CURRENT PROBLEMS IN GRAVITY FIELD APPROXIMATION

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Abstract

Approximation methods are treated from a general standpoint. They are all viewed as being linear combinations of certain base functions. The main difference is whether we do not need or need to solve a system of equations to find the functions or the coefficients of the linear combination. In the latter case a main difference is whether we use more functions than data or conversely.

The most important problems associated with the currently used (or proposed) techniques are judged to be the following. Which errors are committed by not using base functions regular at infinity. How can a theory be developed describing the behaviour of the approximations in data clusters or near data holes. Which error is committed by treating gravity data as being observed at the ellipsoid and not at their correct position at the Earth's surface. How can we explain the large differences between potential coefficient solutions. Which improvements can be obtained by using a correct set of harmonicity for the approximation instead of the set outside a sphere, for example. Is it possible to find models for the statistical distribution for the estimates of errors of predicted gravity field dependent quantities or related parameters.

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1. Introduction

The theory of the solution of the geodetic boundary value problem is well developed. It involves many idealizations, especially with respect to the kind of boundary data to be used. The actual construction of solutions to the boundary value problem is the object of the theory of gravity field approximation. This theory includes many more types of problems, e.g. also problems where the data are not at the boundary, but inside or outside this boundary.

Approximation theory as such has had a limited, but important, use. Existence and uniqueness of solutions have been proved. Also in a few cases it has been possible to prove convergence theorems, which assures us that the results obtained using more and more (regularly distributed) data in the end will bring us to the correct result.

It would naturally be nice to use only methods for which this convergence property is proved. But the practically working geodesist is more interested in other problems, which are related to the actual data distribution and kind of data, and to the costs (in terms of both programming complexity and computational economy). Furthermore, it is sometimes necessary to have one solution instead of having the best or the optimal once in the future.

However, the requirements of having a cm-geoid, the need of studying small time-varying changes, the challenge of handling the immense amount of data from possible future satellite missions and the handling of the noisy gravity gradiometer data, puts forward new demands of an increased precision:

- How may old methods be improved ?
- Where do we (consciously) make approximations or shortcuts, the consequences (errors) of which we hope are small, but we do not really know ?
- How may we in the best possible manner exploit our a-priori knowledge (mathematical and physical) of the Earth's gravity potential and its source, the density distribution ?

In this paper I will try to give an overview over the kind of problems deemed important by those who do construct approximations to the gravity field. Naturally, I have not been able to conduct an investigation questioning everybody in the field. However, at two recent occasions (a workshop at the Ohio State University on gravity field determination in the fall of 1983 and at the Beijing International Summer School on Local Gravity Field Approximation in 1984), sessions were held in order to "collect" such problems. The colleagues who contributed to these sessions are hereby gratefully acknowledged, and I apologize to all who's favorite problem I have not mentioned.

In the following section 2 I will give a brief introduction to the methods used for the construction of approximations. They are all viewed as methods aiming at constructing an approximation to the anomalous gravity potential (T) as a linear combination of certain base functions. The main difference is whether we do not need or need to solve a system of linear equations in order to find the coefficients or the functions. In the latter case the main difference is whether we have more functions than data or conversely. In section 3 I deal with the problems associated with the selection of the functions, and in section 4 the problems associated with the selection of the data is treated. Finally in section 5 problems of treating data errors or associated parameters and of obtaining meaningful error estimates are discussed.

2. Approximation methods

We will here regard all methods, which give an approximation ($\tilde{T}$) to the anomalous potential as approximation methods. This also includes the integration techniques, such as Stokes' integral.

In the numerical implementation they are represented on the form

$$\tilde{T}(P) = \sum_{i=1}^{m} a_i \, f_i(P) \; . \qquad (1)$$

Linear functionals of T, L(T), are obtained by applying L on each of the functions $f_i(P)$. If $f_i(P)$ is Stokes' function and a_i the gravity anomaly in P_i, then

$$f_i(P) = S(\psi_{PP_i}, r) \; ,$$

where ψ_{PP_i} is the spherical distance between P and P_i and r is the radial distance of P from the origin. If deflections of the vertical are needed, the associated linear functionals (derivatives in northern and eastern direction, respectively) are applied on each $f_i(P)$, and Vening-Meinesz's function is obtained. f_i may be the solid spherical harmonics of maximal degree N, i.e. $m = (N+1)^2$. The constants a_i may then be determined in several ways, e.g. by a spherical harmonic analysis of gravity data or by a least-squares technique, using satellite orbit perturbations, or a combination of both, see (Rapp, 1984). Least-squares techniques may also be used with $f_i(P)$ being other functions like potentials of point masses. Such techniques requires a set of equations to be solved with number of unknowns equal to the number of base functions.

The functions $f_i(P)$ may also be selected so that the approximation $\tilde{T}(P)$ agrees exactly with given data (if they have no error), and so that it has the least possible norm in a given space. This is the method of minimum norm collocation. Here the constants a_i are determined by solving a system of equations with dimension equal to the number of observations. The functions $f_i(P)$ are the Riesz representers of the observation functionals L_i.

We may then divide the methods in 3 groups:

(A) the coefficients are known - select f_i,

(B) the functions $f_i(P)$ are selected, totally less than or equal to the number of observations (n). The coefficients are determined by a least-squares principle,

(C) a Hilbert space of dimension m, larger than the number of observations is selected. A requirement of minimum norm and agreement with the data delivers the functions $f_i(P)$.

For all methods it is possible to take into account some kind of information concerning data errors, and error estimates of computed quantities may also be obtained, at least to a certain extend.

3. Choice of base functions or of minimum norm principle

3.0. Mathematical principles

Since T is a solution to the Laplace-equation for which many individual solutions are known, it is natural first to look for solutions $\tilde{T}$ spanned by such functions. A further property is that T is regular at infinity, or at least that the zero and first order coefficients in the spherical harmonic expansion all are very small.

The use of harmonic functions has the drawback, that the equations to be solved for in methods B and C in general will have a full matrix. The use of non-harmonic functions, e.g. finite elements which leads to sparse matrices has been investigated by Meissl (1981). Several problems associated with the use of finite elements are not yet solved.

The condition of regularity at infinity is not fulfilled e.g. if the $f_i(P)$ are potentials of point masses. The use of corresponding constraints on each mass has been proposed, giving one or, more extra equations for each f_i. A simple method may be to subtract out the zero and first order term (r' the distance of P_j from the origin)

$$f_j(P) = \frac{1}{r} \sum_{i=2}^{\infty} \left(\frac{r'}{r}\right)^i \cdot P_i(\cos(\psi_{PP_j})) =$$

$$= \frac{1}{\sqrt{r^2 - 2rr'\cos\psi_{PP_j} + (r')^2}} - \frac{1}{r} - \frac{r'}{r^2}\cos\psi_{PP_j} \quad ,$$

a possibility which seems worthwhile investigating.

In a recent geoid determination for Finland (Vermeer, 1984) point mass potentials were used. The non-fulfilment of the regularity condition could be the reason for the systematic error (a tilt) in the determined geoid. Also Bjerhammars "Dirac"-approach uses base functions with non-zero zero and first order coefficients, see Tscherning (1983).

This point illustrates our implicit use of an a-priori knowledge about the gravity field: We have selected our geodetic coordinate system so that these first coefficients are so small that they for most practical purposes may be regarded as being equal to zero. At least we should constrain our solutions so that we do not arrive at a solution with an unrealistically large zero or first order coefficient This leads to an important general problem treated several times in the following, namely how do we take advantage of other types of a-priori information ?

3.1. Method A: f_i given implicitly by the data

The method uses known coefficients, e.g. mean gravity anomalies or mean geoid undulations. Earlier there were only little discussion of what should be used: Stokes, Hotines or Molodensky's functions, contingently with modifications according to the data collection (spherical cap) area used. Recently Wenzel (1982) has proposed to use optimized kernels, taking into account the degree-variance distribution of T. Also Rummel (1982) has improved the technique, so that point data, and not necessarily mean values need to be used. Extensions of the methods to be used with inhomogeneous data has been proposed e.g. by Sjöberg (1979). There are several problems associated with the general use of these extended methods.

3.2. Method B: f_i predetermined, $m \leq n$

We know from the solution of 1-dimensional approximation problems, that we have to be specially careful in this situation. But the problems with usual polynomial approximation do not seem to discourage geodesists for trying similar techniques in practice. The coefficients are determined so that the differences $L_i(\tilde{T}) - L_i(T)$ fulfils a mean square minimum principle, leading to the solution of a system of equations with m unknowns.

For global applications the base functions must be selected, so that the matrix either may be inverted analytically (it is a diagonal matrix) or that it may be solved very fast with an iterative technique. In both cases the usual spherical harmonics are used. Data are treated in spherical approximation, but ellipsoidal harmonics and ellipsoidal approximation could easily have been used avoiding errors discussed in Rapp (1984).

In order to get a unit matrix (i.e. so that a harmonic analysis may be used following Colombo (1979)) a complete data coverage of the Earths surface with the same kind of data is required. As a consequence of this are areas with no data filled with zero values or values derived from a high order field.

If several data source are used, but still containing quite regularly distributed data, then a diagonal dominant system of equations must be solved. This method has been used lately by Wenzel (1985) who went up to N = 200, where earlier solutions GEM10C (Lerch et al., 1981) and OSU78, OSU81, (Rapp, 1978, 1981) only went up to N = 180. A cut-off seems reasonable when the error estimates of the individual coefficients becomes larger than the coefficients. But what if the error estimates are too pessimistic or too optimistic? At least for Scandinavia not much is gained by going from N = 180 to N = 200 using the solution GPM2 of Wenzel (1985), see Table 1. On the other hand, the solution seems to be slightly better than Rapp's solutions, not only in Scandinavia, but also in other areas where I have tested the coefficients. A possible explanation for this will be given in section 4.3. (GEM10C seems not to have reliable coefficients for high degrees, see Tscherning (1985b)).

When using data like a global and regularly distributed set of mean gravity anomalies, the solution will also satisfy a minimum norm principle, where the norm corresponds quite closely to the usual L^2-norm. This is not the case if the observations are satellite orbit perturbations. The norm is also in this case inherited from the variance-covariance matrix of the observation errors, but the observations are in reality physically strongly correlated. In some way the norm will depend on how many observations of different kinds are used. This is probably the reason why the solutions have to be regularized, by requiring simultaneously a weighted square-sum of the coefficients to be minimized, see Lerch et al. (1977). (The method is the sometimes called least-squares collocation, but it has nothing to do with this method).

The use of norms inherited from the variance-covariance matrices of the observations are one of the possible reasons for the large differences between coefficient sets. In general we need to understand better the impact of using adjustment techniques. Also the differences which seem to be related mainly to the change in the number of base functions need to be understood better. An interesting explanation is given in Remmer (1984).

The same types of problems may occur in local applications, using e.g. point masses or harmonic splines as base functions. Again their dependency is determined partly from the data distribution and partly from the weights associated with the data. Probably by intuition the data used have mainly been regularly distributed in numerical tests, so that the quantity minimalized resembles much a local L^2-norm. It would be interesting to know what happens in data clusters or close to holes. It should not be difficult to investigate this numerically, but how to express this in general mathematical terms ?

3.3. Method C: f_i determined from a minimum principle

The question is then which principle, which norm ? The only mathematical condition for a unique solution is really that the Riesz representers for the observation functionals exist and are linear independent, because they are the base functions f_i. The condition for existence is that they have finite norm in the space.

Since the data we use might include gravity gradients, then at least their associated base functions should have finite norm. However, the functionals do all have finite norm for points in space. The problems occur right at the boundary.

In general it is so, that the maximal value of a quantity will always be smaller than the norm of the functional times the norm of T. So the right kind of norms are in my opinion these which keep a reasonable ratio between the norms of various types of functionals (related to geoid undulations, gravity anomalies and gradients). This necessitates the use of norms minimalizing the 3'rd order derivatives at the boundary or the introduction of a Bjerhammar sphere making the norms of linear functionals associated with derivatives of an arbitrary order bounded. The use of multipoles, located at the boundary, are examples of functions which do not behave well in the above sense. (The use of such functions are discussed e.g. by Hofmann-Wellenhof (1984) and Freden (1983).)

Alternatively, a least squares principle could be used, which has the purpose of minimalizing the mean square error of prediction, $M(\tilde{T}-T)^2$ for T only! It will not be minimalized for other functions! An inherent contradiction here is that this implicitly requires that we already know T. This has been used as an argument against using the principle. And it is right that we do not know T completely, but we know very much about T! At least so much that what we do not know does not seem to harm the solution very much, the main problem being in fact the computation of error estimates, which is a problem for all methods.

It is well known that the use of the principle leads to the least squares collocation method with $f_i(P) = cov(T(P), L_i(T))$.

Most frequently a "M" operator is used, which takes the mean over all possible rotations of the data configurations (see Moritz (1980)). This implies that a set of harmonicity outside a sphere (a Bjerhammar sphere) must be used. This causes problems, exactly because for example the ratio between the norm of a geoid and a gravity anomaly functional will vary quite a lot depending on whether a point near the sphere or at a mountain top is considered, see Tscherning (1985a,Fig. 9).

In the "mixed collocation" method (Sansó and Tscherning, 1982), it is proposed to add blocks, representing the topography and mass anomalies down to the Bjerhammar sphere. The norm used for the (potentials of the) blocks should be of the same kind as the one used for the masses inside the Bjerhammar sphere. The problem of finding (constructing) such norms is not yet solved satisfactory. Furthermore we do not know whether the mixed-collocation set-up will give any improvements in practice. But at least it gives a more satisfactory mathematical model.

The main problem associated with the use of "C" type methods is that a system of equations with as many unknowns as the number of used observations (n) has to be solved. If data are given in regular patterns - and we stick to rotational invariant kernels, then it might be possible to set-up collocation solutions to global problems, see Bose et all. (1983) or to local problems with large amounts of data, see Jekeli (1985). The problem with large data in a local area may also be solved using data selection procedures (see next chapter) or by constructing solutions for overlapping areas. But it is still an unsolved problem of how to fit these local solutions together in the best possible manner. Errors up to 0.2 m has occurred between geoid surfaces in adjacent areas even when using $\frac{1}{2}^{\circ}$ overlapping areas, see Tscherning (1985 , Fig. 4).

Using the empirical covariance function associated with least squares collocation, we should know the behaviour of a solution in data holes. The covariance function should tend to zero in a distance characteristic for the area in question. The behaviour in clusters should also be under control, as long as the condition number for the normal equations used to find the coefficients a_i does not become too unfavourable for the actual computer. However, should a numerical singularity occur, the reason is probably that an observation is being used two times, or that much more data than necessary is being used.

Let us finally touch one question. What happens if harmonic base functions are treated as if they were zero outside a certain distance ? One single experiment in Scandinavia, where the covariances were put equal to zero in a 1^o distance, gave a change in predicted quantities below their estimated errors. We need more investigations related to the use of such "tricks". A lot of computer-money could be saved !

4. Data selection

4.1. General considerations

For all norm minimalization techniques it is so that the maximal value of various types of errors is proportional to the norm of T. In other words: the smoother the function, the better.

We should use the best possible reference field. Not only because of the smoothing, but also because we have a better starting point for the various types of linearizations made, for the up- or downward continuations of data etc., see Rapp (1984). The use of a high degree and order reference field also takes into account corresponding degrees and orders (minus 2 - 3) of the topography.

A further smoothing, by subtracting the topographic effects may then be limited to the "residual" topography, see Forsberg and Tscherning (1981).

The technique has been successfully used in local applications, but it should also improve global solutions. If adjustment methods are used, it should make data errors more easy to detect. For local applications it makes data more correlated, see Forsberg (1984), so that less data is needed in order to get the same quality of a result.

This also points at the problem of how to take advantage of geophysical data. Should our Hilbert spaces include functions defined inside the Earth ? A possible solution is indicated in the "mixed-collocation" method, but more research is needed.

On the other hand, we should in general not select more data than necessary and as little data as possible. The last condition will be fulfilled if we have smoothed the gravity field as much as possible using topographic and density data. Supposing that we have done this, then there are different strategies for data compression, to be discussed in the next section.

4.2. Data selection strategies

Suppose we have selected a class of functions (Method A, B) or a minimum principle (C). The starting point could then be a set of regularly distributed observations and contingently associated base functions. Values could then be predicted in all other points and the residuals inspected. (At this step large outliers due to gross errors could be removed). Then new values associated with the largest residuals could be added, with corresponding new base functions. This strategy has been adopted by several investigators , see Barthelmes and Kautzleben (1983), Goad et al. (1984), Tscherning (1985).

Another method used in order to reduce the computational effort is to use mean values. This is a dangerous method, but several investigators think that this is the only right thing to do, see Weber (1984,p.4). Mean values are generally computed as straightforward means of all data in the area. Since data are given in different heights, it is also difficult to know to which height the mean value should be associated: the mean height or the height of the heighest point ? Also in the case where there maybe are many values available, an erroneous result may be obtained. Such a situation occurs in mountains, where most gravity data are collected in the valleys.

I have tried to use both mean values and point values as close as possible to the middle of the block when predicting deflections of the vertical in Finland. The result (Table 2) showed that the use of point data was slightly better in this case. Note, that Finland has a very regular and dense gravity observation distribution.

4.3. Reduced data or data at the correct height ?

The use of spherical harmonic analysis requires that data are on the same sphere. This is only possible, not involving a hypothesis concerning gravity gradients inside the masses, if an external sphere is used, cf. Rapp

(1984). Alternatively spherical harmonic coefficients may be determined using adjustment techniques with data in the correct heights, i.e. at the Earths surface, see Wenzel (1985). In Fig. 1 is shown how two spherical harmonic coefficient sets OSU 78 and 81 (Rapp, 1978,1981) seem to have a height dependent error for geoid undulations, while the GPM2 solution of Wenzel (1985) does not seem to have such an error. This indicates that it is worthwhile to have the data at their right positions !

In order to verify this, the same spherical harmonic coefficient sets were used to compute point gravity anomalies in Scandinavia. The residuals (errors) show here (cf. Table 3) for three sets a clear dependence of the height ! However this error could be caused by errors in the mean values used to derive the coefficients !

4.4. Original data, or transformed data ?

The problems associated with the use of gravity data at their proper height is a general one. Data are not always associated with points we would like it to be associated, for example so that equations to be solved had elements with a regular structure. Or our equations require gravity data, but we only have geoid heights or conversely.

In some cases we see, that collocation is used first in order to predict the quantities needed in a regular grid for the later use in an integral formula (Engelis et al. 1984), (Weber, 1984). This seems a little ackward, since the main work is already done in the collocation step. Here methods proposed several years ago by Lachapelle (1977) seems more reasonable.

In order to produce a N = 360 spherical harmonic expansion, prediction of mean values of smaller blocks (30') will be needed. Is this really worthwhile, since the combination of a N = 180 solution with a local solution should satisfy most requirements ? It is probably worthwhile, since many users get a very clear and unsophisticated product, when they get a set of spherical harmonic coefficients. But for users wanting the highest precision and reliability a combination of local and global solutions should be preferred, because these solutions may be easier updated and improved.

5. Errors and parameters

It is nearly as worthwhile to have an error estimate of a quantity as having the quantity itself. The basis for this is that we know the error estimates of the observed quantities. However such errors may be of systematic character, e.g. represented by an unknown parameter. Such unexpected systematic errors are very annoying. The experience with sunspot related errors in Doppler derived geoid heights and biased longitudes detected in the course of constructing a geoid for Scandinavia is told in Tscherning (1985a) and will not be repeated here. However it repeats the well-known truth that different data sources should be used to compute the same quantities in order to detect such errors and in order to verify the variances associated with the various data sources. Most frequently the error distributions will have longer tails than could be expected from the internally derived statistics.

The errors of predicted quantities do not only depend on the errors of the observations, but also on their distribution and kind. And then on their physical correlation - the actual variation of the gravity field.

If we make a histogram of gravity anomalies for a larger area, then we realize that they are close to being normally distributed. Error variances of other quantities predicted using a linear prediction formula, should therefore be χ^2-distributed ? Actual comparisons of "true" errors and error estimates (the ratio of which should have a t-distribution) indicates that this is correct, but a theory for the distribution of prediction errors is urgently needed. See Table 4 for examples of the distribution of the ratio between the actual error and the error estimate.

The errors in the observations are accounted for in methods of type (A) by making the functions dependent on a supposed regular noise pattern, see e.g. Sjöberg (1979). For method (B) it is normally the minimalization of the noise variance which leads to the unique solution. For method (C) a weighted sum of the norm of the approximation $\tilde{T}$ and of the noise variance is minimalized. How this weight ratio should be fixed is still a problem not solved satisfactory. If an empirical covariance function is used, the factor is usually equal to 1, and this seems to give good results. But why do we consider the results good ? (A not too small and not too large damping seems to take place).

The problem become even more complicated when parameters have to be estimated. Also the interpretation of the variances associated with the estimates is difficult. However, the numbers which comes out of computer programs seems to be quite reasonable if reasonable parameters are estimated from reasonable data. Reasonable results should not be expected when estimating a 7-parameter datum transformation from only geoid data in a limited region, see e.g. Ezeigbo (1984). Here is also an area where more investigations are urgently needed.

6. Conclusions

Many problems associated with gravity field approximation remain to be solved. Many methods are available with all their advantages and disadvantages. The methods should be refined, tested and compared, tasks which has been assigned to a number of IAG Special Study Group, specifically.

Many methods, or improvements of methods have been proposed, but few have been implemented for production purposes. The methods which have been tested have generally not been tested on a systematic basis, using many data types, error patterns or varying types of gravity field variations. The extremely unrealistic Molodensky Mountain Model (MMM) seems still to represent the ultimate testing object for some geodesists.

We should, however, remember that numerical tests do not prove anything. Unfortunately several times rapid conclusions have been made, which easily could be disproved by a counter example, see e.g. Tscherning (1983).

A better mathematical foundation would solve this problem. But mathematicians seem to be too interested in more abstract problems. However, even when mathematical proofs exist we see geodesists try to prove numerically that it is not true. A good example is least squares collocation, which must give the least mean square error. However methods have been found which give a smaller error than least squares collocation in some cases. Are geodesists born pessimists or optimists, we could ask after this !

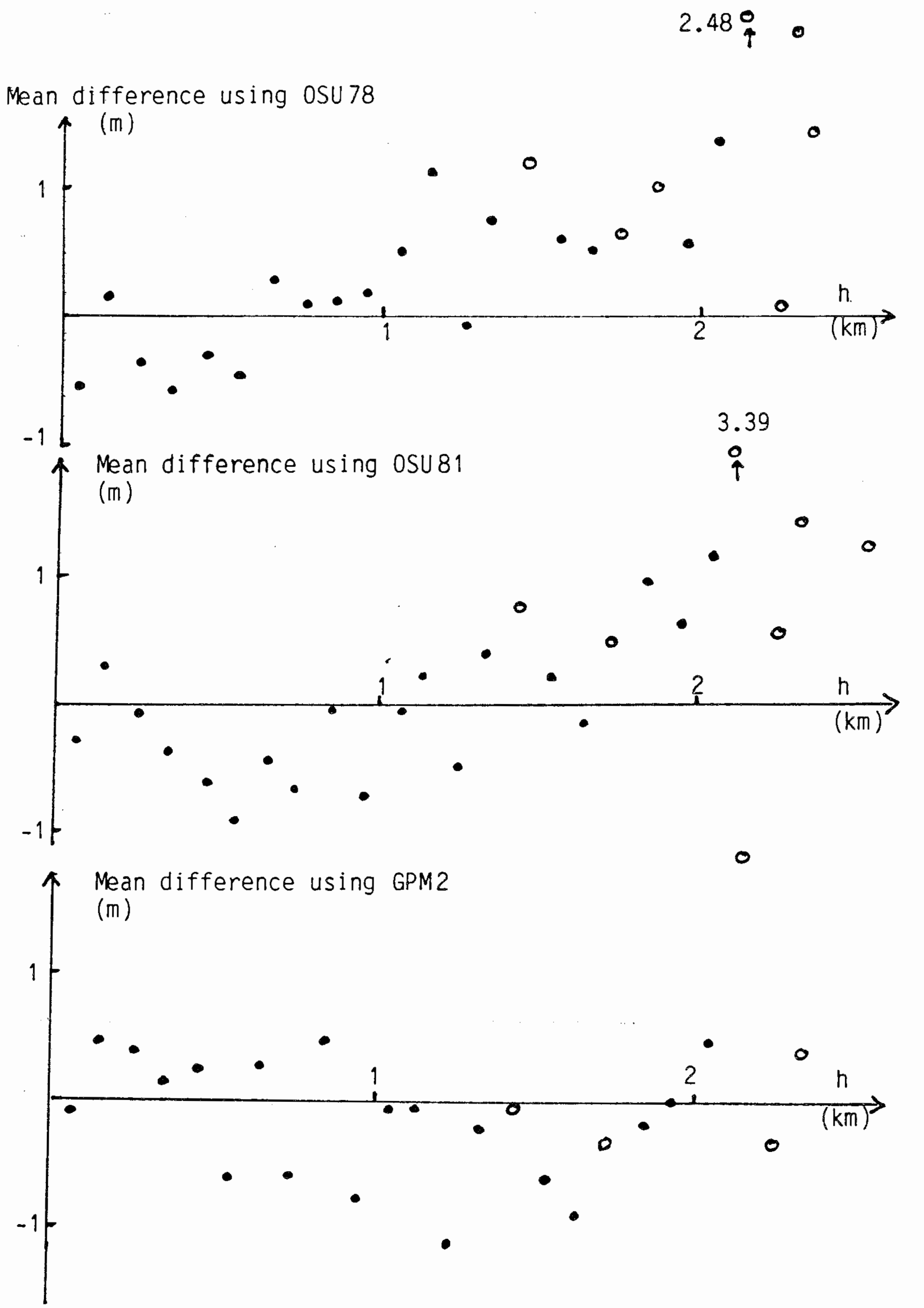

Fig. 1 Mean values of differences between Doppler derived geoid undulations and values computed from OSU78, OSU81 and GPM2 (N = 180), sampled according to station height (h) in 100m intervals. The Doppler derived geoid undulations have been corrected for ionospheric effect using results in Tscherning and Goad (1985). Totally 792, 790 and 806 values used for OSU78, OSU81 and GPM2, respectively because only values with absolute differences smaller than 4m were used. •-more than 10 values in interval, o-less than 10 values.

Table 1. Mean values and standard deviations of differences between point free-air gravity values and values computed from various sets of potential coefficients. Totally 1605 gravity values used, spaces as close as possible to the knots of a 30' by 30' grid covering Scandinavia and Finland. All units mgal.

	Potential coefficient set						
Name	NONE	GEM10C	OSU 78	OSU 81	GPM2	GPM2	GPM2
Maximal degree	0	180	180	180	120	180	200
Mean	0.55	0.74	0.31	-0.87	0.11	0.01	-0.02
St.deviation	30.15	27.19	22.23	24.54	24.30	22.18	22.09

Table 2. Results from prediction of deflections of the vertical in Finland from point gravity data as close a possible to the points in a 10'*20' grid and from mean gravity anomalies over 10'*20' blocks with the grid points as mid-points. A longitude bias of 1.2" has been removed from the prime vertical components. The predictions were done separately for 2 areas south and north of latitude 65 degrees. Least squares collocation used.

Southern part, totally 240 deflection points.

	Meridian component			Prime vertical component		
		Difference observed - predicted			Difference observed - predicted	
	Original data	Predicted from point gravity	Predicted from mean gravity	Original data	Predicted from point gravity	Predicted from mean gravity
	"	"	"	"	"	"
Mean	-1.37	-0.16	-0.15	-0.08	0.29	0.24
Stdv.	2.49	0.77	0.79	2.24	0.93	0.93

Northern part, totally 81 deflection points.

	Meridian component			Prime vertical component		
		Difference observed - predicted			Difference observed - predicted	
	Original data	Predicted from point gravity	Predicted from mean gravity	Original data	Predicted from point gravity	Predicted from mean gravity
	"	"	"	"	"	"
Mean	-2.25	0.02	-0.04	2.12	0.18	-0.08
Stdv.	2.41	0.95	1.01	3.65	0.98	1.01

Table 3. Differences between observed and computed point free-air gravity anomalies sampled according to the height of the observation station. Results shown for 3 different sets of potential coefficients.Same data as used in Table 1.

h is mid point of sampling interval (100 m size). n is number of observations in the interval.

		Potential coefficients used:					
		GEM10C		OSU 81		GPM2 (N=180)	
h	n	mean mgal	stdv. mgal	mean mgal	stdv. mgal	mean mgal	stdv. mgal
-50	55	-12.4	18.4	-9.3	19.2	-4.7	15.9
50	702	-4.5	24.9	-2.9	24.0	-3.9	20.4
150	273	-3.4	17.2	-4.3	18.9	-2.2	15.5
250	215	0.2	17.4	-3.7	16.8	-0.9	14.4
350	99	-2.3	22.4	-6.7	23.1	-3.4	21.3
450	73	-2.1	26.0	-4.0	25.4	-4.1	23.7
550	44	4.8	22.1	2.3	24.6	1.6	22.4
650	25	13.8	25.6	7.3	24.5	7.4	25.5
750	17	10.0	24.7	-1.3	30.4	9.5	29.9
850	18	22.6	21.2	18.2	21.8	20.1	17.6
950	13	31.1	25.4	25.7	18.8	28.2	19.8
1050	11	45.8	22.0	25.1	21.2	25.2	19.1
1150	19	48.9	17.3	35.7	17.7	38.9	18.4
1250	14	63.5	17.7	46.2	21.2	46.8	18.5
1350	9	62.8	24.9	56.7	27.7	44.1	27.8
1450	3	95.0	9.0	72.2	7.5	64.7	14.4
1550	6	96.1	13.4	56.2	17.9	71.1	17.1
1650	5	99.2	17.1	48.2	13.7	57.7	13.3
1750	2	111.1	0.9	70.9	23.8	74.1	4.0

Table 4. Distribution of numerical values |t| of errors of prediction (observed - predicted) divided by the estimated error from least squares collocation using various models for the covariance function. Free-air gravity and meridian component of the deflection of the vertical used from the White Sands test area (New Mexico), see Forsberg and Tscherning (1981). The Tscherning & Rapp(1974) degree-variance model used with varying depth (D) to the Bjerhammar sphere and a varying number of terms (n) set equal to zero. Stdv. is the standard deviation of observed minus predicted values.

Covariance fct. param.		Distribution of |t| for 54 predicted meridian components of defl. vert.				113 predicted free-air gravity anomalies.			
D km	n	Stdv. "	|t|<1 %	1<|t|< 2 %	|t| > 2 %	stdv. mgal	|t|<1 %	1<|t|< 2 %	|t|> 2 %
-1.1	200	1.56	91	7	2				
0.0	200	1.40	87	11	2	4.42	88	10	2
0.0	220	1.40	87	11	2	4.42	88	9	3
1.25	200	1.35	74	22	4	4.18	80	10	10
1.25	220	1.36	74	22	4	4.18	81	9	10
2.50	200	1.34	57	30	13	4.03	72	13	15
* 2.50	220	1.34	60	28	12	4.03	72	14	14
Values from |t|-distribution			68	26	6		68	26	6

* these values give best fit to empirically estimated covariances.
Note, that for larger values of D we have stronger correlations.

References:

Barthelmes, F. and H.Kautzleben: A new method of modelling the gravity field of the Earth by Point Masses. Proc. of the International Association of Geodesy (IAG) Symposia, Vol. 1, pp. 442-448, 1983.

Bose, S.C., G.E.Thobe, J.T.Kouba and R.E.Mortensen: Optimal Global Gravity Field estimation. Proc. of the International Association of Geodesy Symposia, Vol. 1, pp. 449-482, 1983.

Colombo, O.: Optimal estimation from data regularly sampled on a sphere with applications in geodesy. Reports of the Dep. of Geodetic Science, No. 291, The Ohio State University, Columbus, 1979.

Engelis, T., R.H.Rapp and C.C.Tscherning: The Precise Computation of Geoid Undulation Differences with Comparison to Results obtained from the Global Positioning System. Geophys. Res. Letters, Vol. 11, No. 9, pp. 821-824, 1984.

Ezeigbo,C.U.: The problem of Local Geoid and Datum Determination by Means of Least Squares Collocation. Boll. di Geodesia e Sci. Aff., Vol. XLIII, N. 3, pp. 245-272, 1984.
1984.

Forsberg, R.: A Study of Terrain Reductions, Density Anomalies and Geophysical Inversion Methods in Gravity Field Modelling. Reports of the Department of Geodetic Science and Surveying, No. 355, The Ohio State University, Columbus, Ohio, 1984.

Forsberg, R. and C.C.Tscherning: The use of Height Data in Gravity Field Approximation by Collocation. J.Geophys.Res., Vol. 86, No. B9, pp. 7843-7854, 1981.

Freeden,W.: Least Squares Approximation by Linear Combinations of (Multi)poles. Reports of the Dep. of Geodetic Sciences and Surveying, No. 344, The Ohio State University, Columbus, 1983.

Goad, C.C., C.C.Tscherning and M.M.Chin: Gravity Empirical Covariance values for the Continental United States. J.Geophys.Res., Vol. 89, No. B9, pp. 7962-7968, 1984.

Hofmann-Wellenhof, B.: Multipoles and Finite Elements for the Representation of the Potential. Manuscripta Geodaetica, Vol. 9, no. 1/2, pp. 93-109, 1984.

Jekeli, C.: On Optimal Estimation of Gravity from Gravity Gradients at Aircraft Altitude. Rev. of Geophysics, (in print), 1985.

Lachapelle, G.: Estimation of Disturbing Potential Components using a Combined Integral Formulae and Collocation Approach, Manuscripta Geodaetica, Vol. 2, pp. 233-262, 1977.

Lerch, F.J., S.M.Klosko, R.E.Laubscher and C.A.Wagner: Gravity Model Improvement using GEOS-3 (GEM 9 & 10), Goddard Spece Flight Center, X-921-77-246, 1977.

Lerch, F.J., B.Putney, S.Klosko and C.Wagner: Goddard Earth Models for Oceanographic Applications (GEM 10B and 10C). Marine Geodesy, Vol. 5, pp. 145-187, 1981.

Meissl, P.: The use of finite elements in physical geodesy. Reports of the Dep. of Geodetic Science and Surveying, No. 313, The Ohio State University, Columbus, 1981.

Moritz, H.: Advanced Physical Geodesy. H.Wichmann Verlag, Karlsruhe, 1980.

Rapp, R.H.: A Global 1 deg. x 1 deg. Anomaly Field Combining Satellite, Geos-3 Altimeter and Terrestrial Data. Dep. of Geodetic Science Report No. 278, The Ohio State University, Columbus, Ohio, 1978.

Rapp, R.H.: The Earth's gravity field to degree and order 180 using SEASAT altimeter data, terrestrial gravity data, and other data. Reports of the Department of Geodetic Science and Surveying No. 322, The Ohio State University, Columbus, Ohio 1981.

Rapp, R.H.: The determination of high degree potential coefficient expansions from the combination of satellite and terrestrial gravity information. Reports of the Dep. of Geodetic Science and Surveying, No. 361, The Ohio State University, Columbus, 1984.

Remmer, O.: A Theorem in Physical Geodesy and some Consequences. Manuscripta Geodaetica, Vol. 9, pp. 1-19, 1984.

Rummel, R.: Gravity parameter estimation from large data sets using stabilized integral formulas and a numerical integration base on discrete point data. Reports of the Dep. of Geodetic Science and Surveying, No. 339, The Ohio State University, 1982.

Sanso, F. and C.C.Tscherning: Mixed Collocation: A proposal. Quaterniones Geodasiae, Vol. 3, no. 1, pp. 1-15, 1982.

Sjøberg, L.: Integral formulas for heterogeneous data in physical geodesy. Bulletin Geodesique, Vol. 53, no. 4, pp. 297-316, 1979.

Tscherning, C.C.: On the Use and Abuse of Molodensky's Mountain. in: K.P.Schwarz and G.Lachapelle (Ed.): Geodesy in Transition, pp. 133-147, The University of Calgary, Division of Surveying Engineering Publication 60002, 1983.

Tscherning, C.C.: Geoid Modelling using Collocation in Scandinavia and Greenland. Marine Geodesy, Vol. 9, no.1, pp. 1-16, 1985.

Tscherning, C.C.: Local Approximation of the Gravity Potential by Least Squares Collocation. In: K.P.Schwarz (Ed.): Proceedings of the International Summer School on Local Gravity Field Approximation, Beijing, China, Aug. 21 - Sept. 4, 1984. Publ. 60003, Univ. of Calgary, Calgary, Canada, pp. 277-362, 1985a.

Tscherning, C.C.: On the Long-Wavelength Correlation between Gravity and Topography. Proceedings 5th Int. Symp. "Geodesy and Physics of the Earth", G.D.R. Magdeburg, Sept. 23-29, 1984. Part II, pp. 134-142, Veroeffentlichungen des Zentralinstituts fuer Physik der Erde, Nr. 81, Potsdam 1985b.

Tscherning, C.C. and C.C.Goad: Correlation between Timedependent Variations of Doppler-Determined Heights and Sunspot Numbers. J.Geophys.Res.,Vol. 90, No. B6, pp. 4589-4596, 1985.

Tscherning, C.C. and R.H.Rapp: Closed Covariance Expressions for Gravity Anomalies, Geoid Undulations, and Deflections of the Vertical Implied by Anomaly Degree-Variance Models. Reports of the Department of Geodetic Science No. 208, The Ohio State University, Columbus, Ohio, 1974.

Vermeer, M.: Geoid Studies on Finland and the Baltic. Reports of the Finnish Geodetic Institute 84:3, Helsinki 1984.

Weber, G.: Hochaufloesende mittlere Freiluftsanomalien und gravimetrische Lotabweichungen fuer Europa. Wiss. Arb. d. Fachrichtung Vermessungswesen der Universitaet Hannover, Nr. 135, 1984.

Wenzel, H.-G.: Geoid computation by least squares spectral combination using integral kernels. Proceedings of the General Meeting of the IAG, pp. 438-453, 1982.

Wenzel,H.-G.: Hochaufloesende Kugelfunktionsmodelle fuer das Gravitationspotential der Erde. Wiss. Arb. Fachrichtung Vermessungswesen der Universitaet Hannover, (in print), 1985.