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Reports of the Department of Geodetic Science
Report No. 208

CLOSED COVARIANCE EXPRESSIONS FOR GRAVITY ANOMALIES, GEOID UNDULATIONS, AND DEFLECTIONS OF THE VERTICAL IMPLIED BY ANOMALY DEGREE VARIANCE MODELS.

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ABSTRACT

This report first develops a new anomaly degree variance model by considering potential coefficient information to degree 20, and updated values of the point anomaly variance (1795 mgal^2), the 1° block variance (920 mgal^2) and the 5° block variance (302 mgal^2), the variances being given with respect to the Geodetic Reference System 1967. This new model was computed assuming that anomaly information was given on a sphere of radius 6371 km with the radius of the best fitting Bjerhammer sphere found to be 6369.8 km.

This new model and several other models were used to develop closed expressions for the covariance and cross-covariance functions between gravity anomalies, geoid undulations (or height anomalies), and deflections of the vertical. It is shown how these global covariance expressions can be modified for use as local covariances and for use when mean anomalies are being considered. A Fortran subroutine is provided for the determination of the covariance values implied by the recommended anomaly degree variance model.

FOREWORD

This report was prepared by C.C. Tscherning, Research Associate, Department of Geodetic Science, and the Danish Geodetic Institute, and by Richard H. Rapp, Professor, Department of Geodetic Science. This work was sponsored, in part, by the Air Force Cambridge Research Laboratories, Bedford, Massachusetts, under Air Force Contract No. F19628-72-C-0120, The Ohio State University Research Foundation Project No. 3368B1, and in part by the Danish Geodetic Institute. The Air Force Technical Monitor is Mr. Bela Szabo.

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1. Introduction

In carrying out the estimation of gravimetric dependent quantities using the methods of least squares collocation (Moritz, 1972) we need to have an analytical function that can be used to determine the covariance functions for such quantities as anomalies, deflections of the vertical, geoid undulations etc. Generally speaking a numerical covariance function for anomalies can be determined from anomaly data. The resultant function can be considered by determining a model for the anomaly degree variances. Tscherning (1972) has shown how such anomaly degree variance models may be used to determine the covariance models for several gravimetric quantities. Since we need the best estimates of our covariance models for the application of least squares collocation, it is appropriate that we use the latest available data in determining our models. In addition we are now at a stage where refinements in anomaly degree variance modeling, beyond that used by Rapp (1973) can be considered.

The purpose of this report is to describe recent computations made and subsequent analytical work that leads to improved analytical covariance models.

2. Preliminary Equations

In this section some of the relevant formulas to be used in later sections will be presented.

We first consider our covariance function which for the purposes of this report will be considered as stationary and isotropic. Then we can follow the standard definition (Heiskanen and Moritz, 1967) of the anomaly covariance as the mean product (at a given distance) of the anomaly pair Δg_P, Δg_Q. Thus:

$$C(P, Q) = \operatorname{cov}(\Delta g_P, \Delta g_Q) = M(\Delta g_P, \Delta g_Q) \tag{1}$$

On a plane the distance, or anomaly separation is usually specified by some linear distance (such as 20 km). If we deal with data on a sphere we usually considered the distance to be defined as ψ a spherical arc so that we are interested in values of $C(\psi)$. At $\psi=0$, $C(\psi)$ becomes the anomaly variance. For the estimation of $C(\psi)$ from anomaly data given on the surface of a sphere, we can write (Heiskanen and Moritz, 1967, p. 258):

$$C(\psi) = \frac{1}{4\pi} \int_{\lambda=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \Delta g(\theta, \lambda)\, \Delta g(\theta', \lambda') \sin\theta \, d\theta \, d\lambda \, d\alpha \tag{2}$$

where θ is a polar angle (0 at the north pole), λ is the longitude and α is an azimuth.

We will obtain from (2), a point anomaly covariance function if the Δg values are point anomalies or we will obtain a mean anomaly covariance function (for a specific block size) if the Δg values are mean anomalies. In practice the sphere is not completely covered by anomalies so that an expression that may be used to compute the covariance between any two functions f_j and f_k given in blocks on the sphere whose area is A_j and A_k respectively may be written: (Kaula, 1966a, p. I. B. 7).

$$C(\psi) = \frac{\sum A_j A_k f_j f_k}{\sum A_j A_k} \tag{3}$$

In our case $f_j = \overline{\Delta g}(\theta, \lambda)$ and $f_k = \overline{\Delta g}(\theta', \lambda')$ where the overbar signifies a mean anomaly. If the anomalies are given in equal area block (3) becomes:

$$C(\psi) = \frac{\sum f_j f_k}{n} \tag{4}$$

where n is the number of products taken at a given spherical distance ψ. In practice the distance ψ to which a special product at distance ψ_{jk} is determined by the equation:

$$\psi - \frac{\Delta\psi}{2} < \psi_{jk} < \psi + \frac{\Delta\psi}{2} \tag{5}$$

where $\Delta\psi$ is a suitably chosen range. In our numerical results to be discussed later, $\Delta\psi$ was specified to be 1°.

A more fundamental covariance function than that of the gravity anomalies is that of the disturbing potential, K(P, Q). We generally do not estimate K(P, Q) from numerical data, but rather consider the following series representation for it: (Moritz, 1972, p. 88):

$$K(P, Q) = \sum_{\ell=0}^{\infty} \sigma_\ell \left(\frac{R^2}{rr'}\right)^{\ell+1} P_\ell(\cos\psi) \tag{6}$$

where:

σ_ℓ are the degree variances of the anomalous potential;
R is the radius of the Bjerhammar sphere;
r, r′ are the geocentric radii to points P and Q which are separated by a spherical radius ψ.

For convenience we let:

$$s = \frac{R^2}{rr'} \tag{7}$$

In the case that we are dealing with information at the approximate surface of the earth, it is convenient to take $rr' = R_e^2$ where R_e is a mean earth radius. Then:

$$s = \left(\frac{R}{R_e} \right)^2 \tag{8}$$

We then can write:

$$K(P, Q) = \sum_{\ell=0}^{\infty} \sigma_\ell \, s^{\ell+1} P_\ell(\cos\psi) \tag{9}$$

We can also write the anomaly covariances in a series expression as (Moritz, 1972, p. 89):

$$C(P, Q) = \sum_{\ell=0}^{\infty} c_\ell \, s^{\ell+2} P_\ell(\cos\psi) \tag{10}$$

where c_ℓ are the anomaly degree variances. As written, equation (10) would yield a point anomaly covariance. In order to obtain a mean anomaly covariance we can use the β_ℓ functions of Meissl (1970, p. 23) or the q_ℓ functions of Pellinen (1966). Using β_ℓ, the modification of equation (10) yields:

$$\overline{C}(P, Q) = \sum_{\ell=0}^{\infty} \beta_\ell^2 \, c_\ell \, s^{\ell+2} P_\ell(\cos\psi) \tag{11}$$

where P and Q now refer to anomaly blocks. β_ℓ is defined as follows: (Meissl, 1970, p. 24):

$$\beta_\ell = \frac{1}{1-\cos\psi_0} \; \frac{1}{2\ell+1} \left[P_{\ell-1}(\cos\psi_0) - P_{\ell+1}(\cos\psi_0) \right] \tag{12}$$

where ψ_0 is the circular cap radius of the mean anomaly block whose covariance is to be computed. We have (for example):

$$\beta_0 = 1 \tag{12A}$$

$$\beta_1 = \tfrac{1}{2} \sin\psi_0 \cot \frac{\psi_0}{2} \tag{12B}$$

Since we usually deal with rectangular blocks of dimension s°, the corresponding ψ_0° can be found simply by equating the areas of the circular cap and the square blocks. Assuming a plane figure we write (for small blocks only):

$$\psi_0^\circ = s^\circ/\sqrt{\pi} = 0.564 s^\circ \tag{13}$$

As $\psi_0 \to 0$, $\beta_\ell \to 1$

Since gravity anomalies are related to the disturbing potential by the following equation (valid in a spherical approximation which is the case considered here):

$$\Delta g = \frac{-\partial T}{\partial r} - \frac{2}{R} T \tag{14}$$

where T is the disturbing potential, we can relate the anomaly degree variances (c_ℓ) and the degree variances of the anomalous potential (σ_ℓ) by:

$$\sigma_\ell = \frac{R^2}{(\ell-1)^2} c_\ell \tag{15}$$

Analytic models for either σ_ℓ, or c_ℓ have been described by Lauritzen (1973), Tscherning (1972), by Rapp (1973a) and implicitly by Kaula (1966b, p. 98).

The inverse of equation (10) is:

$$c_\ell = \frac{2\ell+1}{2} s^{-(\ell+2)} \int_0^\pi C(\psi) P_\ell (\cos\psi) \sin\psi \, d\psi \tag{16}$$

Equation (16) is written assuming $C(\psi)$ is a point anomaly covariance function referring to a sphere whose radius is R_e. If $C(\psi)$ is a point anomaly covariance function, then (16) with $C(\psi)$ replaced by $\overline{C}(\psi)$ will yield a mean anomaly degree variance $\overline{c}_\ell$, which is related to c_ℓ through the β_ℓ equations;

$$\overline{c}_\ell = \beta_\ell^2 c_\ell \tag{17}$$

Thus, knowing $\overline{C}(\psi)$ we can find $\overline{c}_\ell$ from (16) and c_ℓ from (17) knowing the size of the anomaly blocks to which $\overline{C}(\psi)$ refers. Specifically we can write:

$$c_\ell = \frac{2\ell+1}{2} \frac{1}{\beta_\ell^2 s^{(\ell+2)}} \int_0^\pi \overline{C}(\psi) P_\ell (\cos\psi) \sin\psi \, d\psi \tag{16A}$$

3. Numerical 1° Covariance Functions

We first start our numerical determinations by the estimation of the covariance function for 1° (approximately) equal area anomalies. One degree covariance functions have been previously estimated for ψ values from 0° to 7° by Kaula (1966c) and by

Rapp (1972). The values found in the past studies were based on analyzing 1° anomalies within a 5° equal area anomaly so that product pairs in adjacent 5° blocks were not computed nor were product pairs for distances greater than ψ approximately 7° were considered. In addition, a programming error made the results of Kaula and Rapp somewhat erroneous.

Because of the limitations of previous estimations of the 1° covariance function it was decided that it was appropriate to compute a global 1° covariance function. The starting point was a recent collection of 29960, 1° x 1° equiangular mean free-air anomalies that was obtained by updating a 1° x 1° mean anomaly set supplied by the Defense Mapping Agency - Aerospace Center. The updating was carried out using additional data along the lines of a previous update as described in Rapp (1972). These anomalies were all referred to the gravity formula of the Geodetic Reference System 1967. The 1° x 1° equiangular tape was then converted to a set of 21828 (approximately) equal area anomalies. The subdivisions of these anomalies was such that the latitude increment was 1° while the longitude increment was some integer degree of such size that the block was approximately equal in area to a 1° x 1° block at the equator. The covariances were computed using equation (3) with the $\Delta\psi$ in equation (5) of 1°. The results of this computation are given in Table A of the appendix. In this table the following quantities are given: number of product pairs, average ψ (in degrees), covariance (mgal^2). For further use the 181 values given in Table A were interpolated to determine a covariance at 0.5 degree intervals. This interpolation was carried out using an Aitken-Lagrange interpolation using 20 points as implemented through subroutine DALI (and DATSG) of the IBM System/360 Scientific Subroutine Package (H20-0205-3), Version III. The resultant 361 values are given in Table One, being identified as the unmodified $\overline{C}(\psi)$ values. The plot of this covariance function is shown in Figure One.

From these unmodified $\overline{C}(\psi)$ values we can compute the smoothed anomaly degree variances from equation (16). Such values are shown for degree 0 through 10 in Table Two where s and β are taken to be one (causing a maximum error of less than 5%). In addition values of c_ℓ from the recommended set of potential coefficients given by Rapp (1973b) are given for comparison purposes.

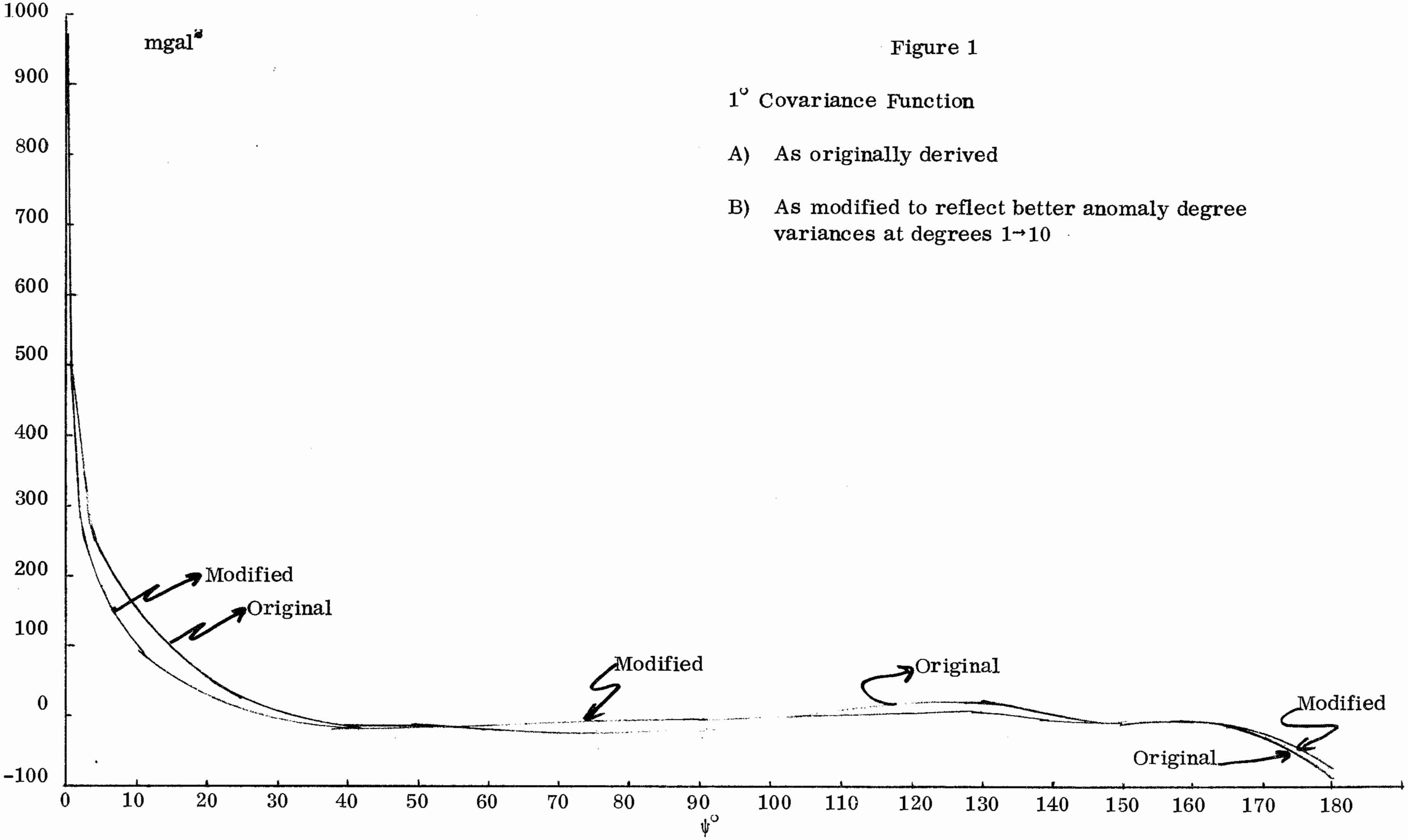


Figure 1

1° Covariance Function

A) As originally derived

B) As modified to reflect better anomaly degree variances at degrees 1→10

Table One
One Degree Anomaly Covariance Function

ψ°	$\overline{C}(\psi)$ Modified	$\overline{C}(\psi)$ Unmodified
0.0	919.66	996.66
0.50	671.64	748.60
1.00	493.43	570.27
1.50	368.24	444.87
2.00	285.35	361.70
2.50	236.09	312.07
3.00	211.42	286.95
3.50	200.69	275.70
4.00	193.37	267.78
4.50	176.88	250.62
5.00	155.86	228.85
5.50	146.38	218.55
6.00	141.39	212.67
6.50	133.39	203.72
7.00	124.92	194.23
7.50	119.86	188.09
8.00	117.38	184.48
8.50	116.89	182.80
9.00	115.10	179.77
9.50	107.02	170.40
10.00	96.52	158.56
10.50	92.45	153.11
11.00	90.54	149.79
11.50	82.86	140.66
12.00	74.57	130.89
12.50	72.40	127.21
13.00	71.77	125.06
13.50	66.85	118.59
14.00	59.75	109.92
14.50	53.61	102.21
15.00	49.30	96.32
15.50	47.01	92.44
16.00	45.95	89.79
16.50	44.04	86.29
17.00	41.61	82.29
17.50	39.27	78.38
18.00	36.96	74.51
18.50	34.87	70.88
19.00	32.97	67.46
19.50	31.09	64.08
20.00	29.33	60.85
20.50	27.89	57.96
21.00	26.35	55.01
21.50	24.11	51.38
22.00	21.59	47.51
22.50	19.81	44.42
23.00	18.06	41.40
23.50	14.99	37.09
24.00	11.78	32.69
24.50	10.79	30.54
25.00	10.51	29.16
25.50	8.90	26.48
26.00	6.63	23.19
26.50	4.67	20.26
27.00	2.99	17.65
27.50	1.65	15.42
28.00	0.47	13.40
28.50	-0.86	11.28
29.00	-2.23	9.16
29.50	-2.86	7.83
30.00	-3.29	6.73
30.50	-4.57	4.84
31.00	-5.84	2.99
31.50	-6.17	2.12
32.00	-6.14	1.65
32.50	-6.46	0.87
33.00	-7.38	-0.48
33.50	-8.88	-2.37
34.00	-10.61	-4.46
34.50	-11.85	-6.03
35.00	-12.71	-7.19
35.50	-13.52	-8.28
36.00	-14.16	-9.17
36.50	-14.36	-9.60
37.00	-14.49	-9.93
37.50	-15.11	-10.74
38.00	-15.73	-11.53
38.50	-15.39	-11.35
39.00	-14.70	-10.81
39.50	-14.94	-11.18
40.00	-15.41	-11.78
40.50	-14.59	-11.08
41.00	-13.60	-10.20
41.50	-13.46	-10.18
42.00	-13.36	-10.19
42.50	-12.28	-9.22
43.00	-11.10	-8.15
43.50	-10.96	-8.13
44.00	-11.32	-8.61
44.50	-11.53	-8.95
45.00	-11.94	-9.49
45.50	-12.80	-10.49
46.00	-13.75	-11.60
46.50	-14.14	-12.15
47.00	-14.26	-12.44
47.50	-14.75	-13.12
48.00	-15.25	-13.81
48.50	-15.06	-13.83
49.00	-14.93	-13.92
49.50	-16.34	-15.56

50.00	-17.93	-17.39
50.50	-18.22	-17.94
51.00	-18.07	-18.06
51.50	-18.45	-18.72
52.00	-19.03	-19.59
52.50	-19.34	-20.21
53.00	-19.19	-20.37
53.50	-18.43	-19.94
54.00	-17.53	-19.37
54.50	-17.29	-19.47
55.00	-17.42	-19.95
55.50	-17.28	-20.17
56.00	-17.00	-20.25
56.50	-16.64	-20.26
57.00	-16.03	-20.02
57.50	-14.87	-19.23
58.00	-13.61	-18.35
58.50	-12.86	-17.98
59.00	-12.44	-17.93
59.50	-12.37	-18.24
60.00	-12.46	-18.71
60.50	-12.51	-19.13
61.00	-12.38	-19.37
61.50	-11.79	-19.14
62.00	-10.88	-18.59
62.50	-9.92	-17.99
63.00	-9.23	-17.64
63.50	-9.17	-17.92
64.00	-9.42	-18.51
64.50	-9.34	-18.75
65.00	-9.14	-18.87
65.50	-9.21	-19.24
66.00	-9.30	-19.63
66.50	-8.92	-19.53
67.00	-8.50	-19.39
67.50	-8.50	-19.65
68.00	-8.82	-20.23
68.50	-9.26	-20.91
69.00	-9.53	-21.42
69.50	-9.29	-21.40
70.00	-8.84	-21.16
70.50	-8.71	-21.24
71.00	-8.73	-21.45
71.50	-9.00	-21.90
72.00	-9.12	-22.20
72.50	-8.62	-21.86
73.00	-7.71	-21.10
73.50	-6.77	-20.31
74.00	-6.07	-19.75
74.50	-5.86	-19.67
75.00	-6.12	-20.06
75.50	-6.72	-20.77
76.00	-7.34	-21.50
76.50	-7.60	-21.87
77.00	-7.61	-21.97
77.50	-7.66	-22.12
78.00	-7.55	-22.09
78.50	-6.83	-21.45
79.00	-5.97	-20.67
79.50	-5.87	-20.64
80.00	-5.91	-20.75
80.50	-5.93	-20.83
81.00	-6.02	-20.98
81.50	-6.71	-21.72
82.00	-7.14	-22.20
82.50	-6.80	-21.90
83.00	-6.47	-21.60
83.50	-6.15	-21.31
84.00	-5.83	-21.02
84.50	-6.01	-21.22
85.00	-6.04	-21.26
85.50	-5.17	-20.39
86.00	-4.16	-19.37
86.50	-3.54	-18.74
87.00	-3.09	-18.26
87.50	-2.75	-17.89
88.00	-2.55	-17.64
88.50	-2.57	-17.60
89.00	-2.51	-17.47
89.50	-2.12	-17.00
90.00	-1.78	-16.56
90.50	-1.80	-16.46
91.00	-1.70	-16.23
91.50	-1.10	-15.48
92.00	-0.52	-14.73
92.50	-0.47	-14.49
93.00	-0.25	-14.07
93.50	0.45	-13.14
94.00	1.03	-12.31
94.50	1.19	-11.88
95.00	1.47	-11.31
95.50	2.13	-10.34
96.00	2.83	-9.30
96.50	3.36	-8.41
97.00	3.82	-7.56
97.50	4.24	-6.73
98.00	4.83	-5.71
98.50	5.65	-4.43
99.00	6.56	-3.03
99.50	7.42	-1.67
100.00	7.97	-0.59
100.50	8.07	0.06
101.00	8.42	0.99
101.50	9.67	2.83
102.00	10.61	4.39
102.50	10.21	4.62
103.00	9.79	4.85
103.50	10.04	5.77
104.00	10.26	6.68
104.50	10.02	7.14

105.00	9.38	7.21
105.50	8.38	6.93
106.00	7.64	6.92
106.50	7.71	7.73
107.00	7.90	8.66
107.50	7.82	9.33
108.00	7.85	10.11
108.50	8.19	11.19
109.00	8.60	12.34
109.50	8.92	13.40
110.00	9.20	14.41
110.50	9.28	15.21
111.00	9.61	16.24
111.50	10.34	17.67
112.00	10.95	18.95
112.50	11.10	19.76
113.00	10.80	20.09
113.50	10.20	20.10
114.00	9.87	20.36
114.50	10.31	21.36
115.00	10.51	22.09
115.50	9.69	21.76
116.00	8.95	21.49
116.50	8.85	21.82
117.00	8.69	22.06
117.50	8.01	21.74
118.00	7.38	21.43
118.50	6.63	20.96
119.00	6.24	20.82
119.50	6.58	21.36
120.00	6.96	21.90
120.50	7.53	22.59
121.00	7.87	23.02
121.50	7.80	22.99
122.00	7.26	22.45
122.50	6.29	21.44
123.00	5.32	20.38
123.50	4.62	19.57
124.00	4.55	19.34
124.50	5.22	19.81
125.00	5.63	20.00
125.50	5.11	19.21
126.00	4.67	18.47
126.50	4.86	18.34
127.00	5.34	18.46
127.50	5.89	18.63
128.00	6.64	18.97
128.50	7.62	19.52
129.00	8.77	20.21
129.50	10.03	21.00
130.00	10.83	21.32
130.50	10.78	20.76
131.00	10.11	19.58
131.50	9.12	18.07
132.00	8.14	16.56
132.50	7.45	15.34
133.00	7.40	14.76
133.50	8.11	14.93
134.00	8.88	15.17
134.50	8.98	14.75
135.00	8.83	14.08
135.50	8.89	13.64
136.00	8.15	12.40
136.50	6.01	9.78
137.00	4.15	7.46
137.50	3.76	6.62
138.00	3.65	6.08
138.50	3.13	5.15
139.00	2.63	4.26
139.50	2.36	3.63
140.00	1.84	2.77
140.50	0.87	1.48
141.00	0.18	0.50
141.50	0.22	0.27
142.00	0.23	0.04
142.50	0.28	-0.13
143.00	0.12	-0.48
143.50	-0.35	-1.12
144.00	-0.87	-1.78
144.50	-1.24	-2.27
145.00	-1.76	-2.88
145.50	-2.56	-3.75
146.00	-3.42	-4.66
146.50	-4.20	-5.48
147.00	-4.92	-6.21
147.50	-5.60	-6.88
148.00	-5.99	-7.25
148.50	-5.94	-7.17
149.00	-6.01	-7.19
149.50	-6.67	-7.79
150.00	-6.72	-7.77
150.50	-5.28	-6.26
151.00	-3.91	-4.81
151.50	-3.45	-4.27
152.00	-3.48	-4.22
152.50	-3.75	-4.40
153.00	-3.97	-4.54
153.50	-4.01	-4.51
154.00	-4.55	-4.98
154.50	-6.07	-6.44
155.00	-6.94	-7.26
155.50	-5.86	-6.15
156.00	-4.85	-5.11
156.50	-4.94	-5.19
157.00	-4.74	-5.00
157.50	-3.69	-3.97
158.00	-3.57	-3.89
158.50	-5.32	-5.70
159.00	-6.65	-7.11
159.50	-6.48	-7.04

160.00	-6.07	-6.75
160.50	-6.19	-7.02
161.00	-7.31	-8.30
161.50	-9.38	-10.55
162.00	-10.89	-12.27
162.50	-10.66	-12.26
163.00	-10.96	-12.81
163.50	-13.54	-15.66
164.00	-15.49	-17.89
164.50	-15.45	-18.15
165.00	-14.70	-17.72
165.50	-14.05	-17.40
166.00	-14.10	-17.80
166.50	-14.96	-19.01
167.00	-15.07	-19.49
167.50	-13.53	-18.33
168.00	-12.14	-17.32
168.50	-12.07	-17.64
169.00	-12.91	-18.88
169.50	-14.92	-21.28
170.00	-17.18	-23.93
170.50	-19.16	-26.30
171.00	-19.96	-27.49
171.50	-19.48	-27.39
172.00	-19.95	-28.23
172.50	-23.66	-32.30
173.00	-27.41	-36.39
173.50	-29.62	-38.93
174.00	-30.53	-40.16
174.50	-30.66	-40.59
175.00	-30.88	-41.08
175.50	-31.99	-42.45
176.00	-33.69	-44.38
176.50	-36.24	-47.14
177.00	-40.40	-51.48
177.50	-47.02	-58.26
178.00	-54.21	-65.58
178.50	-54.64	-66.11
179.00	-41.72	-53.26
179.50	-37.87	-49.46
180.00	-72.83	-84.43

Table Two

Smoothed Anomaly Degree Variances (c_ℓ) As
Computed With 1° Free Air Anomalies
and From a Current Potential Coefficient Set

($mgal^2$)

Degree	c_ℓ from 1° anomaly data	c_ℓ from potential coefficients (Rapp, 1973b)
0	0.07	--
1	2.3	--
2	26.2	7.5
3	58.3	33.9
4	16.0	19.2
5	26.3	21.6
6	36.0	18.9
7	22.8	18.8
8	12.6	10.4
9	20.0	11.1
10	9.3	11.4

If the gravity formula were that of a mean earth ellipsoid, the zeroth degree variance should be zero. This is essentially the case here with the fact that the γ_e and the flattening of the GRS67 are quite close to be current best estimates of these parameters (Rapp, 1974). The anomalies taken on a global scale should have no first degree anomaly degree variance. The non-global 1° anomalies that we have imply through the covariance function a small one of 2.3 $mgal^2$.

The anomaly degree variances from the potential coefficients should be reliable at the lower degrees because of the accurate determination of low degree potential coefficients through satellite orbital analysis. Comparison of these values with that implied by the covariance function indicates poor agreement for degrees 2, 3, 6 and 9. This disagreement may be related to the fact that the 1° anomalies cover only 50% of the earth's surface and we cannot hope to find good low degree information from such limited coverage.

However, for future analysis it is important that we use a 1° covariance function that is characteristic of the real world especially at low degrees. To develop such a covariance function we modify the covariance function computed from the anomalies by imposing on the modified function the c_ℓ values to degree 10 as listed in Table Two (as computed from potential coefficients). To do this we first remove the effect of the $\overline{c}_\ell$ values listed in Table Two and then add back the covariance contribution from the c_ℓ values, in both cases using equation (11) setting β_ℓ and s equal to one. In effect we carry out the following modification to obtain a modified 1° covariance function:

$$\overline{C(\psi)}_m = \overline{C(\psi)}_{ORIG} + \sum_{\ell=1}^{10} (c_{\ell_{(POT\ COEFF)}} - c_{\ell_{(ORIG)}}) P_\ell(\cos\psi) \tag{18}$$

The modified covariance function is shown in Table One being labeled Modified $\overline{C}(\psi)$. This modified covariance function is plotted in Figure One.

Smoothed anomaly degree variances were developed from this modified covariance function where were then converted to the actual degree variances using equation (16A). These results and values of β_ℓ for one degree blocks and $s^{-(\ell+2)}$ are given in Table B of the appendix.

From Table One, using the modified covariance function of the current estimate for the variance of a 1° anomaly is 919.66 $mgal^2$, or a root mean square value of ±30.3 mgals with respect to the gravity formula of the Geodetic Reference System 1967.

4. A Five Degree Anomaly Variance

For purposes of obtaining models of anomaly degree variance using procedures such as described in Rapp (1973a) we need to estimate the variance of the 5° anomalies. This can be done in two ways. The first procedure is by the numerical integration of the 1° modified covariance function according to equation (7-82) of Heiskanen and Moritz (p.270). This leads to an estimate of 305 $mgal^2$. The second procedure is to compute the variance directly from the 5° anomalies. This was done by first predicting 5° equal area anomalies using the methods described in Rapp (1972) but with the more current 1°x 1° set. The variance computed by this procedure from the 1354 predicted anomalies was 298 $mgal^2$. We adopt for further use the variance of 5 degree anomalies as 302 $mgal^2$ with respect to the gravity formula of the Geodetic Reference System 1967.

5. The Point Anomaly Variance

The value of C_0 is an important quantity as it is a scaling factor for many representations of the point anomaly covariance function. C_0 has been treated as both a local or regional quantity, or a global quantity. On a regional basis C_0 is the variance of the point anomalies in some defined area. Thus, it will change from area to area. The global C_0 value is considered to be representative of the gravity field of the whole earth. The estimation of C_0 on a global basis is not straight forward since we do not have global gravity coverage. The only global point covariance function numerically estimated is that given by Kaula (1959) where he used gravity data that was current to 1958. During the 16 years since the compliation of gravity data as used by Kaula, a considerable amount of additional data has become available. Thus, a new computation of global point covariance seems appropriate and is needed. Such a computation can only be done through some organization that has access to the gravity data holdings. For this report we do not have the facilities or funds to carry out a computation of a point covariance function. However, we can use several procedures to determine C_0, the quantity so fundamental to the analytical representation of a point covariance function.

5.1 Method One

One method to estimate C_0 is to consider the relationship between a point covariance function (C(d)) and the variance ($G^2_{s^\circ}$) of anomalies given in blocks of size s°. One convenient relationship is given by Hirvonen (1962):

$$G^2_{s^\circ} = \int_0^{\sqrt{2}} W\, C(d)\, dr \qquad (19)$$

where $d = r s^\circ$

$$W = (2\pi - 8r + 2r^2) r \quad \text{when } 0 < r < 1$$

$$W = (2\pi - 4 - 2r^2 + 8\sqrt{r^2 - 1} \;\; -8 \tan^{-1}\sqrt{r^2 - 1})\; r \quad \text{where } 1 < r < \sqrt{2}$$

If we represent C(d) in the form of:

$$C(d) = C_0\, f(d) \qquad (20)$$

we can solve (19) and (20) for C_0:

$$C_0 = \frac{G^2_{s^\circ}}{\int_0^{\sqrt{2}} W f(d)\, dr} \equiv \frac{G^2_{s^\circ}}{I} \qquad (21)$$

The value of I can be obtained for various representative f(d).

Many representations of the point covariance function have been suggested. Many of these representations are summarized in papers by Groten (1966), Lauer (1971), and Jordan (1972). For the purposes of this paper we have used three models. These are:

$$(1) \quad C(d) = C_0 \underbrace{e^{-\alpha d}}_{f_1(d)} \tag{22}$$

$$(2) \quad C(d) = C_0 \underbrace{\left(1 - \frac{d}{2c_2}\right) e^{-d/c_2}}_{f_2(d)} \tag{23}$$

$$(3) \quad C(d) = C_0 \underbrace{(1+d(a_1 + d(a_2 + d(a_3 + d(a_4 + d(a_5))))))}_{f_3(d)} \tag{24}$$

The c_1 and c_2 values were obtained from fitting the Kaula (1959) point covariance curve to a distance of 1.5°. We found $c_1 = 0^\circ.897$ and $c_2 = 1^\circ.88$. Beyond a distance of 1.5°, the point covariance would not be represented well by equations (22) and (23) with the above constants. The constants of equation (24) were obtained by a least squares polynomial fit using the Kaula point covariance function to 8°. We found:

a_1 = -.9816195
a_2 = .4894498
a_3 = -.1149583
a_4 = .0126057
a_5 = -.000523222

For these models, the root mean square fit to the observed covariance function was ±30 mgal^2, ±75 mgal^2, and ±28 mgal^2 for models 1, 2 and 3 respectively. For $s^\circ = 1^\circ$, values of I (computed by numerical integration), and C_0(taking G_s^2 = 919.66 from Table One) are given for each of the models in Table Three.

Table Three

Estimation of C_0 from 1°
Anomaly Block Variances

Model	I	C_0
1, equation (22)	.64185	1433 mgal^2
2, equation (23)	.66491	1383 mgal^2
3, equation (24)	.62595	1469 mgal^2

Using weights based on the root mean square fits to the point covariance curve, the estimated C_0 from this analysis is 1447 mgal^2.

5.2 Method Two

A more direct method for determining C_0 is through the analysis of the actual point gravity anomalies. Such an analysis is not a straight forward one since the anomaly data is not uniformly distributed over the earth. Since certain areas (such as land areas) have, in general, denser anomaly coverage than ocean areas, and since free-air anomalies are correlated with land elevations or ocean depth, special care needs to be taken in the analysis of a set of point gravity anomalies for C_0.

In our analysis we basically considered a point variance by elevation range, and then converted these individual variances into a global estimate of C_0 by forming a weighted mean with weights being based on the percentage of the earth's surface lying within the elevation range.

As the first step in this procedure the Defense Mapping Agency Aerospace Center considered a set of 2,253,122 point free-air anomalies whose elevation or depth was known. Elevation ranges of 100 meter increment were chosen. For all anomalies falling within each range, the mean anomaly, the mean square anomaly and the mean elevation from the points, was determined. The mean square anomaly was computed as the sum of the square of the anomalies with the elevation range divided by the number of anomalies within the range. In subsequent discussions this quantity will be referred to as the variance of the range. This terminology is not specifically correct as a variance is usually defined with respect to a quantity whose mean is zero. In fact, the anomaly mean within a range will not be zero, but it will be zero or close to it on a global basis. This data by ranges is shown in Table Four.

In order to form a global estimate of C_0, we now need to know how elevations are distributed on the actual earth. To do this we considered mean elevations in 1654 5° equal area blocks and 64800, $1^\circ \times 1^\circ$ mean elevations. From this data the percentage of the earth's surface within a given elevation range could be found. The results found for the 5° and 1° data are shown as the last two columns in Table Four. The 5° results are shown as a matter of interest only, as the 5° subdivision is too large for the purposes needed here. We should note that all 0.0's given in Table Four with the exception of the mean anomaly for the 100 to 200 meter range indicate no data was available for the quantity. The 1° subdivision is also not sufficiently small for the most accurate work as can be seen from the fact that certain elevation ranges for which there was point elevations data were not represented in the data from the 1° mean elevation data.

The weighted variance (or C_0) was then determined as follows:

$$C_0 = \frac{\sum_i P_i (C_0)_i}{\sum_i P_i} \tag{25}$$

Table Four
Anomaly Variance and Related Information by Elevation Range

Elevation Range (meters)		No. of point Anomalies	Point anom. mean (mgals)	Mean sq. anomaly ($mgal^2$)	Average of pt. elevations (meters)	Percentage of earth's surface within range	
						1° data	5° data
-14100	-14000	0	0.0	0.0	0.0	0.002	0.0
-11200	-11100	1	-213.1	45411.6	-11113.0	0.0	0.0
-10800	-10700	5	-300.8	90650.3	-10750.6	0.0	0.0
-10700	-10600	1	-277.3	76895.3	-10674.0	0.0	0.0
-10600	-10500	2	-285.0	81253.5	-10592.0	0.0	0.0
-10500	-10400	5	-282.4	79833.0	-10425.2	0.0	0.0
-10400	-10300	13	-270.6	74429.2	-10353.8	0.0	0.0
-10300	-10200	8	-290.3	85035.7	-10228.4	0.0	0.0
-10200	-10100	12	-283.4	80914.0	-10149.4	0.0	0.0
-10100	-10000	9	-282.3	80402.3	-10065.4	0.0	0.0
-10000	-9900	22	-279.8	79255.9	-9947.7	0.0	0.0
-9900	-9800	7	-276.4	76443.4	-9858.6	0.0	0.0
-9800	-9700	11	-273.5	77105.5	-9743.8	0.0	0.0
-9700	-9600	19	-260.5	70629.0	-9645.9	0.0	0.0
-9600	-9500	16	-267.7	73083.8	-9546.7	0.0	0.0
-9500	-9400	21	-248.8	64161.0	-9446.1	0.0	0.0
-9400	-9300	27	-241.9	63265.9	-9347.8	0.0	0.0
-9300	-9200	17	-259.2	68675.0	-9247.8	0.0	0.0
-9200	-9100	22	-231.9	56997.3	-9147.5	0.0	0.0
-9100	-9000	24	-243.2	61142.7	-9041.5	0.0	0.0
-9000	-8900	30	-236.8	57878.9	-8956.7	0.0	0.0
-8900	-8800	24	-242.4	61454.6	-8853.9	0.0	0.0
-8800	-8700	36	-222.6	52069.2	-8764.4	0.0	0.0
-8700	-8600	28	-227.8	54300.5	-8667.9	0.0	0.0
-8600	-8500	43	-222.1	51648.5	-8551.8	0.0	0.0
-8500	-8400	56	-225.5	52099.5	-8440.7	0.0	0.0
-8400	-8300	386	-249.9	63345.2	-8360.1	0.0	0.0
-8300	-8200	130	-220.6	50153.0	-8259.2	0.0	0.0
-8200	-8100	156	-214.7	47255.5	-8148.5	0.0	0.0
-8100	-8000	239	-221.0	51087.8	-8048.0	0.0	0.0
-8000	-7900	277	-231.6	57988.7	-7952.1	0.0	0.0
-7900	-7800	310	-211.2	47870.1	-7856.2	0.002	0.0
-7800	-7700	220	-207.7	46967.6	-7759.5	0.0	0.0
-7700	-7600	210	-204.2	45255.4	-7654.5	0.002	0.0
-7600	-7500	265	-205.3	46034.8	-7551.2	0.002	0.0
-7500	-7400	313	-214.8	51871.4	-7446.8	0.0	0.0
-7400	-7300	417	-180.2	35771.0	-7337.9	0.008	0.0
-7300	-7200	478	-168.6	31597.3	-7250.2	0.003	0.0
-7200	-7100	506	-158.3	28832.5	-7151.5	0.002	0.0
-7100	-7000	359	-180.4	38391.0	-7054.5	0.014	0.0
-7000	-6900	407	-154.1	27775.9	-6949.7	0.007	0.0
-6900	-6800	358	-160.3	30243.8	-6851.3	0.002	0.0
-6800	-6700	473	-131.8	21597.5	-6744.5	0.005	0.0
-6700	-6600	465	-119.5	18588.8	-6658.1	0.009	0.0
-6600	-6500	530	-119.5	18717.8	-6546.2	0.024	0.0
-6500	-6400	706	-105.5	15090.3	-6454.0	0.010	0.0
-6400	-6300	796	-93.0	12701.3	-6349.6	0.027	0.0
-6300	-6200	978	-77.2	9847.8	-6246.4	0.054	0.0
-6200	-6100	1311	-57.7	6841.8	-6146.7	0.153	0.0
-6100	-6000	2596	-32.4	4002.0	-6049.2	0.321	0.0

-6000	-5900	4024	-27.3	3215.6	-5946.8	0.292	0.120
-5900	-5800	7166	-22.5	1786.2	-5847.2	0.502	0.181
-5800	-5700	8170	-17.8	1319.7	-5750.8	0.697	0.403
-5700	-5600	9542	-14.2	1287.4	-5652.4	0.983	0.562
-5600	-5500	10061	-13.0	1143.5	-5550.1	1.449	0.833
-5500	-5400	12429	-11.7	1106.4	-5450.9	1.170	1.059
-5400	-5300	14512	-12.0	1113.8	-5350.6	1.478	1.688
-5300	-5200	14791	-12.2	1245.1	-5248.2	2.130	1.620
-5200	-5100	15930	-13.6	1243.7	-5147.5	3.033	2.111
-5100	-5000	16687	-12.7	1231.6	-5051.1	2.766	1.738
-5000	-4900	16185	-11.9	1370.6	-4950.5	2.222	2.816
-4900	-4800	16178	-14.6	1401.8	-4849.1	2.395	2.085
-4800	-4700	16412	-13.9	1350.3	-4750.2	1.815	2.459
-4700	-4600	16153	-13.2	1433.0	-4651.0	2.027	2.300
-4600	-4500	16000	-12.2	1243.0	-4551.8	2.115	1.613
-4500	-4400	14261	-12.3	1431.1	-4452.9	2.102	2.059
-4400	-4300	12350	-10.6	1550.4	-4352.9	2.013	2.866
-4300	-4200	12568	-12.1	1588.8	-4251.1	2.578	2.051
-4200	-4100	11377	-15.1	2402.4	-4149.5	2.366	2.514
-4100	-4000	11111	-11.0	1709.1	-4050.4	2.435	2.053
-4000	-3900	11122	-8.7	1476.6	-3949.8	1.649	2.129
-3900	-3800	10887	-10.3	1705.2	-3851.0	1.930	2.204
-3800	-3700	9948	-7.6	1732.4	-3751.4	1.626	1.929
-3700	-3600	9982	-7.4	1861.9	-3650.7	1.404	2.398
-3600	-3500	10272	-7.9	2086.7	-3550.7	1.647	1.299
-3500	-3400	9574	-6.8	2061.3	-3451.0	1.282	2.092
-3400	-3300	9899	-7.1	1974.4	-3350.6	1.220	1.449
-3300	-3200	11356	-5.0	1920.2	-3250.4	1.339	1.818
-3200	-3100	11342	-5.0	1884.6	-3151.1	1.219	1.167
-3100	-3000	11490	-7.6	2039.3	-3050.1	1.673	1.091
-3000	-2900	12194	-10.2	2075.9	-2950.7	0.829	0.914
-2900	-2800	13157	-10.6	2138.0	-2848.7	0.765	0.967
-2800	-2700	13830	-9.4	2080.9	-2748.9	0.874	0.714
-2700	-2600	13583	-12.9	2353.4	-2651.8	0.453	0.837
-2600	-2500	19490	-2.2	2107.7	-2544.1	0.570	0.558
-2500	-2400	10858	-17.7	3254.9	-2451.0	0.909	1.047
-2400	-2300	10449	-15.4	3024.0	-2351.8	0.421	0.789
-2300	-2200	9355	-15.0	3135.1	-2251.8	0.439	1.332
-2200	-2100	12206	-9.5	2133.6	-2147.7	1.018	0.593
-2100	-2000	12676	-2.8	1803.5	-2048.7	0.559	0.615
-2000	-1900	12963	1.2	1713.2	-1951.6	0.288	0.538
-1900	-1800	11371	-0.3	2125.1	-1851.6	0.682	0.322
-1800	-1700	9321	0.0	2522.1	-1751.1	0.330	0.659
-1700	-1600	9475	5.2	2308.1	-1649.8	0.302	0.488
-1600	-1500	10452	6.1	1964.9	-1549.9	0.563	0.790
-1500	-1400	10102	4.9	2023.2	-1453.0	0.241	0.673
-1400	-1300	10371	5.4	2321.7	-1350.6	0.252	0.547
-1300	-1200	10581	3.8	2407.0	-1252.6	0.504	0.721
-1200	-1100	9917	6.1	2397.7	-1151.8	0.251	0.431
-1100	-1000	9722	7.8	2246.5	-1051.8	0.439	0.892
-1000	-900	9816	11.5	2221.2	-949.4	0.383	0.766
-900	-800	11381	9.2	2640.5	-848.5	0.331	0.720
-800	-700	10731	11.2	2043.0	-752.6	0.288	0.661
-700	-600	9038	14.2	2294.5	-650.6	0.400	0.570
-600	-500	10338	16.9	2636.4	-547.9	0.459	0.604

-500	-400	12816	15.4	2470.5	-447.7	0.394	0.779
-400	-300	16341	12.9	2490.7	-348.5	0.696	0.371
-300	-200	19910	12.9	1938.9	-249.7	0.772	0.528
-200	-100	37357	13.6	1756.5	-141.3	1.073	1.091
-100	0	85482	8.7	1713.2	-49.1	3.151	1.925
0	100	404177	3.1	1345.0	40.6	3.557	2.825
100	200	227862	0.0	807.1	147.8	3.961	3.477
200	300	172980	-0.6	801.3	245.2	3.431	3.821
300	400	106121	-0.8	970.9	347.6	2.960	2.447
400	500	86419	0.5	1054.6	448.2	2.424	2.855
500	600	51225	3.3	1345.5	546.7	1.812	2.197
600	700	35994	1.4	1580.4	647.9	1.497	0.990
700	800	29210	-2.2	1654.3	748.6	1.210	1.272
800	900	26750	1.8	1540.9	849.5	1.109	0.717
900	1000	23329	2.4	1540.5	948.9	1.054	1.277
1000	1100	23078	5.6	1416.9	1048.9	0.889	0.898
1100	1200	26348	-0.6	1193.7	1154.8	0.773	0.734
1200	1300	26176	0.7	1214.8	1251.9	0.689	0.559
1300	1400	30036	-1.2	930.2	1348.5	0.506	0.538
1400	1500	23156	1.4	1165.4	1448.3	0.451	0.234
1500	1600	17911	1.6	1557.4	1548.4	0.368	0.058
1600	1700	15296	3.1	1671.9	1648.0	0.272	0.443
1700	1800	12868	7.3	1610.2	1749.1	0.238	0.129
1800	1900	11550	10.5	1842.9	1849.7	0.219	0.061
1900	2000	12138	12.3	1683.3	1951.7	0.183	0.183
2000	2100	13163	12.2	1638.8	2049.5	0.171	0.173
2100	2200	10544	19.1	1886.8	2146.8	0.153	0.058
2200	2300	8208	33.7	2766.9	2247.9	0.091	0.183
2300	2400	4939	37.4	3644.2	2346.1	0.078	0.0
2400	2500	4006	42.5	4236.7	2450.8	0.062	0.0
2500	2600	3547	50.8	5041.9	2547.8	0.068	0.0
2600	2700	2661	55.4	5229.0	2647.4	0.075	0.061
2700	2800	2150	57.5	6384.3	2748.3	0.044	0.058
2800	2900	1721	58.3	7472.7	2846.7	0.033	0.067
2900	3000	1331	74.4	8846.3	2947.7	0.035	0.058
3000	3100	1098	78.4	10269.7	3048.9	0.037	0.067
3100	3200	869	87.3	11715.8	3147.4	0.030	0.0
3200	3300	771	88.8	11760.3	3249.9	0.022	0.0
3300	3400	654	94.5	13293.2	3348.9	0.033	0.0
3400	3500	596	80.5	11545.6	3449.6	0.026	0.125
3500	3600	362	103.0	16509.2	3549.2	0.022	0.0
3600	3700	585	82.6	10158.4	3660.7	0.027	0.0
3700	3800	566	91.5	11237.4	3743.8	0.022	0.0
3800	3900	680	93.5	10670.7	3844.9	0.031	0.0
3900	4000	406	102.6	15102.7	3944.7	0.028	0.0
4000	4100	281	105.8	13003.0	4052.3	0.040	0.0
4100	4200	234	117.9	16503.9	4148.8	0.034	0.061
4200	4300	149	134.6	23227.6	4242.6	0.024	0.0
4300	4400	208	151.4	26309.9	4344.9	0.024	0.0
4400	4500	136	114.6	16198.7	4447.5	0.029	0.061
4500	4600	101	137.4	20794.3	4548.0	0.031	0.0
4600	4700	87	149.5	24966.3	4637.7	0.027	0.0
4700	4800	20	163.6	29051.9	4736.6	0.034	0.0
4800	4900	8	198.3	43451.1	4834.6	0.029	0.051
4900	5000	4	111.5	46485.9	4961.1	0.037	0.0

5000	5100	1	252.5	63756.2	5018.6	0.037	0.0
5100	5200	4	82.2	18374.4	5163.8	0.028	0.061
5200	5300	1	268.4	72038.6	5235.8	0.037	0.0
5300	5400	0	0.0	0.0	0.0	0.020	0.0
5400	5500	0	0.0	0.0	0.0	0.016	0.0
5500	5600	0	0.0	0.0	0.0	0.012	0.0
5700	5800	0	0.0	0.0	0.0	0.004	0.0
5800	5900	0	0.0	0.0	0.0	0.002	0.0
5900	6000	0	0.0	0.0	0.0	0.002	0.0
7000	7100	0	0.0	0.0	0.0	0.002	0.0
8900	9000	0	0.0	0.0	0.0	0.002	0.0

where $(C_0)_i$ is the variance for each of the elevation ranges and P_i is the percentage of the earth's surface area having that elevation range as estimated from the 1° mean elevation data. Values of C_0 as estimated from (25) using all the data, and data from just the positive and negative elevations are given in Table Five.

Table Five

Estimates of C_0

Method	C_0 (mgal^2)
Kaula (1959)	1201
Table Three	1447
Equation (25), all data	1795
Equation (25), negative elevations	1772
Equation (25), positive elevations	1860
Based on all anomalies without consideration of elevation ranges	1644

For our future needs we select the $C_0 = 1795$ mgal^2 as the best estimate. A truer value may even be larger than this as certain high variance values found in certain elevation ranges are not represented in the 1795 figures as our elevation data was not sufficiently detailed to tell us what percentage of the earth's surface lies within these elevation ranges. The 1795 value should be more reliable than the value of 1447 estimated from Table Three, as a certain smoothing has taken place in deriving the Table Three estimates. In addition, it was necessary to make assumptions on the shape of the covariance curve in deriving the values for Table Three.

6. Anomaly Degree Variance Modeling

At this point we will develop a model for the anomaly degree variance which in turn will prove of value in deriving a closed expression for the covariance function of the disturbing potential and other gravimetric quantities. The basic procedures for this modeling have been discussed by Rapp (1973a). However, we introduce for this paper the s term and the β_ℓ term.

We first postulate an anomaly degree variance model of the following form:

$$c_\ell = \frac{A(\ell - 1)}{(\ell - 2)(\ell + B)} \tag{25A}$$

This model had originally been suggested by Tscherning. Best estimates for the A and B parameters are to be found subject to the following data:

1. Anomaly Degree Variances Determined From Potential Coefficients

The values of c_ℓ that are used here are for $\ell = 3$ to 20 are those found from the least squares collocation solution for potential coefficients as described in Rapp (1973b). These values are given in Table Five.

Table Five

Anomaly Degree Variances From Potential Coefficients (Rapp, 1973b) ($mgal^2$)

ℓ	c_ℓ	ℓ	c_ℓ
3	33.9	12	4.8
4	19.2	13	11.7
5	21.6	14	5.5
6	18.9	15	7.3
7	18.8	16	6.5
8	10.4	17	5.7
9	11.1	18	10.7
10	11.4	19	11.0
11	8.4	20	8.9

No formal standard deviations were attached to these values of c_ℓ.

These values of c_ℓ can be directly used with (25 A).

2. Anomaly Block and Point Variances

We have previously determined the block variances for 1° and 5° equal area blocks. These values can be related to c_ℓ values through equation (11) which is rewritten for the variance (i.e. $\psi = 0$) as:

$$\overline{C}(\psi = 0) = \sum_{\ell=0}^{\infty} \beta_\ell^2 c_\ell s^{\ell+2} \tag{26}$$

Equation (26) is also valid for point anomalies recalling that in this case β_ℓ equals one.

In (26) the summation is started from $\ell = 0$ but in fact we are trying to model c_ℓ from degree 3. Thus, we carry out the summation to degree 3 but we must modify our point and block variances by essentially removing the c_2 value. From Rapp (1973b) $c_2 = 7.5$ mgal². The modified data is shown in Table Six.

Table Six

Modified* Point and Block Variances
For Anomaly Degree Variance Fitting

Size	Modified Variance
Point	1788 mgal^2
1°	912 "
5°	295 "

*to refer to a complete second degree field

The adjustment procedure was carried out by first trying to determine best estimates of A and B for equation (25) by using the data of Table Five and the block variances of Table Six. The value of β_ℓ needed in (26) was computed using a ψ value determined from equation (13). Tests indicated the summation to ∞ in (26) could safely be replaced by a summation to (4) $(180^\circ)/\theta^\circ$ or to $720/\theta^\circ$. Various runs were made with different s values to determine a proper value such that the summation to ∞ (or in practice a high number such as 50,000 or 100,000) would come close to the modified point variance of 1788 mgal^2. (It was found that for an accuracy of 0.1 mgals it was sufficient to carry out the point anomaly summation to $\ell = 16000$ while for a 0.001 mgal accuracy the summation should be carried to about $\ell = 30000$).

For theoretical reasons to be seen later, the B unknown in equation (25A) should be an integer. To produce such an unknown we first made an adjustment letting A and B adjust freely. The resultant B found was 24.03. We then repeated the adjustment, fixing B at 24 exactly. In this adjustment the two block variances were given weights of 1/100. All anomaly degree variances except for degree 3 and 4 were given weights of 1/.64. At degree 3 a weight of 1/.08 was used while at degree 4 a weight of 1/.16 was used. These weight assignments were made only to assure a reasonable fit to the data and were not based on relative accuracy considerations of the data.

We give in Table Seven the parameters of the final model.

Table Seven

Parameters of Anomaly Degree Variance Model

$A = 425.28\ \text{mgal}^2$

$B = 24$ (exact)

$s = 0.999617$

We give in Table Eight a comparison of the anomaly degree variances from Table Five and those as computed from Equation (25A) using the A and B values given in Table Seven.

Table Eight

Anomaly Degree Variances ($mgal^2$)

	Original Table 5	Equation (25A)		Original	Equation (25A)
3	33.9	31.5	12	4.8	13.0
4	19.2	22.8	13	11.7	12.5
5	21.6	19.6	14	5.5	12.1
6	18.9	17.7	15	7.3	11.7
7	18.8	16.5	16	6.5	11.4
8	10.4	15.5	17	5.7	11.1
9	11.1	14.7	18	10.7	10.8
10	11.4	14.1	19	11.0	10.5
11	8.4	13.5	20	8.9	10.2

The root mean square difference between the original and adjusted values was ± 4.0 $mgal^2$. The 1° residual block variance from the adjusted model is 841 $mgal^2$ with the 5° residual block variance being 360 $mgal^2$ as compared to the corresponding values of 912 $mgal^2$ and 295 $mgal^2$ as given in Table Six. By summing (26) with $\beta_\ell = 1$ to a sufficiently high degree (50000) the point variance implied by this model is 1788 $mgal^2$. If we wished, at this point, the covariance functions implied by this new anomaly degree variance model could be computed by substitution of the model into equation (10) or (11). This type of computation will be postponed until the discussion of the closed covariance function expressions.

7. Relationship Between the Covariance Function of the Anomalous Potential and Covariance Functions of Gravity Anomalies or Deflections of the Vertical

As explained e.g. in Moritz (1972, p. 97), covariance functions of quantities related to the anomalous potential can be derived from the covariance function of the anomalous potential K(P, Q). The covariance between two quantities A and B, derived by applying a certain operation on T can be found by applying the same operation on K(P, Q). Moritz calls this fact "the law of propagation of covariances". We have above used the law to derive (15), and thereby the relation between K(P, Q) and C(P, Q). In the following we will derive the relationship between K(P, Q) and the covariances of or between the height anomaly ζ, the free-air gravity anomaly Δg and the two deflection components ξ and η.

We will use the same notation for the covariance functions as used in Moritz (1972), i.e. cov(A, B) for the covariance of the two quantities A and B. The relationship between the gravity anomaly and the anomalous potential is given above in (14). For the three other quantities we have the well known relations:

$$\zeta = T/\gamma \, , \tag{27}$$

$$\xi = -\frac{1}{\gamma \cdot r} \cdot \frac{\partial T}{\partial \varphi} \quad \text{and} \tag{28}$$

$$\eta = -\frac{1}{\cos\varphi \cdot \gamma \cdot r} \cdot \frac{\partial T}{\partial \lambda} \, , \tag{29}$$

where γ is the reference gravity, r the distance from the center of the Earth, φ the latitude and λ the longitude. It will for most purposes be sufficient to work in spherical approximation. But we will not restrict ourselves to consider only points on the surface of the Earth.

On the surface of the Earth r is substituted by a mean Earth radius (R_e), γ by a mean gravity value (G), and φ by the geocentric latitude. For a point outside (or inside) the surface of the Earth, we will substitute for r the radius of a sphere e.g. including the same volume as an ellipsoid confocal with the adopted reference ellipsoid and passing through the considered point. (Thus, we will still call this quantity r). The reference gravity can then be substituted by kM/r^3 and φ again with the proper geocentric latitude. (In practice φ is just treated as if it was equal to the geocentric latitude).

We will introduce a more compact notation for the partial derivative with respect to an independent variable e.g. r:

$$D_r = \frac{\partial}{\partial r} \, ,$$

and for the second order partial derivative with respect to r and t:

$$D^2_{rt} = \frac{\partial^2}{\partial r \partial t}$$

The formulae (27), (14), (28) and (29) becomes then:

$$\zeta = T/G \tag{30}$$

$$\Delta g = -D_r T - \frac{2}{r} T \tag{31}$$

$$\xi = -\frac{1}{G \cdot r} D_\varphi T \quad \text{and} \tag{32}$$

$$\eta = -\frac{1}{G \cdot r \cdot \cos\varphi} D_\lambda T . \tag{33}$$

Using the law of propagation of covariances given by Moritz (1972, p. 97) applied to equations (30) - (33) we find:

$$\mathrm{cov}\,(T_P,\ T_Q) = K\,(P,Q) \tag{34}$$

$$\mathrm{cov}(\Delta g_P, \Delta g_Q) = C(P,Q) = D_r D_{r'}\, K(P,Q) + \frac{2}{r} \cdot D_{r'} K(P,Q) + \frac{2}{r'} D_r\, K(P,Q) + \frac{4}{rr'}\, K(P,Q), \tag{35}$$

$$\mathrm{cov}(\Delta g_P, \zeta_Q) = (-D_r K(P,Q) - \frac{2}{r} K(P,Q)) \cdot \frac{1}{G'}\,, \tag{36}$$

$$\mathrm{cov}(\zeta_P\,, \zeta_Q) = K(P,Q) / (G \cdot G')\,, \tag{37}$$

$$\mathrm{cov}(\xi_P\,, \zeta_Q) = -D_\varphi K(P,Q) / (G \cdot G' \cdot r), \tag{38}$$

$$\mathrm{cov}(\eta_P\,, \zeta_Q) = -D_\lambda K(P,Q) / (G' G \cdot r \cdot \cos\varphi) \tag{39}$$

$$\mathrm{cov}(\xi_P\,, \xi_Q) = D_\varphi D_{\varphi'}\, K(P,Q) \,/\, (G' \cdot G \cdot r \cdot r') = D^2_{\varphi\varphi'}\, K(P,Q) / (G \cdot G' \cdot rr'), \tag{40}$$

$$\mathrm{cov}(\xi_P\,, \eta_Q) = D^2_{\varphi\lambda} K(P,Q) / (G' \cdot r' \cdot \cos\varphi' \cdot r \cdot G), \tag{41}$$

$$\mathrm{cov}(\eta_P\,, \eta_Q) = D^2_{\lambda\lambda'}\, K(P,Q) / (G' \cdot G \cdot rr' \cos\varphi \cdot \cos\varphi'), \tag{42}$$

$$\mathrm{cov}(\Delta g_P, \xi_Q) = -D_{\varphi'}\, (\mathrm{cov}(\Delta g_P\,,\ \zeta_Q)) / r' = D_{\varphi'}\, (D_r K(P,Q) + \frac{2}{r} K(P,Q)) / (G' r') \tag{43}$$

$$\mathrm{cov}(\Delta g_P, \eta_Q) = -D_{\lambda'} (\mathrm{cov}(\Delta g_P\,, \zeta_Q)) / (r' \cdot \cos\varphi') = D_{\lambda'} (D_r\, K(P,Q) + \frac{2}{r}\, K(P,Q)) / (G' r' \cos\varphi'), \tag{44}$$

where the quantities marked with an apostrophe refer to Q and the unmarked quantities refer to P.

The covariances involving the deflections components ((38) - (44)) are most easily expressed (and computed) as derivatives with respect to the cosine of the spherical distance ψ between P and Q. (We will from now on only regard isotropic covariance functions K(P, Q), i.e. so that (9) always is valid and hence K(P, Q) only depends on ψ, r and r').

Putting $t = \cos\psi$, $D_t K(P,Q) = K'$ and $D_t^2 K(P,Q) = K''$ we get:

$$D_\varphi K = D_\varphi t \cdot K'$$

$$D_\lambda K = D_\lambda t \cdot K'.$$

Hence

$$D^2_{\varphi\varphi'} K = D_\varphi t \cdot D_{\varphi'} t \cdot K'' + D^2_{\varphi\varphi'} t \cdot K' \tag{45}$$

$$D^2_{\varphi\lambda'} K = D_\varphi t \cdot D_{\lambda'} t \cdot K'' + D^2_{\varphi\lambda'} t \cdot K') \tag{46}$$

$$D^2_{\lambda\lambda'} K = D_\lambda t \cdot D_{\lambda'} t \cdot K'' + D^2_{\lambda\lambda'} t \cdot K'. \tag{47}$$

$$-D_{\varphi'} (\mathrm{cov}(\Delta g_P, T_Q)) = D_{\varphi'} t \cdot (D^2_{rt} K(P,Q) + \frac{2}{r} D_t K(P,Q)) \tag{48}$$

$$-D_{\lambda'} (\mathrm{cov}(\Delta g_P, T_Q)) = D_{\lambda'} t \cdot (D^2_{rt} K(P,Q) + \frac{2}{r} D_t K(P,Q)) \tag{49}$$

Note, the common factors K' and K'' in (47), (48) and (49), i.e., the three covariance functions $\mathrm{cov}(\xi_P, \xi_Q)$, $\mathrm{cov}(\xi_P, \eta_Q)$ and $\mathrm{cov}(\eta_P, \eta_Q)$ can easily be computed at the same time. The covariance functions (38) - (44) are used in actual prediction computations involving deflections either as observed quantities or as quantities to be predicted. These covariance functions are not anymore isotropic. Then for theoretical discussions it is more convenient to regard the covariances, where one or both of the quantities are either the longitudinal (ℓ) or the transverse component (m) of the deflection of the vertical. This type of covariance function will be isotropic and will have a simple relation to K(P, Q).

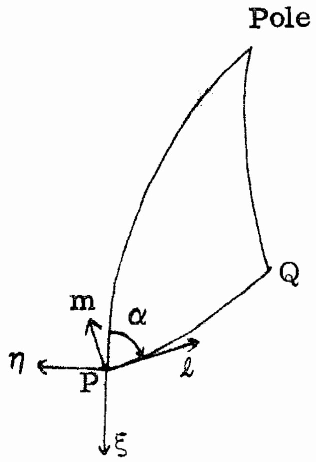


Figure 2.

Spherical triangle (Pole, Q, P) with the deflection components (ξ,η) and (ℓ, m) shown as vectors.

In Moritz (1972) the relationships between K(P, Q) and the covariance functions are expressed in terms of derivatives with respect to ψ. We will express the relations in terms of derivatives with respect to $t = \cos\psi$.

Let the azimuth between P and Q be α. Then we have (cf. figure 2):

$$\begin{aligned} \ell_P &= \cos\alpha\cdot(-\xi_P) + \sin\alpha\cdot(-\eta_P) \text{ and} \\ m_P &= \sin\alpha\cdot(-\xi_P) - \cos\alpha\cdot(-\eta_P) \end{aligned} \tag{50}$$

Using (38) and (39) and the law of propagation of covariances, we get:

$$\operatorname{cov}(\ell_P, \zeta_Q) = (\cos\alpha\cdot D_\varphi t\cdot K' + \sin\alpha\cdot D_\lambda t\cdot K'\cdot\frac{1}{\cos\varphi})/(G\cdot G'\cdot r) \text{ and}$$

$$\operatorname{cov}(m_P, \zeta_Q) = (\sin\alpha\cdot D_\varphi t\cdot K' - \cos\alpha\cdot D_\lambda t\cdot K'\cdot\frac{1}{\cos\varphi})/(G\cdot G'\cdot r).$$

Because

$$t = \sin\varphi\cdot\sin\varphi' + \cos\varphi\cdot\cos\varphi'\cdot\cos(\lambda' - \lambda)$$

we have

$$D_\varphi t = \cos\varphi\cdot\sin\varphi' - \sin\varphi\cdot\cos\varphi'\cdot\cos(\lambda' - \lambda) = \sin\psi\cdot\cos\alpha \text{ and}$$

$$D_\lambda t = \cos\varphi\cdot\cos\varphi'\sin(\lambda' - \lambda) = \cos\varphi\cdot\sin\psi\cdot\sin\alpha,$$

hence

$$\operatorname{cov}(\ell_P, \zeta_Q) = \sin\psi\cdot K'/(G\cdot G'\cdot r) \text{ and} \tag{51}$$

$$\mathrm{cov}(m_P, \zeta_Q) = 0 \tag{52}$$

For the covariance with the gravity anomaly we get in the same way (using the law of propagation of covariations and (14))

$$\mathrm{cov}(\ell_P, \Delta g_Q) = +(D_r \quad K' + \frac{2}{r} K') \cdot \sin\psi/(G \cdot r) \tag{53}$$

$$\mathrm{cov}(m_P, \Delta g_Q) = 0 \tag{54}$$

The expressions for $\mathrm{cov}(\ell_P, \ell_Q)$, $\mathrm{cov}(\ell_P, m_Q)$ and $\mathrm{cov}(m_P, m_Q)$ are derived in a very simple way in Moritz (1972, p. 109). We repeat the results expressed as derivatives of t.

$$\mathrm{cov}(\ell_P, \ell_Q) = -D_\psi^2 K/(G \cdot G' \cdot r \cdot r') = (t \cdot K' - \sin^2\psi \cdot K'')/(G \cdot G' \cdot r \cdot r') \tag{55}$$

$$\mathrm{cov}(\ell_P, m_Q) = 0 \quad \text{and} \tag{56}$$

$$\mathrm{cov}(m_P, m_Q) = -D_\psi K/(\sin\psi \cdot G \cdot G' \cdot r \cdot r') = K'/(G \cdot G' \cdot r \cdot r'). \tag{57}$$

From the formulae (51) - (57) several intersting consequences of the imposed isotropic property can be seen. The deflection components at P are independent of the height anomaly and the gravity anomaly in P. The transverse component of the deflection in P, m_P is independent of ζ_Q, Δg_Q and ℓ_Q. For ξ_P this implies, that ξ_P is independent of η_Q for $\varphi = \varphi'$ and η_P independent of ξ_Q for $\lambda = \lambda'$.

Finally we will conclude that the basic quantities to be computed in the evaluation of the expressions (34) - (49) and (51) - (57) are K, K′, K″, $D_r K' + \frac{2}{r} K$ and $\mathrm{cov}(\Delta g_P, \Delta g_Q)$.

8. Closed covariance function expressions.

In this section we will consider different models for the degree-variances and explain how closed expressions for corresponding covariance functions can be obtained. We will distinguish between different types of degree-variances and hence between different covariance functions models. Thus we will still consider only isotropic models. A subscript k will be used to distinguish between the models. Then we can define $\sigma_{k,\ell}(A,B)$ to be the degree-variances of degree ℓ in the k'th degree-variance model, i.e. so that the corresponding covariance function becomes:

$$\mathrm{cov}_k(A,B) = \left(\frac{R}{r}\right)^{I} \left(\frac{R}{r'}\right)^{J} \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(A,B)\, s^{\ell+1} P_\ell(t) \;, \tag{58}$$

where I and J are either 0 or 1. (Note, that for I=J=1 we have $\left(\frac{R}{r}\right)\cdot\left(\frac{R}{r'}\right) = s$).

For the already introduced quantities c_ℓ and σ_ℓ we then have:

$$c_\ell = \sigma_{k,\ell}(\Delta g, \Delta g) \text{ and}$$

$$\sigma_\ell = \sigma_{k,\ell}(T, T).$$

The corresponding covariance functions become, using (9) and (10)

$$\mathrm{cov}_k(T_P, T_Q) = \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T, T) s^{\ell+1} P_\ell(t) \text{ and} \tag{59}$$

$$\begin{aligned} \mathrm{cov}_k(\Delta g_P, \Delta g_Q) &= \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(\Delta g, \Delta g) s^{\ell+2} P_\ell(t) \\ &= \left(\frac{R}{r}\right)\left(\frac{R}{r'}\right) \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(\Delta g, \Delta g) s^{\ell+1} P_\ell(t). \end{aligned} \tag{60}$$

The relationship (15) becomes:

$$\sigma_{k,\ell}(T, T) = \frac{R^2}{(\ell-1)^2}\, \sigma_{k,\ell}(\Delta g, \Delta g) \tag{61}$$

In the following we will also consider the degree-variances $\sigma_{k,\ell}(\Delta g, T)$ of the covariance function $\mathrm{cov}_k(\Delta g_P, T_Q)$ which is related to the covariance (36) by:

$$\mathrm{cov}_k(\Delta g_P, T_Q) = \mathrm{cov}(\Delta g_P, \zeta_Q) \cdot G'$$

Using (36) and (59) we get:

$$\begin{aligned} \mathrm{cov}_k(\Delta g_P, T_Q) &= -D_r\left(\sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T, T) s^{\ell+1} P_\ell(t)\right) - \frac{2}{r}\left(\sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T, T) s^{\ell+1} P_\ell(t)\right) \\ &= \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T, T) \frac{(\ell-1)}{r}\, s^{\ell+1} P_\ell(t) \\ &= \frac{R}{r} \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T, T) \frac{(\ell-1)}{R}\, s^{\ell+1} P_\ell(t). \end{aligned} \tag{62}$$

Hence, using (58) we see that I = 1 and J = 0 and that

$$\sigma_{k,\ell}(\Delta g, T) = \sigma_{k,\ell}(T, T) \cdot \frac{(\ell-1)}{R} \qquad \text{and} \tag{63}$$

$$\mathrm{cov}_k(\Delta g_P\,,\ T_Q) = \frac{R}{r}\sum_{\ell=0}^{\infty} \sigma_{k,\ell}(\Delta g, T)s^{\ell+1}P_\ell(t). \tag{64}$$

(Note, that the introduced notation can't be used for covariance-functions involving deflections. These covariance functions can be expressed as the sums of series in $P'_\ell(t)$ and $P''_\ell(t)$ (apostrophe mean differentiation with respect to t), and not on the form (58) as a series in $P_\ell(t)$ and $s^{\ell+1}$.)

Five different models of the anomaly degree variances will be discussed below, i.e., k will take on values 1, 2, ... 5.

In Tscherning (1972), analytic models have been described for covariance function having anomaly degree-variances equal to:

$$\sigma_{1,\ell}(\Delta g, \Delta g) = A_1(\ell-1)^2,\ \ \ell>1 \tag{65}$$

$$\sigma_{2,\ell}(\Delta g, \Delta g) = A_2(\ell-1)/\ell,\ \ \ell>1 \text{ and} \tag{66}$$

$$\sigma_{3,\ell}(\Delta g, \Delta g) = A_3(\ell-1)/(\ell-2),\ \ \ell>2, \tag{67}$$

where A_1, A_2, and A_3 (and below A_4 and A_5) are positive constants of dimension mgal^2. These types of models have been further considered by Rapp (1972a).

$$\sigma_{4,\ell}(\Delta g, \Delta g) = A_4\ \frac{(\ell-1)}{(\ell-2)(\ell+B)} \qquad \text{and} \tag{68}$$

$$\sigma_{5,\ell}(\Delta g, \Delta g) = A_5\frac{(\ell-1)}{(\ell-2)\ (\ell+B+\beta\ell^2)}\ ,\ \ \ell>2 \tag{69}$$

For $i+j = \frac{1}{\beta}$ and $i\cdot j = B/\beta$ we can write (69):

$$\sigma_{5,\ell}(\Delta g, \Delta g) = \frac{A_5}{\beta}\cdot\frac{(\ell-1)}{(\ell-2)(\ell+i)(\ell+j)}\ . \tag{70}$$

As indicated above, the covariance functions corresponding to models 1, 2 and 3 can be represented by closed expressions. (By closed expression we mean expressions which only contain a finite number of terms). This is also true for model 4 and 5, provided we place some restrictions on B or i and j. First of all the resulting degree-variances have to be greater than or equal to zero for ℓ greater than 2. Hence, B and i, j will have to be greater than -2. And the technique used below for the derivation will imply that we have to restrict B and i, j to integer values and that we also will have to require that i is unequal to j and that all three quantities are greater than -1.

We will not consider the covariance functions derived using the model (65) because the anomaly degree-variances are unrealistic. Thus, the model leads to very simple closed expressions for the covariance functions, which can be found, e.g. in Tscherning (1972).

The technique we will use for the derivation of the closed covariance expressions is very simple. The covariance functions can be split into components which, upon multiplication by appropriate constants will yield the covariance function. These components can be expressed as:

$$F = \sum_{\ell=0}^{\infty} s^{\ell+1} P_\ell(t) \text{ and } \tag{71}$$

$$F_i = \sum_{\ell=0}^{\infty} \frac{1}{\ell+i} s^{\ell+1} P_\ell(t) \quad \text{for } i>0 \tag{72}$$

$$F_i = \sum_{\ell=-i+1}^{\infty} \frac{1}{\ell+i} s^{\ell+1} P_\ell(t) \quad \text{for } i \le 0, \quad \text{and} \tag{73}$$

as the first and second derivatives of F or F_i with respect to t,

$$F', F'', F_i', F_i''.$$

We have for example (using (59) (61) and (67)):

$$\text{cov}_3(T_P, T_Q) = A_3 \sum_{\ell=3}^{\infty} \frac{R^2}{(\ell-1)(\ell-2)} s^{\ell+1} P_\ell(t)$$

$$= A_3 \cdot R^2 \sum_{\ell=3}^{\infty} \left(\frac{1}{\ell-2} - \frac{1}{\ell-1}\right) s^{\ell+1} P_\ell(t)$$

$$= A_3 \cdot R^2 (F_{-2} - (F_{-1} - s^3 P_2(t))).$$

The closed expression for the function F can be derived using the well known formula (Heiskanen and Moritz, 1967, eq. 1-80):

$$F = s \cdot \sum_{\ell=0}^{\infty} s^{\ell} P_{\ell}(t) = \frac{s}{\sqrt{1-2st+s^2}} . \tag{74}$$

The denominator will be one of the basic quantities in the following derivations, so we will use:

$$\begin{aligned} L &= \sqrt{1-2st+s^2}\,, \\ M &= 1 - L - s \cdot t \quad \text{and} \\ N &= 1 + L - st\,. \end{aligned} \tag{75}$$

We then have:

$$F = \frac{s}{L} .$$

The functions F_i can be derived by multiplying F or $\frac{1}{L}$ by an appropriate power of s and integrating the expression with respect to s.

Using:

$$\int_0 s^{\ell+i-1}\, ds = \frac{s^{\ell+i}}{\ell+i} , \quad \ell + i > 0$$

we see, that by integrating

$$\frac{s^{i-1}}{L} = \sum_{\ell=0}^{\infty} s^{\ell+i-1} P_{\ell}(t) \tag{76}$$

we should be able to find F_i. We have by (72)

$$s^{i-1} F_i = \sum_{\ell=0}^{\infty} \frac{s^{\ell+i}}{\ell+i} P_{\ell}(t) \text{ for } i>0 \text{ and by (73):} \tag{77}$$

$$s^{i-1} F_i = \sum_{\ell=0,\,\ell\neq -i}^{\infty} \frac{s^{\ell+i}}{\ell+i} P_\ell(t) - \sum_{\ell=0}^{i-1} \frac{s^{\ell+i}}{\ell+i} P_\ell(t), \quad i \leq 0 . \tag{78}$$

The integrals:

$$\int \frac{s^i}{L}\, ds, \quad i = -2,\ -1,\ 0,\ 1,\ 2 \tag{79}$$

can be found in integral tables as Gradshteyn-Ryzhik (abbreviated below to G.R.), (1965).

From these basic integrals, F_i can be computed using recursion formulae. We will first consider negative powers of s.

Using G.R. 2.268 we get:

$$\int \frac{ds}{s^i L} = -\frac{L}{(i-1)s^{i-1}} + \frac{(2i-3)\cdot t}{(i-1)} \int \frac{ds}{s^{i-1}L} - \frac{i-2}{(i-1)} \int \frac{ds}{s^{i-2}L}, \quad i > 0, \tag{80}$$

and hence

$$\int \frac{ds}{s^2 L} = -\frac{L}{s} + t \cdot \int \frac{ds}{s \cdot L} + a_{-2} \tag{81}$$

$$\begin{aligned} \int \frac{ds}{s^3 L} &= -\frac{L}{2s^2} + \frac{3}{2} t \left(-\frac{L}{s} + t \int \frac{ds}{s \cdot L} \right) - \frac{1}{2} \int \frac{ds}{s \cdot L} + a_{-3} \\ &= -\frac{3ts+1}{2s^2} \cdot L + P_2(t) \int \frac{ds}{s \cdot L} + a_{-3} \end{aligned} \tag{82}$$

where a_{-2}, a_{-3} are integration constants. From G.R. 2.266 we have:

$$\begin{aligned} \int \frac{ds}{s \cdot L} &= -\ln \frac{2 - 2ts + 2 \cdot L}{s} + a_{-1} = \ln \frac{2}{1 - ts + L} + \ln(s) + \ln(4) + a_{-1} \\ &= \ln \frac{2}{N} + \ln(s) + \ln(4) + a_{-1} \end{aligned} \tag{83}$$

The constant a_{-1} is determined requiring (78) to be zero for s equal to zero.

$$s^{-1}F_0 = \int \frac{ds}{s \cdot L} - \int \frac{ds}{s} = \ln\frac{2}{N} + \ln(4) + a_{-1} ,$$

hence $a_{-1} = -\ln(4)$ and then:

$$s^{-1} F_0 = \ln\frac{2}{N} . \tag{84}$$

Then we can compute F_{-1} and F_{-2}.

$$s^{-2} F_{-1} = \sum_{\ell=2}^{\infty} \frac{s^{\ell-1}}{\ell-1} P_\ell(t) = -\frac{L}{s} + t \cdot \ln\frac{2}{N} - \int\frac{ds}{s^2} + a_{-2}$$

$$= \frac{1-L}{s} + t \cdot \ln\frac{2}{N} + a_{-2} = \frac{1-ts-L}{s} + t \cdot \ln\frac{2}{N} \quad \text{or} \tag{85}$$

$$F_{-1} = s \cdot (M + ts \cdot \ln\frac{2}{N}), \tag{86}$$

$$s^{-3} F_{-2} = \sum_{\ell=3}^{\infty} \frac{s^{\ell-2}}{\ell-2} P_\ell(t) = \int \left(\frac{1}{s^3} + \frac{t}{s^2} + \frac{P_2(t)}{s} \right) ds + \int\frac{ds}{s^3 L} + a_{-3}$$

$$= \frac{1}{2s^2} + \frac{t}{s} - P_2(t) \cdot \ln(s) + (P_2(t) \cdot (\ln\frac{2}{N} + \ln(s)) - \frac{3ts+1}{2s^2} L + a_{-3})$$

$$= (1 + 2ts - (3ts+1) \cdot L)/(2s^2) + P_2(t) \cdot \ln\frac{2}{N} + a_{-3}.$$

The constant a_{-3} can now be determined. Because $\ln\frac{2}{N}$ is zero for s equal to zero, we must have:

$$\lim_{s \to 0} \frac{1 + 2ts - (3ts+1) \cdot L}{2s^2} = -a_{-3}$$

The limit can be determined using the rule of l'Hospital two times. Note first, that $D_s L = (s-t)/L$. We then get

$$\lim_{s \to 0} \frac{1+2st-(3ts+1)\cdot L}{2s^2} = \lim_{s \to 0} \frac{-3t \cdot L-(3ts+1)(s-t)/L+2t}{4s}$$

$$= \lim_{s \to 0} (-3t(s-t)/L-(6ts+1-3t^2)/L+(3ts^2+s-3t^2s-t)(s-t)/L^3)/4$$

and hence:

$$a_{-2} = -\frac{7t^2-1}{4} = -\frac{3}{2}t^2+\frac{1-t^2}{4}.$$

We then get:

$$F_{-2} = s((-3t^2s^2+2ts+1-(3ts+1)\cdot L)/2+(P_2(t)\cdot \ell n\frac{2}{N}+\frac{1-t^2}{4})\cdot s^2)$$

$$= s((1-ts-L)(3ts+1)/2+s^2(P_2(t)\cdot \ell n\frac{2}{N}+\frac{1-t^2}{4})) \tag{87}$$

$$= s(M\cdot(3ts+1)/2+s^2(P_2(t)\cdot \ell n\frac{2}{N}+(1-t^2)/4)).$$

For the evaluation of covariance functions involving deflections, we have to compute F'_0, F''_0, F'_{-1}, F''_{-1}, F'_{-2}, and F''_{-2} (where again the apostrophe means differentiation with respect to $t = \cos\psi$).

We will first compute some auxiliary quantities:

$$D_t L = -\frac{s}{L}, \quad D_t(\frac{1}{L}) = \frac{s}{L^3},$$

$$D_t M = -s+\frac{s}{L} = s\frac{(1-L)}{L}$$

$$D_t N = -s-\frac{s}{L} = -s(1+L)/L.$$

Hence from (84) we get by differentiation:

$$F_0' = s \cdot D_t \ln \frac{2}{N} = \frac{s^2(1+L)}{L \cdot N} = s^2 \left(\frac{1}{L \cdot N} + \frac{1}{N}\right) \tag{88}$$

$$F_0'' = s^2\left(\frac{-N(-s/L) - L(-s(1+L)/L)}{(L \cdot N)^2} + \frac{s(1+L)/L}{N^2}\right) \tag{89}$$

$$= s^3\left(\frac{N + L(1+L)}{L^3 \cdot N^2} + \frac{1+L}{L \cdot N^2}\right) = s^3\left(\frac{N+L}{L^3 \cdot N^2} + \frac{2+L}{L \cdot N^2}\right)$$

For F_{-1} we get using (86)

$$\begin{aligned} F_{-1}' &= s(D_t M + F_0 + t \cdot F_0') \\ &= s(s(1-L)/L + F_0 + t \cdot s^2 \left(\frac{1}{L \cdot N} + \frac{1}{N}\right)) \\ &= s^2 \left((1-L)/L + \ln \frac{2}{N} + t \cdot s\left(\frac{1}{L \cdot N} + \frac{1}{N}\right)\right), \end{aligned} \tag{90}$$

$$\begin{aligned} F_{-1}'' &= s(D_t^2 M + 2F_0' + t \cdot F_0'') \\ &= s\left(\frac{s^2}{L^3} + 2s^2\left(\frac{1}{L \cdot N} + \frac{1}{N}\right) + t \cdot s^3\left(\frac{N+L}{L^3 \cdot N^2} + \frac{2+L}{L \cdot N^2}\right)\right) \\ &= s^3\left(\frac{1}{L^3} + \frac{2(1+L)}{L \cdot N} + ts\left(\frac{1}{L^3 \cdot N} + \frac{(1+L)^2}{(L \cdot N)^2}\right)\right) \end{aligned} \tag{91}$$

and for F_{-2} we get by (87)

$$\begin{aligned} F_{-2}' &= s\,[\,D_t M \cdot (3ts+1)/2 + M \cdot \frac{3s}{2} + s^2(3t \cdot \ln \frac{2}{N} + P_2(t) \cdot D_t \ln \frac{2}{N} - t/2)] \\ &= s^2[(3ts+1)(1-L)/(2L) + \frac{3}{2}\,M + s(3t \cdot \ln \frac{2}{N} + P_2(t) \cdot s\left(\frac{1}{LN} + \frac{1}{N}\right) - t/2)] \\ &= s^2[\,\tfrac{1}{2}\left((3ts+1)/L + 2 - 7ts - 3L\right) + s\left(3t \cdot \ln \frac{2}{N} + P_2(t) \cdot s\left(\frac{1+L}{L \cdot N}\right)\right)]. \end{aligned} \tag{92}$$

$$F''_{-2} = s^2\Big[\tfrac{1}{2}\Big(\frac{L\cdot 3s-(3ts+1)(-s/L)}{L^2} - 7s + \frac{3s}{L}\Big) + s\Big(3\ln\frac{2}{N} + 6ts\,\frac{1+L}{L\cdot N}$$

$$+P_2(t)\cdot s\Big(-\frac{-s(1+L)}{N^2\cdot L} - \frac{(-s/L)\cdot N - L(s(1+L)/L)}{(L\cdot N)^2}\Big)\Big)\Big] \tag{93}$$

$$= s^3\Big[\Big(\frac{6}{L} + \frac{3ts+1}{L^3} - 7\Big)\tfrac{1}{2} + 3\ln\frac{2}{N} + 6ts\,\frac{1+L}{L\cdot N} + P_2(t)\cdot s^2\Big(\Big(\frac{1+L}{L\cdot N}\Big)^2 + \frac{1}{L^3 N}\Big)\Big].$$

The closed expressions for F_i, $i > 0$ can be found using another recursion formula, G.R. 2.263. We will treat this case in a more general way, because in this case we want to derive expressions not only for i = 1, 2 and 3 but for i = 1 to ∞. It is also necessary to have a recursion formula well suited for actual computations.

We have (using G.R. 2.263):

$$\int \frac{s^i ds}{L} = \frac{s^{i-1}\cdot L}{i} + \frac{(2i-1)t}{i}\int \frac{s^{i-1}}{L}\,ds - \frac{(i-1)}{i}\int \frac{s^{i-2}}{L}\,ds \quad \text{or} \tag{94}$$

$$\frac{1}{s^{i-1}}\int \frac{s^i ds}{L} = (L + (2i-1)t\cdot\frac{1}{s^{i-1}}\int\frac{s^{i-1}ds}{L} - \frac{(i-1)}{s}\cdot\frac{1}{s^{i-2}}\int\frac{s^{i-2}\,ds}{L}).\frac{1}{i} \tag{95}$$

Realizing that:

$$F_i = \frac{1}{s^{i-1}}\int_0 \frac{s^{i-1}}{L}\,ds \qquad \text{for } i>0$$

we get

$$F_{i+1} = (L + (2i-1)t\cdot F_i - \frac{(i-1)}{s}\cdot F_{i-1})\cdot\frac{1}{s\cdot i}\,. \tag{96}$$

Fortunately we can use the recursion formula for the computation of $D_t F_i = F_i'$ and $D_t^2 F_i = F_i''$ as well.

Differentiating (96) we have:

$$F'_{i+1} = (D_t L + (2i-1)(F_i + t\cdot F_i') - \frac{(i-1)}{s}\cdot F'_{i-1})\cdot\frac{1}{i\cdot s} \quad \text{and} \tag{97}$$

differentiating one time more:

$$F''_{i+1} = \left(D_t^2 L + (2i-1)(2F_i' + t \cdot F_i'') - \frac{(i-1)}{s} \cdot F''_{i-1}\right) \cdot \frac{1}{i \cdot s} \qquad (98)$$

with $D_t L = -\frac{s}{L}$ and $D_t^2 L = -\frac{s^2}{L^3}$.

As in the case where i was less than or equal to zero, we must now compute the first two terms in the recursion formula, i.e.

$$F_1,\ F_1',\ F_1'',\ F_2,\ F_2' \text{ and } F_2''$$

Using G.R. 2.2641 we get

$$F_1 = \int \frac{ds}{L} + a_1 = \ell n\,(2 \cdot L + 2 \cdot s - 2t) + a_1$$

and hence, by (96)

$$F_2 = \frac{1}{s}\left[\int \frac{s\,ds}{L} + a_2\right] = \frac{1}{s}\,[L + t\int \frac{ds}{L} + a_2]$$

Computation of the limites of the integrals for s → 0 give us the integration constants:

$$a_1 = -\ell n(2-2t) \quad \text{and}$$

$$a_2 = -t\,\ell n\,(2-2t) - 1.$$

Hence

$$F_1 = \ell n\left(\frac{L+s-t}{1-t}\right) = \ell n\left(1 + \frac{2s}{1-s+L}\right), \qquad (99)$$

which can be verified by multiplying the numerator and the denominator by (1-s+L). The last expression for F_1 is the best suited for numerical use, because it avoids dividing by zero for $\psi = 0$.

For F_2 we get:

$$F_2 = \frac{1}{s}\,(L + t \cdot F_1 + a_2) = \frac{1}{s}\,(L - 1 + t \cdot F_1). \qquad (100)$$

The first and second derivatives of F_1 and F_2 becomes:

$$F_1' = \frac{(1-s+L)}{(1+s+L)} \cdot \frac{(-2s)(-s/L)}{(1-s+L)^2} = \frac{2s^2}{(1+s+L)(1-s+L)L} \tag{101}$$

$$= \frac{2s^2}{(1+L-ts)\cdot L\cdot 2} = \frac{s^2}{(1+L-ts)\cdot L} = \frac{s^2}{L\cdot N}\,.$$

$$F_1'' = s^2\left[\frac{-N(-s/L) - L(-s/L-s)}{L^2N^2}\right] = s^2\left[\frac{1+L-ts+L+L^2}{L^3N^2}\right] \tag{102}$$

$$= s^3\left[\frac{1+L}{L^2N^2} + \frac{1}{N\cdot L^3}\right]$$

$$F_2' = \frac{1}{s}\,(-s/L + t\cdot F_1' + F_1) = -\frac{1}{L} + \frac{ts}{L\cdot N} + F_1/s \tag{103}$$

$$F_2'' = \frac{1}{s}\,(-s^2/L^3 + 2F_1' + t\cdot F_1'') \tag{104}$$

We will now derive the relations between the functions F_i and the covariance models 2, 3, 4 and 5.

Model 2.

Using (59), (61), (66), we get:

$$\mathrm{cov}_2(T_P, T_Q) = K_2(P,Q) = \sum_{\ell=0}^{\infty} \sigma_{2,\ell}(T,T)\cdot s^{\ell+1} P_\ell(t)$$

$$= \sum_{\ell=2}^{\infty} \frac{R^2}{(\ell-1)^2}\,\sigma_{2,\ell}(\Delta g, \Delta g)\, s^{\ell+1} P_\ell(t)$$

$$= A_2\cdot R^2 \sum_{\ell=2}^{\infty} \frac{1}{\ell\cdot(\ell-1)}\, s^{\ell+1} P_\ell(t)$$

$$= A_2\cdot R^2 \left(\sum_{\ell=2}^{\infty}\left(\frac{1}{\ell-1} - \frac{1}{\ell}\right) s^{\ell+1} P_\ell(t)\right)$$

and by (73), (84) and (86)

$$\mathrm{cov}_2(T_P, T_Q) = A_2 R^2 [(F_{-1} + ts^2 - F_0] = A_2 R^2 [s(M + ts \cdot \ell n \frac{2}{N} + ts) - s \cdot \ell n \frac{2}{N}]$$

$$= A_2 R^2 \cdot s[1 - L + (ts - 1)\ell n \frac{2}{N}]. \tag{105}$$

In the same way, we get using (64), (63), (66) and (84)

$$\mathrm{cov}_2(T_P, \Delta g) = \frac{R}{r'} \sum_{\ell=0}^{\infty} \sigma_{2,\ell} (\Delta g, T) s^{\ell+1} P_\ell(t) = A_2 \cdot R \cdot \left(\frac{R}{r'}\right) \cdot \sum_{\ell=2}^{\infty} \frac{s^{\ell+1}}{\ell} P_\ell(t)$$

$$= A_2 \cdot \frac{R^2}{r} \cdot (F_0 - ts^2) = A_2 \cdot \frac{R^2}{r} \cdot s(\ell n \frac{2}{N} - ts) \tag{106}$$

and by (60), (66), (73), (74) and (84):

$$\mathrm{cov}_2(\Delta g_P, \Delta g_Q) = A_2 \sum_{\ell=2}^{\infty} \frac{\ell - 1}{\ell} s^{\ell+2} P_\ell(t)$$

$$= A_2 \cdot s \left(\sum_{\ell=2}^{\infty} s^{\ell+1} P_\ell(t) - \sum_{\ell=2}^{\infty} \frac{s^{\ell+1}}{\ell} P_\ell(t) \right) \tag{107}$$

$$= A_2 s (F - s - ts^2 - (F_0 - s^2 t)) = A_2 s^2 \left(\frac{1}{L} - 1 - \ell n \frac{2}{N} \right)$$

The covariance functions involving deflections of the vertical will, as mentioned above contain K_2', K_2'' and $-D_r K_2' - \frac{2}{r} K_2'$.

Differentiating (105) gives:

$$K_2' = A_2 R^2 (F_{-1}' + s^2 - F_0') \quad \text{and} \tag{108}$$

$$K_2'' = A_2 R^2 (F_{-1}'' - F_0'') \tag{109}$$

Because $-D_r K_2 - \frac{2}{r} K_2' = \mathrm{cov}_2(\Delta g_P, T_Q)$ we get by differentiating (106):

$$-D_r K_2' - \frac{2}{r} K_2' = A_2 \frac{R^2}{r} (F_0' - s^2) \tag{110}$$

Combining the three last equations with (55), (57), (51), and (53) we get the following equations, which can be evaluated using equations (88) - (91).

$$\begin{aligned} \mathrm{cov}_2(\ell_P, \ell_Q) &= (t \cdot K_2' - \sin^2\psi \cdot K_2'')/(G \cdot G' \cdot r \cdot r') \\ &= A_2 \cdot \frac{R^2}{r \cdot r'} (t(F_{-1}' - s^2 - F_0') - \sin^2\psi \cdot (F_{-1}'' - F_0''))/(G \cdot G') \end{aligned} \tag{111}$$

$$\mathrm{cov}_2(m_P, m_Q) = K_2'/(G \cdot G' \cdot r \cdot r') = A_2 \frac{R^2}{r \cdot r'} (F_{-1}' - s^2 - F_0')/(G \cdot G') \tag{112}$$

$$\mathrm{cov}_2(\ell_P, \zeta_Q) = \sin\psi \cdot K_2'/(G \cdot G' \cdot r) = A_2 \frac{R^2}{r} (F_{-1}' - s^2 - F_0') \cdot \sin\psi/(G \cdot G') \tag{113}$$

$$\mathrm{cov}_2(\ell_P, \Delta g_Q) = A_2 \cdot \frac{R^2}{r \cdot r'} \cdot (F_0' - s^2) \cdot \sin\psi \cdot \frac{1}{G} = \frac{A_2 \cdot s}{G} \cdot \sin\psi (F_0' - s^2) \tag{114}$$

Model 3.

From (59), (61) and (67) we get:

$$\begin{aligned} \mathrm{cov}_3(T_P, T_Q) = K_3(P, Q) &= \sum_{\ell=3}^{\infty} \frac{R^2}{(\ell-1)^2} \cdot \sigma_{3,\ell}(\Delta g, \Delta g) s^{\ell+1} P_\ell(t) \\ &= A_3 R^2 \sum_{\ell=3}^{\infty} \frac{1}{(\ell-1)(\ell-2)} s^{\ell+1} P_\ell(t) = A_3 R^2 \sum_{\ell=3}^{\infty} \left(\frac{1}{\ell-2} - \frac{1}{\ell-1} \right) s^{\ell+1} P_\ell(t) \end{aligned}$$

and then using (73):

$$\mathrm{cov}_3(T_P, T_Q) = A_3 R^2 \cdot [F_{-2} - (F_{-1} - s^3 P_2(t))] \tag{115}$$

For the covariances between the gravity anomaly and the anomalous potential we get using (62), (63), (67) and (73):

$$\mathrm{cov}_3(T_P, \Delta g_Q) = \sum_{\ell=3}^{\infty} \left(\frac{R}{r'}\right) \cdot \frac{R}{\ell-1} \sigma_{3,\ell}(\Delta g, \Delta g)\, s^{\ell+1} P_\ell(t)$$

$$= A_3 \frac{R^2}{r'} \sum_{\ell=3}^{\infty} \frac{1}{\ell-2} s^{\ell+1} P_\ell(t) \qquad (116)$$

$$= A_3 \frac{R^2}{r'} \cdot F_{-2}.$$

And for $\mathrm{cov}_3(\Delta g_P, \Delta g_Q)$ we get using (60), (67), (71), (73) and (74):

$$\mathrm{cov}_3(\Delta g_P, \Delta g_Q) = A_3 \sum_{\ell=3}^{\infty} \frac{\ell-1}{\ell-2} s^{\ell+2} P_\ell(t) = A_3 \cdot s \left[\sum_{\ell=3}^{\infty} s^{\ell+1} P_\ell(t)\right.$$

$$\left. + \sum_{\ell=3}^{\infty} \frac{1}{\ell-2} s^{\ell+1} P_\ell(t)\right] \qquad (117)$$

$$= A_3 \cdot s \left[\frac{s}{L} - s - s^2 t - s^3 P_2(t) + F_{-2}\right]$$

The formula (115) becomes using (86) and (87):

$$\mathrm{cov}_3(T_P, T_Q) = A_3 R^2 \left[-s(M + ts \cdot \ln \frac{2}{M}) + s^3 P_2(t) + s(M(3ts+1)/2\right.$$

$$\left. + s^2 (P_2(t) \cdot \ln \frac{2}{N} + (1-t^2)/4))\right] \qquad (118)$$

$$= A_3 R^2 \cdot \left[s^3 (P_2(t)(1 + \ln \frac{2}{N}) + \frac{\sin^2\psi}{4}) - s^2 \cdot t \ln \frac{2}{M} + s(3ts-1) \cdot \frac{M}{2}\right]$$

This is the correct version of the formula given by Lauritzen (1973, p. 82), in which the quantities here called M and N have been interchanged and the R^2 factor is missing.

Explicit expressions can be written down for (116) and (117) as well, using (86) and (87). But generally it is easier to compute the values of (86) and (87) separately and then evaluate the covariances using (115)-(117).

The derivatives necessary for the evaluation of the covariances involving deflections ((38) - (44) and (51) - (57)) becomes by differentiating (115) and (116):

$$K'_3 = A_3 R^2 [\, F'_{-2} + 3s^3 t - F'_{-1}] \tag{119}$$

$$K''_3 = A_3 R^2 [\, F''_{-2} + 3s^3 - F''_{-1}] \quad \text{and} \tag{120}$$

$$-D_r K'_3 - \frac{2}{r} K'_3 = D_t \operatorname{cov}_3 (\Delta g_P ,\ T_Q) = \frac{A_3 \cdot R^2}{r} \cdot F'_{-2} , \tag{121}$$

which then can be evaluated using the formula for F'_{-1}, F'_{-2}, F''_{-1}, and F''_{-2}, (90) - (93). Combining the three last equations with (55), (57), (51), and (53) we get:

$$\begin{aligned}\operatorname{cov}_3 (\ell_P ,\ \ell_Q) &= (t \cdot K'_3 - \sin^2\psi \cdot K''_3)/(G \cdot G' \cdot r \cdot r') \\ &= A_3 \cdot s \cdot [t(F'_{-2} + 3ts^3 - F'_{-1}) - \sin^2\psi (F''_{-2} + 3s^3 - F''_{-1})] \cdot \frac{1}{G \cdot G'} ,\end{aligned} \tag{122}$$

$$\operatorname{cov}_3 (m_P ,\ m_Q) = K'_3/(G \cdot G' \cdot r \cdot r') = A_3 \circ s(F'_{-2} + 3ts^3 - F'_{-1}), \tag{123}$$

$$\operatorname{cov}_3 (\ell_P ,\ \zeta_Q) = \sin\psi \cdot K'_3/(G \cdot G' \cdot r) = A_3 \frac{R^2}{r \cdot G \cdot G'} (F'_{-2} + 3ts^3 - F'_{-1}) \cdot \sin\psi \tag{124}$$

$$\operatorname{cov}_3 (\ell_P ,\ \Delta g_Q) = \sin\psi (-D_{r'} K' - \frac{2}{r'} K')/(G \cdot r) = A_3 \cdot s \cdot \sin\psi \cdot F'_{-2} \cdot \frac{1}{G} \tag{125}$$

Model 4. Using again (59) and (61) and now (68) we get

$$\begin{aligned}\operatorname{cov}_4 (T_P ,\ T_Q) = K_4 (P, Q) &= \sum_{\ell=3}^{\infty} \frac{R^2}{(\ell-1)^2} \sigma_{4,\ell} (\Delta g,\ \Delta g) \cdot s^{\ell+1} P_\ell (t) \\ &= A_4 \cdot R^2 \cdot \sum_{\ell=3}^{\infty} \cdot \frac{1}{(\ell-1)(\ell-2)(\ell+B)} \cdot s^{\ell+1} \cdot P_\ell(t).\end{aligned} \tag{126}$$

Unfortunately we will now have to introduce one more notation related to the degree-variances. We will define:

$$\tau_{k,\ell} (T, T) = \sigma_{k,\ell} (T, T) \cdot \frac{1}{A_k \cdot R^2} , \tag{127}$$

$$\tau_{k,\ell}(\Delta g, T) = \sigma_{k,\ell}(\Delta g, T) \cdot \frac{1}{A_k \cdot R} \qquad \text{and} \tag{128}$$

$$\tau_{k,\ell}(\Delta g, \Delta g) = \sigma_{k,\ell}(\Delta g, \Delta g) \cdot \frac{1}{A_k} \quad . \tag{129}$$

All the quantities (127) - (129) are unitless quantities, and we have e.g. using (127), (61) and (68):

$$\tau_{4,\ell}(T,T) = \frac{R^2}{(\ell-1)^2}\, \sigma_{4,\ell}(\Delta g, \Delta g) \cdot \frac{1}{A_4 \cdot R^2} = \frac{1}{(\ell-1)(\ell-2)(\ell+B)}$$

This quantity can be partitioned as follows:

$$\tau_{4,\ell}(T,T) = \frac{1}{\ell+B}\left[\frac{1}{\ell-2} - \frac{1}{\ell-1}\right] = \frac{1}{B+2}\left(\frac{1}{\ell-2} - \frac{1}{\ell+B}\right) - \frac{1}{B+1}\left(\frac{1}{\ell-1} - \frac{1}{\ell+B}\right)$$

$$= \frac{1}{(B+2)(B+1)}\left[\frac{B+1}{\ell-2} - \frac{B+2}{\ell-1} + \frac{1}{\ell+B}\right] .$$

hence using (126), (127), (72) and (73) we get:

$$\begin{aligned} \mathrm{cov}_4(T_P, T_Q) = K_4(P,Q) &= \frac{A_4 \cdot R^2}{(B+2)(B+1)}\left[\sum_{\ell=3}^{\infty} \frac{B+1}{\ell-2}\, s^{\ell+1} P_\ell(t) - \sum_{\ell=3}^{\infty} \frac{B+2}{\ell-1}\, s^{\ell+1} P_\ell(t) \right. \\ &\qquad \left. + \sum_{\ell=3}^{\infty} \frac{1}{\ell+B}\, s^{\ell+1} P_\ell(t)\right] \\ &= \frac{A_4 \cdot R^2}{(B+2)(B+1)}\left[(B+1)\cdot F_{-2} - (B+2)(F_{-1} - s^3 P_2(t)) \right. \\ &\qquad \left. + F_B - \frac{s}{B} - \frac{s^2 t}{B+1} - \frac{s^3 P_2(t)}{B+2}\right] \end{aligned} \tag{130}$$

Correspondingly we get using (128), (68) and (63):

$$\tau_{4,\ell}(\Delta g, T) = \frac{1}{(\ell-2)(\ell+B)} = \frac{1}{B+2}\left[\frac{1}{\ell-2} - \frac{1}{\ell+B}\right]$$ and hence using (64), (72)

and (73):

$$\mathrm{cov}_4(\Delta g_P, T_Q) = A_4 \frac{R^2}{r}\left[\sum_{\ell=3}^{\infty} \frac{1}{\ell-2} s^{\ell+1} P_\ell(t) - \sum_{\ell=3}^{\infty} \frac{1}{\ell+B} s^{\ell+1} P_\ell(t)\right] \cdot \frac{1}{B+2}$$

$$= \frac{A_4 R^2}{r \cdot (B+2)} \left[F_{-2} - \left(F_B - \frac{s}{B} - \frac{s^2 t}{B+1} - \frac{s^3 P_2(t)}{B+2}\right)\right]. \tag{131}$$

For $\tau_{4,\ell}(\Delta g, \Delta g)$ we get in a similar way using (129) and (68):

$$\tau_{4,\ell}(\Delta g, \Delta g) = \frac{\ell-1}{(\ell-2)(\ell+B)} = \frac{\ell-2+1}{(\ell-2)(\ell+B)} = \frac{1}{\ell+B} + \left(\frac{1}{\ell-2} - \frac{1}{\ell+B}\right)\frac{1}{B+2}$$

$$= \frac{B+1}{(B+2)(\ell+B)} + \frac{1}{(B+2)(\ell-2)} = \frac{1}{(B+2)}\left(\frac{B+1}{\ell+B} + \frac{1}{\ell-2}\right)$$

and hence using (60), (72) and (73)

$$\mathrm{cov}_4(\Delta g_P, \Delta g_Q) = \frac{A_4}{(B+2)}\left(\sum_{\ell=3}^{\infty} \frac{B+1}{\ell+B} s^{\ell+2} P_\ell(t) + \sum_{\ell=3}^{\infty} \frac{1}{\ell-2} s^{\ell+2} P_\ell(t)\right)$$

$$= \frac{A_4 \cdot s}{(B+2)} \left[(B+1)\left(F_B - \frac{s}{B} - \frac{s^2 t}{B+1} - \frac{s^3 P_2(t)}{B+2}\right) + E_2\right] \tag{132}$$

We will now differentiate (130) and (131) getting the formula necessary for the computation of the covariances involving deflections;

$$K'_4 = \frac{A_4 \cdot R^2}{(B+2)(B+1)} \left[(B+1) F'_{-2} - (B+2)(F'_{-1} - 3ts^3) + F'_B - \frac{s^2}{B+1} - \frac{3s^3 t}{B+2} \right] \tag{133}$$

$$K''_4 = \frac{A_4 \cdot R^2}{(B+2)(B+1)} \left[(B+1) F''_{-2} - (B+2)(F''_{-1} - 3s^3) + F''_B - \frac{3s^3}{B+2} \right] \tag{134}$$

$$-D_r K' - \frac{2}{r} K'_4 = D_t(\mathrm{cov}(\Delta g_P, T_Q)) = \frac{A_4 R^2}{r \cdot (B+2)} \left[F'_{-2} - \left(F'_B - \frac{s^2}{B+1} - \frac{3s^3 t}{B+2}\right)\right] \tag{135}$$

The formula (133)-(135) can be evaluated using (90)-(93) and the recursion formula (97) and (98) with the "initial values" given by (101)-(104).

By using (133)-(135) we can write down the covariance functions (55), (57), (51), and (53). We get:

$$\mathrm{cov}_4(\ell_P, \ell_Q) = (t \cdot K_4' - \sin^2\psi K_4'')/(G \cdot G' \cdot r \cdot r') \tag{136}$$

$$= \frac{A_4 \cdot s}{(B+2)(B+1)} \Big[t \cdot \Big((B+1) F_{-2}' - (B+2)(F_{-1}' - 3ts^3) + F_B' - \frac{s^2}{B+1} - \frac{3s^3 t}{B+2} \Big)$$

$$\div \sin^2\psi \Big((B+1) F_{-2}'' - (B+2)(F_{-1}'' - 3s^3) + F_B'' - \frac{3s^3}{B+2} \Big) \Big] \cdot \frac{1}{G \cdot G'} \, ,$$

$$\mathrm{cov}_4(m_P, m_Q) = K_4'/(G \cdot G' \cdot r \cdot r') \tag{137}$$

$$= \frac{A_4 \cdot s}{(B+2)(B+1) \cdot G \cdot G'} \Big[(B+1) F_{-2}' - (B+2)(F_{-1}' - 3ts^3) + F_B' - \frac{s^2}{B+1} - \frac{3s^3 t}{B+2} \Big]$$

$$\mathrm{cov}_4(\ell_P, \zeta_Q) = \sin\psi \cdot K_4'/(G \cdot G' \cdot r) \tag{138}$$

$$= \frac{A_4 \cdot R^2}{(B+2)(B+1) r \cdot G \cdot G'} \cdot \Big[(B+1) F_{-2}' - (B+2)(F_{-1}' - 3ts^3) + F_B'$$

$$- \frac{s^2}{B+1} - \frac{3s^3 t}{B+2} \Big]$$

and finally:

$$\mathrm{cov}_4(\ell_P, \Delta g_Q) = \sin\psi \cdot (-D_{r'} K_4' - \frac{2}{r} K_4') \cdot \frac{1}{G \cdot r} \tag{139}$$

$$= \frac{A_4 \cdot \sin\psi \cdot s}{(B+2) \cdot G} \Big[F_{-2}' - (F_B' - \frac{s^2}{B+1} - \frac{3s^3 t}{B+2}) \Big]$$

Model 5. Using (127), (61) and (69) we get:

$$\tau_{5,\ell}(T, T) = \frac{R^2}{(\ell-1)} \sigma_{5,\ell}(\Delta g, \Delta g) \frac{1}{A_5 \cdot R^2} = \frac{1}{(\ell-1)(\ell-2)(\ell+i)(\ell+j)}$$

$$= \frac{1}{j-i} \Big[\frac{1}{(\ell-1)(\ell-2)(\ell+i)} - \frac{1}{(\ell-1)(\ell-2)(\ell+j)} \Big]$$

$$= \frac{1}{j-i}\left[\left(\frac{1}{(\ell-2)(i+2)} - \frac{1}{(\ell-1)(i+1)} + \frac{1}{(\ell+i)(i+1)(i+2)}\right)\right.$$

$$\left. - \left(\frac{1}{(\ell-2)(j+2)} - \frac{1}{(\ell-1)(j+1)} + \frac{1}{(\ell+j)(j+1)(j+2)}\right)\right]$$

$$= \frac{1}{j-i}\left[\frac{j+2-i-2}{(\ell-2)(i+2)(j+2)} + \frac{i+1-j-1}{(\ell-1)(j+1)(i+1)} + \frac{1}{(\ell+i)(\ell+1)(i+2)}\right.$$

$$\left. - \frac{1}{(\ell+j)(j+1)(j+2)}\right]$$

$$= \frac{1}{(\ell-2)(i+2)(j+2)} - \frac{1}{(\ell-1)(i+1)(j+1)} + \left[\frac{1}{(\ell+i)(i+1)(i+2)}\right.$$

$$\left. - \frac{1}{(\ell+j)(j+1)(j+2)}\right]\frac{1}{j-i} ,$$

and by (128), (63) and (69)

$$\tau_{5,\ell}(\Delta g, T) = \frac{1}{(\ell-2)(\ell+i)(\ell+j)} = \frac{1}{j-i}\left[\frac{1}{\ell-2}\cdot\frac{1}{\ell+i} - \frac{1}{\ell-2}\cdot\frac{1}{\ell+j}\right]$$

$$= \frac{1}{j-i}\left[\frac{1}{i+2}\left(\frac{1}{\ell-2} - \frac{1}{\ell+i}\right) - \frac{1}{j+2}\left(\frac{1}{\ell-2} - \frac{1}{\ell+j}\right)\right]$$

$$= \frac{1}{\ell-2} + \frac{1}{j-i}\left[\frac{1}{(j+2)(\ell+j)} - \frac{1}{(\ell+i)(i+2)}\right]$$

and finally by (129) and (69)

$$\tau_{5,\ell}(\Delta g, \Delta g) = \frac{\ell-1}{(\ell-2)(\ell+i)(\ell+j)} = \frac{1}{j-i}\left[\frac{i+1}{i+2}\cdot\frac{1}{\ell+i} + \frac{1}{(\ell-2)(i+2)}\right.$$

$$\left. - \frac{j+1}{j+2}\cdot\frac{1}{\ell+j} - \frac{1}{(\ell-2)(j+2)}\right]$$

$$= \frac{1}{(\ell-2)(i+2)(j+2)} + \frac{1}{j-i}\left[\frac{i+1}{i+2}\cdot\frac{1}{\ell+i} - \frac{j+1}{j+2}\cdot\frac{1}{\ell+j}\right]$$

and hence using (59), (72) and (73) we get:

$$\begin{aligned}
\mathrm{cov}_5(T_P, T_Q) = K_5(P, Q) = A_5\cdot R^2\Bigg[&\frac{1}{(i+2)(j+2)}\sum_{\ell=3}^{\infty}\frac{1}{\ell-2}s^{\ell+1}P_\ell(t)\\
&-\frac{1}{(i+1)(j+1)}\sum_{\ell=3}^{\infty}\frac{1}{\ell-1}s^{\ell+1}P_\ell(t)\\
&+\frac{1}{j-i}\left(\frac{1}{(i+1)(i+2)}\sum_{\ell=3}^{\infty}\frac{1}{\ell+i}s^{\ell+1}P_\ell(t)\right.\\
&\left.-\frac{1}{(j+1)(j+2)}\sum_{\ell=3}^{\infty}\frac{1}{\ell+j}s^{\ell+1}P_\ell(t)\right)\Bigg]
\end{aligned} \qquad (140)$$

$$\begin{aligned}
= A_5R^2\Bigg[&\frac{1}{(i+2)(j+2)}F_{-2} - \frac{1}{(i+1)(j+1)}(F_{-1} - s^3P_2(t))\\
&+\frac{1}{j-i}\left(\frac{1}{(i+1)(i+2)}\left(F_i - \frac{s}{i} - \frac{s^2t}{i+1}\quad\frac{s^3P_2(t)}{i+2}\right) - \frac{1}{(j+1)(j+2)}\left(F_j - \frac{s}{j}\right.\right.\\
&\left.\left.-\frac{s^2t}{j+1} - \frac{s^3P_2(t)}{j+2}\right)\right)\Bigg],
\end{aligned}$$

by(62)

$$\begin{aligned}
\mathrm{cov}_5(\Delta g_P, T_Q) = A_5\frac{R^2}{r}\Bigg[&\sum_{\ell=3}^{\infty}\frac{1}{\ell-2}s^{\ell+1}P_\ell(t) + \frac{1}{j-i}\left(\frac{1}{j+2}\sum_{\ell=3}^{\infty}\frac{1}{\ell+j}s^{\ell+1}P_\ell(t)\right.\\
&\left.-\frac{1}{i+2}\sum_{\ell=3}^{\infty}\frac{1}{\ell+i}s^{\ell+1}P_\ell(t)\right)\Bigg]
\end{aligned} \qquad (141)$$

$$=A_5 \frac{R^2}{r}\left[F_{-2} + \frac{1}{j-i}\left(\frac{1}{j+2}\left(F_j - \frac{s}{j} - \frac{s^2 t}{j+1} - \frac{s^2 P_2(t)}{j+2}\right)\right.\right.$$

$$\left.\left.- \frac{1}{i+2}\left(F_i - \frac{s}{i} - \frac{s^2 t}{i+1} - \frac{s^3 P_2(t)}{i+2}\right)\right)\right] \qquad \text{(141) cont'd}$$

and by (60)

$$\mathrm{cov}_5(\Delta g_P, \Delta g_Q) = A_5\left[\frac{1}{(i+2)(j+2)}\sum_{\ell=3}^{\infty}\frac{1}{\ell-2}\, s^{\ell+2} P_\ell(t)\right.$$

$$\left.+\frac{1}{j-i}\left(\frac{i+1}{i+2}\sum_{\ell=3}^{\infty}\frac{1}{\ell+i}\, s^{\ell+2} P_\ell(t) - \frac{j+1}{j+2}\sum_{\ell=3}^{\infty}\frac{1}{\ell+j}\, s^{\ell+2} P_\ell(t)\right)\right] \qquad (142)$$

$$=A_5 \cdot s\left[\frac{1}{(i+2)(j+2)}\, F_{-2} + \frac{1}{j-i}\left(\frac{i+1}{i+2}\left(F_i - \frac{s}{i} - \frac{s^2}{i+1} - \frac{s^3 P_2(t)}{i+2}\right)\right.\right.$$

$$\left.\left.- \frac{j+1}{j+2}\left(F_j - \frac{s}{j} - \frac{s^2 t}{j+1} - \frac{s^3 P_2(t)}{j+2}\right)\right)\right].$$

The covariances (140)-(142) can then be evaluated using (86), (87) and the recursion formula (96) with "initial values" (99) and (100). As in the other models it is necessary to compute K_5', K_5'' and $-D_r K_5' - \frac{2}{r}K_5'$ to find the expressions for the covariance functions involving deflections of the vertical. The formulae can be derived by differentiating (140) and (141) and later evaluated using the proper recursion formula exactly as explained in model 4.

Note in the equations (140), (141) and (142) the denominators are equal to j-i, i+2, j+2, i+1, j+1. The occurence of these and similar quantities are the reason for the above mentioned restrictions on i and j (and B).

The above described expressions for the closed covariance functions can also be used in cases, where a set of empirical degree-variances are used in connection with degree-variances defined through one of the models (65)-(69). In this case, the basic covariance function cov (T_P, T_Q) is represented by, e.g.

$$\sum_{\ell=0}^{n} \hat{\sigma}_{\ell}(T,T)s^{\ell+1}P_{\ell}(t)+\sum_{\ell=n+1}^{\infty}\sigma_{k,\ell}(T,T)s^{\ell+1}P_{\ell}(t) \tag{143}$$

where $\hat{\sigma}_{\ell}(T,T)$ are the empirically determined degree-variances as would be computed from equation (15). We will distinguish between the above mentioned covariance functions $\mathrm{cov}_k(A,B)$ and this new type of covariance function by a subscript E, i.e., $\mathrm{cov}_E(T_P,T_Q)$ is equal to expression (143). We rewrite (143):

$$\begin{aligned}\mathrm{cov}_E(T_P,T_Q)&=\sum_{\ell=0}^{n}(\hat{\sigma}_{\ell}(T,T)-\sigma_{k,\ell}(T,T))s^{\ell+1}P_{\ell}(t)+\sum_{\ell=0}^{\infty}\sigma_{k,\ell}(T,T)s^{\ell+1}P_{\ell}(t)\\&=\sum_{\ell=0}^{n}(\hat{\sigma}_{\ell}(T,T)-\sigma_{k,\ell}(T,T))s^{\ell+1}P_{\ell}(t)+\mathrm{cov}_k(T_P,T_Q).\end{aligned} \tag{144}$$

Noting, that the relations (61) and (63) are valid for empirical degree-variances as well, we find using (60), (63), and the relations (34) - (37), (51), (53), (55) and (57)

$$\mathrm{cov}_E(\Delta g_P,T_Q)=\sum_{\ell=0}^{n}(\hat{\sigma}_{\ell}(\Delta g,T)-\sigma_{k,\ell}(\Delta g,T))\cdot\frac{R}{r}s^{\ell+1}P_{\ell}(t)+\mathrm{cov}_k(\Delta g_P,T_Q), \tag{145}$$

$$\mathrm{cov}_E(\Delta g_P,\Delta g_Q)=\sum_{\ell=0}^{n}(\hat{\sigma}_{\ell}(\Delta g,\Delta g)-\sigma_{k,\ell}(\Delta g,\Delta g))s^{\ell+2}P_{\ell}(t)+\mathrm{cov}_k(\Delta g_P,\Delta g_Q) \tag{146}$$

$$\begin{aligned}\mathrm{cov}_E(\ell_P,\ell_Q)&=\Big(\sum_{\ell=0}^{n}(\hat{\sigma}_{\ell}(T,T)-\sigma_{k,\ell}(T,T))\,s^{\ell+1}(t\cdot P_{\ell}'(t)-\sin^2\psi\cdot P_{\ell}''(t)\Big)/(G\cdot G'\cdot r\cdot r')\\&\quad+\mathrm{cov}_k(\ell_P,\ell_Q)\end{aligned} \tag{147}$$

$$\begin{aligned}&=\Big(t\cdot\sum_{\ell=0}^{n}(\hat{\sigma}_{\ell}(T,T)-\sigma_{k,\ell}(T,T))s^{\ell+1}P_{\ell}'(t)-\sin^2\psi\sum_{\ell=0}^{n}(\hat{\sigma}_{\ell}(T,T)-\sigma_{k,\ell}(T,T))\\&\quad\cdot s^{\ell+1}P_{\ell}''(t)\Big)/(G\cdot G'\cdot r\cdot r')+\mathrm{cov}_k(\ell_P,\ell_Q)\end{aligned}$$

$$\mathrm{cov}_E\,(m_P, m_Q) = \Big(\sum_{\ell=0}^{n} (\hat{\sigma}_\ell(T,T) - \sigma_{k,\ell}(T,T))\,s^{\ell+1} P'_\ell(t)\Big) / (G \cdot G' \cdot r \cdot r') \tag{148}$$

$$+\mathrm{cov}_k\,(m_P, m_Q)$$

$$\mathrm{cov}_E\,(\ell_P, \ell_Q) = \sin\psi \Big(\sum_{\ell=0}^{n} (\hat{\sigma}_\ell(T,T) - \sigma_{k,\ell}(T,T))\,s^{\ell+1} P_\ell(t)\Big) / (G \cdot G' \cdot r) + \mathrm{cov}(\ell_P, \ell_Q) \tag{149}$$

and finally

$$\mathrm{cov}_E(\ell_P, \Delta g_Q) = \sin\psi \Big(\sum_{\ell=0}^{n} (\hat{\sigma}_\ell(\Delta g, T) - \sigma_{k,\ell}(\Delta g, T))\,s^{\ell+1} \cdot \frac{R}{r'} \cdot P'_\ell(t)\Big) / (G \cdot r) \tag{150}$$

$$+\mathrm{cov}_k(\ell_P, \Delta g_Q)$$

where $P'_\ell(t)$ and $P''_\ell(t)$ are the ℓ'th order Legendre polynomial differentiated with respect to t one and two times respectively.

We now define ϵ_ℓ through the following equations:

$$\epsilon_\ell(T, T) = \hat{\sigma}_\ell(T, T) - \sigma_{k,\ell}(T, T) \tag{151}$$

$$\epsilon_\ell(\Delta g, T) = \hat{\sigma}_\ell(\Delta g, T) - \sigma_{k,\ell}(\Delta g, T) \text{ and} \tag{152}$$

$$\epsilon_\ell(\Delta g, \Delta g) = \hat{\sigma}_\ell(\Delta g, \Delta g) - \sigma_{k,\ell}(\Delta g, \Delta g) \tag{153}$$

We can then see, that the covariance functions (144)-(150) involves the summation of finite series.

$$\sum_{\ell=0}^{n} \epsilon_\ell(T, T)\,s^{\ell+1} P_\ell(t),\ \sum_{\ell=0}^{n} \epsilon_\ell(T, \Delta g)\,s^{\ell+1} P_\ell(t),\ \sum_{\ell=0}^{n} \epsilon_\ell(\Delta g, \Delta g)\,s^{\ell+1} P_\ell(t) \tag{154}$$

$$\sum_{\ell=0}^{n} \epsilon_\ell(T, T)\,s^{\ell+1} P'_\ell(t),\ \sum_{\ell=0}^{n} \epsilon_\ell(T, \Delta g)\,s^{\ell+1} P'_\ell(t) \tag{155}$$

and

$$\sum_{\ell=0}^{n} \epsilon_{\ell}(T,T)s^{\ell+1}P''_{\ell}(t) \tag{156}$$

Recursion algorithms for the summation of those three types of series will be given in the next section.

Using the above developed expressions (144)-(150) it is possible to compute covariance functions of and between height anomalies, gravity anomalies and deflections of the vertical corresponding to the recommended model for the anomaly degree-variances. This is possible because we have selected the value B in Table Seven (p. 22) equal to the integer 24.

Using the empirical determined value of $\hat{\sigma}_2(\Delta g, \Delta g) = 7.5\ \mathrm{mgal}^2$ (cf. Table Two) we can then, for example, write down an expression for the covariance functions of the anomalous potential:

$$\mathrm{cov}_E(T_P, T_Q) = 7.5 \cdot 10^{-10} \cdot R^2 \cdot s^3 \cdot P_2(t) + A \cdot 10^{-10} \cdot R^2 \sum_{\ell=3}^{\infty} \frac{1}{(\ell-1)(\ell-2)(\ell+24)} s^{\ell+1} P_{\ell}(t)$$

$$= 7.5 \cdot 10^{-10} \cdot R^2 \cdot s^2 \cdot P_2(t) + \mathrm{cov}_4(T_P, T_P),$$

where the factor 10^{-10} is used to convert the covariance into units of m^4/sec^4, supposing R in units of meters.

In a similar way we can write down the expressions for the covariance functions, $\mathrm{cov}_E(\Delta g_P, \zeta_Q)$, $\mathrm{cov}_E(\Delta g_P, \Delta g_Q)$, $\mathrm{cov}_E(\Delta g_P, \ell_Q)$, $\mathrm{cov}_E(\zeta_P, \ell_Q)$, $\mathrm{cov}_E(\ell_P, \ell_Q)$ and $\mathrm{cov}_E(m_P, m_Q)$.

We have computed values of the covariances for varying spherical distance ψ and for P and Q lying on the surface of the Earth and 500 km above the surface of the Earth respectively. See tables 9 and 10 and figures 3-9.

The radius of the Bjerhammar sphere, R, has been determined as:

$$R = \sqrt{s_{\text{table 7}}} \cdot R_e = (0.999617)^{\frac{1}{2}} \cdot 6371.0\,\mathrm{km} = 6369.8\,\mathrm{km}.$$

The quantities r and r' are computed by adding the actual height above the reference ellipsoid (here 0 and 500 km) to the adopted mean Earth radius, R_e.

The subroutine presented in the appendix has been used for the computation of the given values.

Table 9

Covariance between various quantities computed using the anomaly degree variances of model 4 and $\sigma_0(\Delta g, \Delta g) = 7.5 \text{mgal}^2$ at the surface of the sphere approximating the earth ($R_e = 6371$km).

<u>Covariances Between</u>

ψ	$\Delta g_P, \Delta g_Q$	$\Delta g_P, \ell_Q$	$\Delta g_P, \zeta_Q$	ℓ_P, ℓ_Q	m_P, m_Q	ℓ_P, ζ_Q	ζ_P, ζ_Q
	mgal^2	mgal·arc sec	mgal·m	arc sec^2	arc sec^2	arc sec·m	m^2
0° 0.0'	1795.0	0.0	452.3	45.3	45.3	0.0	926.1
0 30.0	801.8	67.3	434.8	19.2	27.1	7.3	925.0
1 0.0	572.7	59.9	417.7	14.1	21.7	11.7	922.4
1 30.0	452.6	54.2	402.3	11.5	18.7	15.1	918.8
2 0.0	375.5	49.8	388.3	9.8	16.7	18.0	914.3
2 30.0	320.9	46.2	375.4	8.6	15.2	20.4	909.1
3 0.0	279.9	43.3	363.3	7.7	14.0	22.6	903.3
3 30.0	247.6	40.9	352.0	7.0	13.1	24.6	896.9
4 0.0	221.6	38.8	341.3	6.4	12.3	26.4	890.0
5 0.0	181.9	35.4	321.3	5.5	11.0	29.6	874.9
6 0.0	152.8	32.8	303.0	4.8	10.0	32.4	858.2
8 0.0	112.8	28.9	269.8	3.7	8.6	36.9	820.7
10 0.0	86.1	26.2	240.2	2.9	7.5	40.5	778.9
12 0.0	66.9	24.0	213.2	2.3	6.7	43.3	733.7
14 0.0	52.2	22.3	188.3	1.7	6.1	45.4	685.9
16 0.0	40.6	20.8	165.1	1.2	5.5	47.0	636.0
18 0.0	31.1	19.4	143.4	0.7	5.0	48.0	584.8
20 0.0	23.2	18.2	123.2	0.3	4.6	48.6	532.7
22 0.0	16.5	17.0	104.2	-0.0	4.2	48.7	480.2
24 0.0	10.9	15.9	86.5	-0.4	3.9	48.5	427.7
26 0.0	6.0	14.8	69.9	-0.7	3.5	47.9	375.7
28 0.0	1.8	13.8	54.5	-1.0	3.2	47.0	324.5
30 0.0	-1.7	12.7	40.2	-1.2	3.0	45.8	274.4
35 0.0	-8.6	10.3	9.2	-1.8	2.4	41.7	156.2
40 0.0	-12.9	7.9	-15.2	-2.2	1.8	36.3	50.9
45 0.0	-15.4	5.6	-33.3	-2.4	1.4	30.1	-38.7
50 0.0	-16.3	3.5	-45.6	-2.5	1.0	23.4	-110.8
55 0.0	-15.9	1.7	-52.6	-2.5	0.7	16.6	-164.7
60 0.0	-14.5	0.0	-54.8	-2.4	0.4	10.0	-200.5
65 0.0	-12.4	-1.3	-53.0	-2.2	0.1	3.9	-219.2
70 0.0	-9.8	-2.4	-47.9	-1.9	-0.1	-1.5	-222.2
75 0.0	-6.9	-3.2	-40.3	-1.5	-0.2	-6.1	-211.8
80 0.0	-4.0	-3.6	-31.0	-1.2	-0.3	-9.7	-190.3
85 0.0	-1.1	-3.8	-20.9	-0.8	-0.4	-12.3	-160.4
90 0.0	1.6	-3.8	-10.6	-0.4	-0.4	-13.9	-124.9
95 0.0	3.9	-3.5	-0.8	-0.1	-0.5	-14.5	-86.5
100 0.0	5.7	-3.0	7.9	0.2	-0.5	-14.3	-47.5
105 0.0	6.9	-2.4	15.2	0.5	-0.4	-13.3	-10.2
110 0.0	7.6	-1.7	20.7	0.7	-0.4	-11.7	23.7
115 0.0	7.7	-0.9	24.1	0.8	-0.3	-9.7	52.7
120 0.0	7.2	-0.1	25.5	0.9	-0.3	-7.5	76.0
125 0.0	6.2	0.6	24.9	0.9	-0.2	-5.2	93.0
130 0.0	4.7	1.2	22.5	0.8	-0.1	-2.9	103.9
135 0.0	2.9	1.7	18.5	0.7	-0.0	-0.9	108.9
140 0.0	0.8	2.1	13.4	0.6	0.0	0.9	108.9
145 0.0	-1.4	2.3	7.4	0.4	0.1	2.2	104.7
150 0.0	-3.7	2.4	1.1	0.2	0.2	3.0	97.5
155 0.0	-5.8	2.3	-5.2	0.1	0.3	3.4	88.7
160 0.0	-7.7	2.0	-11.0	-0.1	0.3	3.4	79.3
165 0.0	-9.3	1.6	-15.9	-0.2	0.4	2.9	70.7
170 0.0	-10.5	1.1	-19.7	-0.3	0.4	2.1	63.8
175 0.0	-11.3	0.6	-22.1	-0.4	0.4	1.1	59.4
180 0.0	-11.5	0.0	-22.9	-0.4	0.4	0.0	57.8

Table 10

Covariances between various quantities computed using the anomaly degree variances of model 4 and $\hat{\sigma}_2(\Delta g, \Delta g) = 7.5\ \text{mgal}^2$ at a height 500 km above the earth ($r = R_e + 500$ km).

	Covariances Between						
ψ	$\Delta g_P, \Delta g_Q$ mgal^2	$\Delta g_P, \ell_Q$ mgal·arc sec	$\Delta g_P, \zeta_Q$ mgal·m	ℓ_P, ℓ_Q arc sec^2	m_P, m_Q arc sec^2	ℓ_P, ζ_Q arc sec·m	ζ_P, ζ_Q m^2
0° 0.0'	64.1	0.0	162.2	4.3	4.3	0.0	648.4
0 30.0	64.0	0.6	162.1	4.3	4.3	1.2	648.3
1 0.0	63.7	1.3	161.8	4.3	4.3	2.5	647.7
1 30.0	63.2	1.9	161.4	4.2	4.3	3.7	646.8
2 0.0	62.5	2.5	160.7	4.2	4.3	4.9	645.5
2 30.0	61.7	3.1	159.9	4.1	4.2	6.2	643.9
3 0.0	60.7	3.7	158.9	4.1	4.2	7.4	642.0
3 30.0	59.5	4.2	157.7	4.0	4.2	8.5	639.7
4 0.0	58.3	4.7	156.4	3.9	4.2	9.7	637.0
5 0.0	55.5	5.7	153.4	3.7	4.1	11.9	630.7
6 0.0	52.4	6.5	149.8	3.6	4.0	14.0	623.2
8 0.0	46.0	7.7	141.5	3.1	3.9	17.9	604.5
10 0.0	39.7	8.5	132.1	2.7	3.7	21.3	581.7
12 0.0	33.7	8.9	121.9	2.3	3.5	24.2	555.2
14 0.0	28.3	9.2	111.3	1.9	3.3	26.6	525.5
16 0.0	23.5	9.2	100.5	1.5	3.1	28.6	493.4
18 0.0	19.1	9.1	89.8	1.2	2.9	30.2	459.2
20 0.0	15.3	8.9	79.2	0.8	2.8	31.3	423.4
22 0.0	11.9	8.7	68.9	0.5	2.6	32.1	386.4
24 0.0	8.9	8.4	58.8	0.3	2.4	32.6	348.8
26 0.0	6.3	8.0	49.2	0.0	2.2	32.8	310.7
28 0.0	3.9	7.6	40.1	-0.2	2.1	32.6	272.7
30 0.0	1.9	7.2	31.4	-0.4	1.9	32.2	235.0
35 0.0	-2.2	6.0	11.9	-0.9	1.6	30.2	143.9
40 0.0	-5.0	4.8	-4.1	-1.2	1.3	27.1	60.4
45 0.0	-6.7	3.5	-16.5	-1.5	1.0	23.1	-12.6
50 0.0	-7.6	2.3	-25.4	-1.6	0.7	18.5	-73.2
55 0.0	-7.8	1.3	-31.0	-1.7	0.5	13.8	-120.2
60 0.0	-7.3	0.3	-33.5	-1.6	0.3	9.0	-153.3
65 0.0	-6.5	-0.5	-33.4	-1.5	0.1	4.5	-172.8
70 0.0	-5.3	-1.2	-31.0	-1.3	0.0	0.4	-179.7
75 0.0	-3.9	-1.7	-27.0	-1.1	-0.1	-3.2	-175.4
80 0.0	-2.5	-2.0	-21.7	-0.9	-0.2	-6.1	-161.7
85 0.0	-1.0	-2.2	-15.6	-0.6	-0.3	-8.3	-140.5
90 0.0	0.3	-2.2	-9.3	-0.4	-0.3	-9.8	-113.9
95 0.0	1.5	-2.0	-3.1	-0.2	-0.3	-10.6	-84.1
100 0.0	2.5	-1.8	2.6	0.1	-0.3	-10.8	-52.9
105 0.0	3.2	-1.4	7.5	0.2	-0.3	-10.3	-22.1
110 0.0	3.6	-1.0	11.3	0.4	-0.3	-9.4	6.7
115 0.0	3.7	-0.6	13.9	0.5	-0.3	-8.2	32.4
120 0.0	3.6	-0.2	15.3	0.5	-0.2	-6.7	54.1
125 0.0	3.1	0.3	15.5	0.6	-0.2	-5.1	71.3
130 0.0	2.5	0.6	14.6	0.5	-0.1	-3.5	83.9
135 0.0	1.6	0.9	12.7	0.5	-0.1	-2.1	92.1
140 0.0	0.6	1.2	10.0	0.4	-0.0	-0.8	96.1
145 0.0	-0.4	1.3	6.7	0.3	0.0	0.3	96.8
150 0.0	-1.5	1.3	3.2	0.2	0.1	1.0	94.9
155 0.0	-2.5	1.3	-0.4	0.1	0.1	1.4	91.3
160 0.0	-3.5	1.1	-3.7	-0.0	0.1	1.6	86.8
165 0.0	-4.2	0.9	-6.5	-0.1	0.2	1.4	82.4
170 0.0	-4.8	0.7	-8.7	-0.2	0.2	1.1	78.6
175 0.0	-5.2	0.3	-10.0	-0.2	0.2	0.6	76.1
180 0.0	-5.3	0.0	-10.5	-0.2	0.2	0.0	75.3

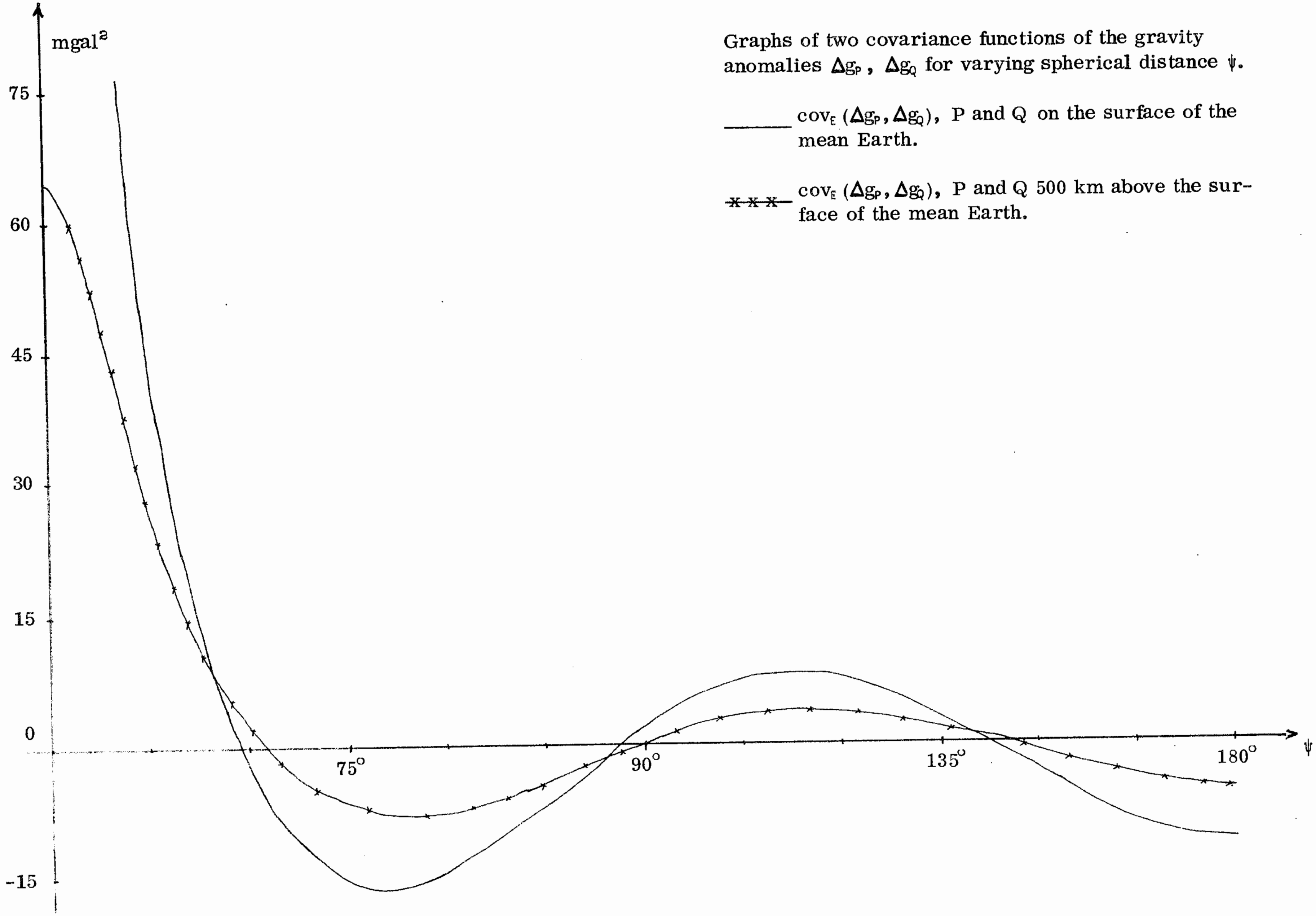


Figure 3

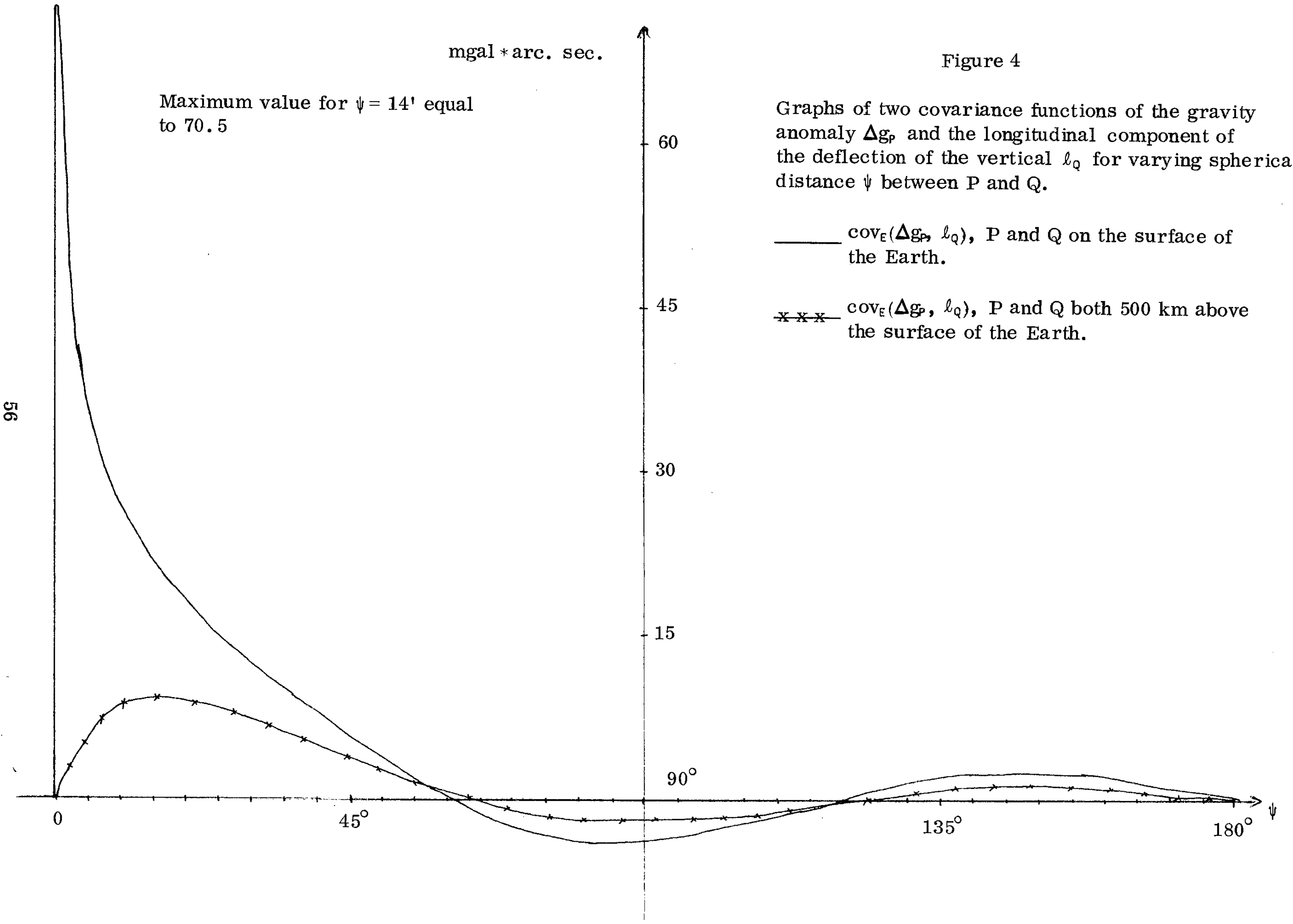
Graphs of two covariance functions of the gravity anomalies Δg_P , Δg_Q for varying spherical distance ψ.

——— $\mathrm{cov}_E(\Delta g_P, \Delta g_Q)$, P and Q on the surface of the mean Earth.

-x-x-x- $\mathrm{cov}_E(\Delta g_P, \Delta g_Q)$, P and Q 500 km above the surface of the mean Earth.

Figure 4

Graphs of two covariance functions of the gravity anomaly Δg_P and the longitudinal component of the deflection of the vertical ℓ_Q for varying spherical distance ψ between P and Q.



Maximum value for $\psi = 14'$ equal to 70.5

——— $\text{cov}_E(\Delta g_P, \ell_Q)$, P and Q on the surface of the Earth.

-x-x-x- $\text{cov}_E(\Delta g_P, \ell_Q)$, P and Q both 500 km above the surface of the Earth.