d\Omega$$

The norm of the internal solid harmonics becomes:

(26) $$||V^i_{nm}||^2 = \frac{1}{4\pi}\iiint \frac{r^{2n+1}}{R^{2n}} S_{nm}(\theta,\lambda)^2 d\Omega = \int_0^R \frac{r^{2n+3}}{R^{2n}} dr = \frac{R^4}{2n+4}$$

and hence, it will have the reproducing kernel:

(27) $$K(P,Q) = \sum_{\ell=0}^{\infty} \left(\frac{r \cdot r'}{R^2}\right)^{\ell} \frac{(2\ell+4)(2\ell+1)}{R^4} P_{\ell}(\cos\psi).$$

We will then apply the identity operator to the distribution (9) deleting the constant $(4\pi k)$:

$$(28)\quad I(\Delta V_{ij}^{\circ})(P) = \iiint r \cdot K(P,Q) \cdot \Delta V_{ij}^{\circ}(P)\, d\Omega = \rho_{ij}(P)$$

and using Green's identity

$$(29)\quad \rho_{ij}(Q) = \iiint \Delta(r \cdot K(P,Q)) V_{ij}^{\circ}(P) d\Omega + \iint_{\omega} (r \cdot K(P,Q) \frac{\partial}{\partial n} V_{ij}^{\circ}(P) .$$

$$- \frac{\partial}{\partial n} (r \cdot K(P,Q)) V_{ij}^{\circ}(P))\, d\omega .$$

Now, $K(P,Q)$ is a function in r and a quantity F_{ℓ} independent of r:

$$(30)\quad K(P,Q) = \sum_{\ell=0}^{\infty} r^{\ell} F_{\ell}$$

First, using the expression for the Laplace operator in spherical coordinates Heiskanen and Moritz (1967, eq. (1-41)):

$$\Delta(r \cdot K(P,Q)) = \frac{\partial^2}{\partial r^2} (r \cdot K(P,Q)) + \frac{2}{r} \frac{\partial}{\partial r} (r \cdot K(P,Q)) + r \cdot R(\theta, \lambda),$$

where $R(\theta,\lambda)$ are the terms resulting from the differentiation of $K(P,Q)$ with respect to θ and λ. Then, by applying (30) and becuase $K(P,Q)$ is harmonic in P:

$$\Delta(r \cdot K(P,Q)) = \frac{\partial}{\partial r} (K(P,Q) + r \frac{\partial}{\partial r} K(P,Q)) + \frac{2}{r}(K(P,Q) + r \frac{\partial}{\partial r} K(P,Q)) + r \cdot R(\theta,\lambda)$$

$$= 2 \frac{\partial}{\partial r} K(P,Q) + r \frac{\partial^2}{\partial r^2} K(P,Q) + \frac{2}{r} K(P,Q) + r(\frac{2}{r} \frac{\partial}{\partial r} K(P,Q)) + r \cdot R(\theta,\lambda)$$

$$= 2 \frac{\partial}{\partial r} K(P,Q) + \frac{2}{r} K(P,Q) + r(\frac{\partial^2}{\partial r} K(P,Q) + \frac{2}{r} \frac{\partial}{\partial r} K(P,Q) + R(\theta,\lambda))$$

$$= 2 \sum_{\ell=0}^{\infty} (\ell+1) \cdot r^{\ell-1} F_{\ell}$$

$$(36)\quad K(P,Q)=\sum_{\ell=m}^{\infty}\frac{(2\ell+1)(2\ell-n+3)}{R^{3-n}}\frac{(r\cdot r')^{\ell-n}}{R^{2\ell}}P_\ell(t),$$

where $m \geq 0$ and $2m > n-3$

Now, applying the identity operator on $\Delta(V^\circ_{ij})$ for $2i > n-3$:

$$I(\Delta V^\circ_{ij})(Q)=\int_\Omega r^n K(P,Q)\Delta V^\circ_{ij}(P)\,d\Omega=\rho^n_{ij}(Q)\,(-4\pi k).$$

Note that $\Delta(r^n K(P,Q))=0$ (evaluated in P). Using Green's identity we see that

$$(37)\quad (-4\pi k)\rho^n_{ij}(P)=\iint_\omega \left(r^n K(P,Q)\frac{\partial}{\partial n}V_{ij}(Q)-V_{ij}(Q)\frac{\partial}{\partial n}(r^n K(P,Q))\right)d\omega$$

$$=\frac{(2i-n+3)(-i-1)}{R^{3-n}}\frac{r^{i-n}}{R^i}\bar{S}_{ij}(\theta,\lambda)-\frac{(2i-n+3)i}{R^{3-n}}\frac{r^{i-n}}{R^\ell}\bar{S}_{ij}(\theta,\lambda)$$

$$=-\frac{(2i-n+3)(2i+1)}{R^{3-n}}\frac{r^{i-n}}{R^i}\bar{S}_{ij}(\theta,\lambda)$$

It can easily be shown using equation (1), that $\rho^n_{ij}(P)$ really reproduces $V^\circ_{ij}(Q)$ for $2i>n-3$.

By requiring the density functions (multiplied by a positive function of r) to be harmonic functions, we can proceed as above. As a final example of a simple relationship, the norm (35) may be used separately for each 2n on each of the 2n+1 dimensional subspace of degree n. The reproducing kernel becomes:

$$(38)\quad K(P,Q)=\sum_{\ell=0}^{\infty}(2\ell+1)3\cdot\frac{1}{R^3}P_\ell(t)\approx\infty,$$

and all $\rho_{nm}(P)$ are independent of r, but we will no longer have a reproducing kernel Hilbert space.

Finally, it is possible to use density functions which fulfill a condition like

$$(39) \quad \Delta (f_i(r) \cdot \rho(r, \theta, \lambda)) = 0$$

with different functions, $f_i(r)$, for r varying in different intervals between 0 and R. In this way we may take into consideration the apparently discontinuous variations in the density of the Earth.

4. Implications for the use of least squares collocation in Physical Geodesy.

Suppose that we chose a specific Hilbert space of density functions, with a reproducing kernel, which can be used to map the distributions (9) into density functions.

The reproducing kernel can be used as a covariance function of point density values, and we will then be able to predict unknown density values from known values. This covariance function can also be used to produce a covariance function for the external potential (and a cross-covariance function), cf. e.g. E. Grafarend (1970). (For a discussion of the relation between covariance functions and reproducing kernel see e.g. Parzen (1959), Lauritzen (1973) or Tscherning (1973)).

Let us denote the operator (1) by N_Q (N for Newton):

$$N_Q(\rho) = \int_\Omega \frac{1}{\|P-Q\|}\,\rho(P)\,d\Omega \tag{40}$$

The covariance between two density values ρ_P and ρ_Q is then $\operatorname{cov}(\rho_P, \rho_Q) = K(P,Q)$. The covariance between a potential value in P_1 outside Ω and ρ_Q is:

$$\operatorname{cov}(V_{P_1}, \rho_Q) = N_{P_1} K(P,Q) \tag{41}$$

and between two potential values:

$$\operatorname{cov}(V_{P_1}, V_{Q_1}) = N_{Q_1}(N_{P_1}(K(P,Q))). \tag{42}$$

Example 2. The covariance function (15),

$$K(P,Q) = \sum_{\ell=0}^{\infty} \frac{(2\ell+3)(2\ell+1)}{R^3} \left(\frac{r \cdot r'}{R^2}\right)^{\ell} P_\ell(t).$$

$$\operatorname{cov}(V_{P_1}, \rho_Q) = \int_\Omega \frac{1}{L} K(P,Q)\,d\Omega = \sum_{\ell=0}^{\infty} \frac{4\pi k}{2\ell+1} \frac{r_Q^{\ell}}{r_1^{\ell+1}} P_\ell(\cos\psi_1) \tag{43}$$

and

$$\operatorname{cov}(V_{P_1}, V_{Q_1}) = \int_\Omega \frac{1}{L} \operatorname{cov}(V_{P_1}, \rho_Q)\,d\Omega = (4\pi k)^2 R^3 \sum_{\ell=0}^{\infty} \frac{R^{2\ell}}{(r_1 \cdot r_1')^{\ell+1}} \cdot \frac{1}{(2\ell+1)(2\ell+3)} P_\ell(t). \tag{44}$$

(the distance of P_1 and Q_1 from the origin is r_1 and r_1' respectively).

It is therefore possible to combine values of density anomalies, gravity anomalies and other gravimetric quantities in least squares collocation using the covariance models described here. (For a discussion of least squares collocation, see e.g. Moritz, 1972). The resulting covariance functions will not be easy to handle in actual computations. For example, all the covariance functions described here approach infinity for $r \to R$.

Covariance functions of the following kind:

$$(45) \quad K(P,Q) = \sum_{\ell=3}^{\infty} \frac{R^{-3}}{a \cdot \ell^2 + b\ell + c} \left(\frac{r \cdot r'}{R^2} \right)^{\ell} P_{\ell}(t), \quad (a, b, c, \text{ constants})$$

would be very useful, thus producing a covariance function

$$(46) \quad \operatorname{cov}(V_{P_1}, V_{Q_1}) = \sum_{\ell=3}^{\infty} \frac{R^3}{(2\ell+1)^2 (2\ell+3)^2 (a\ell^2 + b\ell + c)} \left(\frac{r \cdot r'}{R^2} \right)^{\ell} P_{\ell}(t)$$

for the potential values outside Ω.

Evaluation of the volume integral part of (29) gives:

$$(31)\quad \iiint \Delta(r\cdot K(P,Q))V^{e}_{ij}(P)\,d\Omega = \int_0^R \frac{(2i+2)(2i+4)}{R^4\cdot r}\left(\frac{r\cdot r'}{R^2}\right)^i \bar{S}_{ij}(\theta',\lambda')\frac{R^i}{r^{i+1}}\, r^2 dr$$

$$= \frac{(r')^i}{R^{i+3}}(2i+2)(2i+4)\,\bar{S}_{ij}(\theta',\lambda')$$

and for the surface integral (using (30)):

$$(32)\quad \iint \left(\frac{\partial}{\partial n}V^{e}_{ij}(P)\cdot r\cdot K(P,Q) - \frac{\partial}{\partial n}(r\cdot K(P,Q))\cdot V^{e}_{ij}(P)\right)d\omega$$

$$= \iint_{\omega}\left(\sum_{\ell=0}^{\infty} R^{\ell+1} F_{\ell}\cdot(-i-1)\cdot\frac{1}{R^2}\bar{S}_{ij}(\theta,\lambda) - \sum_{\ell=0}^{\infty}(\ell+1)R^{\ell}F_{\ell}\cdot\frac{1}{R}\bar{S}_{ij}(\theta,\lambda)\right)d\omega$$

$$= \iint_{\sigma}\left(\left(\sum_{\ell=0}^{\infty}\left(\frac{r'}{R}\right)^{\ell}\frac{(2\ell+4)}{R^5}\sum_{m=-\ell}^{m}\bar{S}_{\ell m}(\theta,\lambda)\bar{S}_{\ell m}(\theta',\lambda')\right)\bar{S}_{ij}(\theta,\lambda)(-i-1)\right.$$

$$\left. -\left(\sum_{\ell=0}^{\infty}\left(\frac{r'}{R}\right)^{\ell}\frac{(2\ell+4)(\ell+1)}{R^5}\sum_{m=-\ell}^{\ell}\bar{S}_{\ell m}(\theta,\lambda)\bar{S}_{\ell m}(\theta',\lambda')\right)\cdot\bar{S}_{ij}(\theta,\lambda)\right)\cdot R^2 d\sigma$$

$$= -\frac{(2i+2)(2i+4)}{R^3}\left(\frac{r'}{R}\right)^i \bar{S}_{ij}(\theta',\lambda'),$$

with σ being the surface of the unit sphere.

Hence, adding both terms ((31) + (32))

$$\rho_{ij}(P) = 0$$

(in fact an effective spreading of the density).

We now have one simple algorithm which we can use for the assignment of density distributions to harmonic functions. Thus, the density functions are harmonic and all but one of the basic density functions $\rho_{ij}(P)$ are zero at the origin. Such a model may be of interest in some studies. Let us note a few facts which made the procedure work:

(a) The Laplace operator was shifted from V_{nm}^{e} to the other factor in (18) using Green's equations, and the Laplace operator applied on this gave zero, because the factor was a harmonic function.

(b) For the integration of ρ_{nm} in (23), the factor (2n+3) disappears because we are integrating the function $r^{-(2n+2)}$.

Thus, we could regard functions expanded as a series in

(33) $r^{-n} V_{\ell m}^{i}(P)$,

i.e. functions which are harmonic after a multiplication with r^n,

(34) $\Delta(r^n \cdot f(r, \theta, \lambda)) = 0$.

We will then use a norm, (which would not cause a meaningful spreading for harmonic functions for n = 1)

(35) $$\|\rho\|^2 = \int_{\Omega} r^n \rho^2 \, d\Omega$$

(This will be a norm, since r^n is non-negative).

The functions $\frac{1}{r^n} V_{\ell m}^{i}(P)$ will form an orthogonal base for this space. Computing the normalization factor for these function:

$$\left\| \frac{1}{r^n} V_{\ell m}^{i} \right\|^2 = \iiint_{\Omega} r^n \left(\frac{1}{r^n} V_{\ell m}^{i}(P) \right)^2 d\Omega = \iiint_{\Omega} \frac{r^{2\ell-n}}{R^{2\ell}} S_{\ell m}(\theta, \lambda)^2 r^2 dr d\sigma$$

$$= \frac{R^{-n+3}}{2\ell - n + 3}$$

The reproducing kernel is then:

5. Applications

Using these simple algorithms for the unique assignment of density functions $\rho^n_{\ell m}$ to the harmonics $V^o_{\ell m}$ we may assign a density model to a given approximation to the potential of the Earth (or another celestial body).

Using the representation (5) we get:

$$(47)\quad \rho(P) = kM \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} \cdot \rho^n_{\ell m}(P) \qquad 2i<n-3.$$

The selection of a proper one to one relationship must be done based on geophysical studies, cf. e.g. Kaula (1968, Chapter 2) or Jeffreys, (1970).

When the relationship has been determined, we may compute a density function, which will reproduce the external potential field as given by a limited number of potential coefficients, e.g. of degree ≤ 20. This density function may be regarded as a density reference function. The data to be dealt with in least squares collocation will hence be density anomalies with respect to this reference density function.

The density function will have the total mass M inside a sphere, e.g. with radius equal to a mean Earth sphere. We may regard this sphere as a sufficient representation of the reference ellipsoid. But still we need a way to model the masses external to the reference figures.

Such a modelling can be made in many ways. But a straightforward procedure would be a representation of the external masses as a density layer on the reference sphere. The density layer could e.g. be proportional to the height times the mean rock density.

It is not likely that the reference density function will represent the mass discontinuities inside the sphere, produced by the presence of the oceans. This discontinuity could be represented by a negative density layer on the sphere, so that we in this way will have a density function defined on the whole sphere.

The density function will be an element of a Hilbert space of functions on the sphere and it can be expanded in surface harmonics.

$$(48)\quad \rho_s = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} b_{\ell m}\, \bar{S}_{\ell m}(\theta,\lambda).$$

It will produce an external potential,

$$(49) \quad V_s(P) = N_P(\rho_s) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{R^{\ell}}{r^{\ell+1}} (2\ell+1) b_{\ell m} \cdot k \cdot 4\pi \cdot \bar{S}_{\ell m}(\theta, \lambda).$$

We may then (in the same way as explained above for volume density functions) adopt a covariance function for surface density values. This covariance function will produce a covariance functions for the external potential field. We should then be able to introduce topographic information in a least square collocation determination of the external potential field.

Conclusion

The above discussed procedures have not yet been numerically tested, but the author is convinced that their simplicity will assure that they can be used in practice.

Further developments in the field will depend upon a cooperation between geodesists, geophysists and mathematicians.

References

Grafarend, Erik: The Gaussian structure of the gravity field, Studia Geophys. et Geod. 14 (1970), 159-167.

Heiskanen,W. A. and H. Moritz: Physical Geodesy, W.H. Freeman and Co., San Francisco and London, 1967.

Jeffreys, Sir Harold: The Earth, 5th Ed., Cambridge, 1970.

Kaula, W.M.: An Introduction to Planetary Physics, John Wiley and Sons, 1968.

Lauritzen, S.A.: The Probabilistic Background of Some Statistical Methods in Physical Geodesy, Danish Geodetic Institute, Meddelélse No.48, 1973.

Meschkowski, H.: Hilbertsche Räume und Kernfunktionen, Berlin, 1962.

Moritz, H.: Advanced Least-Squares Methods, Dept. of Geodetic Science, The Ohio State University, Report No. 175, June, 1972.

Moritz, H.: Ellipsoidal Mass Distributions, Dept. of Geodetic Science, The Ohio State University, Report No. 206, 1973.

Parzen, E.: Statistical Inference on Time Series by Hilbert Space Methods. Techn. Rep. No. 23, University of Stanford, 1959.

Tscherning, C.C.: On the Relation between the Variation of the Degree-Variances and the Variation of the Anomalous Potential, Bollettino di Geodesia e Science Affini, Anno XXXII, N.3, 1973.

Yosida, K.: Functional Analysis, 3'th ed., Springer 1971.