C.C. Tscherning
Department of Geodetic Science
Ohio State University
Columbus, Ohio 43210

Problems and Results in Least Squares Collocation

December, 1973

Abstract

The main problem in the application of least squares collocation for the determination of approximations to the anomalous potential (T) of the Earth is (1) the incomplete knowledge of the covariance functions related to T, (2) the representation of the data (e.g. mean gravity anomalies) by not too complicated linear functionals applied on T and (3) the extensive computations implied by the collocation method. The problems have to a certain extent been solved for collocation problems, where potential coefficients, height anomalies, gravity anomalies and deflections of the vertical are used. The covariance functions are represented by expressions derived from a reproducing kernel. The functionals corresponding to the mentioned type of data (except mean gravity anomalies) are represented by sufficient accuracy using spherical approximation. (Mean gravity anomalies are represented by point gravity anomalies lying in a certain distance above the surface of the Earth.) When potential coefficients and $1^\circ \times 1^\circ$ mean gravity anomalies are used, collocation can be used to determine an improved reference potential. Point anomalies lying 1° apart may in this case be regarded as independent, i.e. the amount of computation can be reduced considerably. Results of computational experiments, showing that T can be approximated very well using collocation, are presented.

Introduction

There is a long tradition and much practical experience in the use of least squares methods in geodetic science. We use least squares adjustment in geometrical geodesy for the determination of station coordinates or in gravimetric geodesy to adjust a gravity network. Generally, we solve problems where the number of unknowns are less than the number of observations. But it is well known that unique solutions only can be obtained by e.g. fixing the coordinates of one station or requiring some minimum condition to be fulfilled.

In physical geodesy, we want to determine the coordinates of one point: the anomalous potential T, which can be regarded as a point in some space of harmonic functions. We face an adjustment problem where the number of unknowns always will be greater than the number of observations. For this reason, least squares methods have only been used to "predict" the value of e.g. a gravity anomaly at a point from known anomalies in other points [5] p. 251.

The basic idea behind the prediction method is the use of a linear prediction formula (presented here for gravity anomalies):

$$(1) \qquad \widetilde{\Delta g}_P = \sum_{i=1}^{n} a_i \Delta g_i ,$$

where Δg_i are anomalies measured in points P_i and $\widetilde{\Delta g}_P$ is the predicted value of the gravity anomaly at the point of prediction P.

The coefficients are determined by solving a set of linear equations:

$$(2) \qquad \{C_{ij}\} \{a_j\} = \{C_{iP}\} ,$$

where $\{C_{ij}\}$ is the $n \times n$ covariance matrix of the measured quantities and $\{C_{iP}\}$ the n-vector of covariances between the measured quantities and Δg_P.

The same kind of prediction formula can be written down for other quantities, e.g. the value of the anomalous potential at the point of prediction P, predicted from some measured quantities, x_i:

$$(3) \qquad \widetilde{T}(P) = \sum_{i=1}^{n} b_i x_i ,$$

$$(4) \qquad \{C(x_i, x_j)\} \{b_j\} = \{C(T_P, x_i)\}.$$

(We have here used the notation $C(x_i, x_j)$ and $C(T_P, x_i)$ for the covariance between the quantities x_i, x_j and T_P, x_i respectively. We will in the following use the same notation for the covariance between other quantities.)

Instead of solving equations (2) and (4) with the right hand sides equal to $\{C_{iP}\}$, $\{C(T_P, x_i)\}$ respectively, we could solve the equations with respect to the observations:

(5) $$\{C(x_i, x_j)\}\{c_j\} = \{x_i\}$$

giving this prediction formula:

(6)
$$\tilde{T}(P) = \sum_{j=1}^{n} c_j \cdot C(T_P, x_j) \quad \text{or}$$
$$\tilde{\Delta g}_P = \sum_{j=1}^{n} c_j \cdot C(\Delta g_P, x_j) \ .$$

It can be shown that the covariance function $C(T_P, x_j)$, regarded as a function of P is a harmonic functions. So (6) is in fact an approximation formula for the anomalous potential. The constants c_j can be regarded as the coordinates of the function $\tilde{T}$ in the finite dimensional space spanned by the n harmonic functions, $C(T_P, x_j)$.

The approximation $\tilde{T}$ has the property that the computation ("prediction") of an already known value, m_j will reproduce the value exactly. The approximation $\tilde{T}$ will furthermore have the least possible norm in a certain (Hilbert) space of harmonic functions.

In the theory of differential equations, a solution method which results in a function fulfilling the differential-equation, agreeing exactly with boundary values in a finite set of points and which has the least possible norm in called collocation.

The original prediction method did not consider measuring errors. But Krarup [7] and Moritz [15] have extended the approximation and corresponding prediction theory so that errors can be taken into account. Moritz has further extended the theory, so that not only approximations which simultaneously are solutions to a differential-equation but also very general kinds of approximation problems can be considered. He calls this method "Least Squares Collocation".

We will now discuss some of the problems with and the results obtained from least squares collocation in physical geodesy, i.e., when we are approximating the solution to an elliptic partial differential equation. We will not discuss the inclusion of measuring errors, because this step is quite straightforward after the problems of "exact" collocation have been solved. We will first consider problems in the theoretical model and later the more practical problems arising from the fact that (5) gives us an equation for each of the thousands of observations available.

The mathematical models behind collocation.

In the statistical models leading to the concept of covariance, the anomalous potential T is associated with a stochastic process. T is an element of a sample space of harmonic functions H with probability measure Φ. The fundamental random variables, X_P, are the mappings which relate a function in the sample space H, to the value of the function in the point P, i.e., $X_P(T) = T(P)$. The stochastic process is formed by all the random variables X_P, where P is a point in the set of harmonicity. The covariance function is a function of two variables P and Q and its value is the value of the covariance between the random variables X_P and X_Q:

$$(7) \qquad C(T_P, T_Q) = \int_H X_P \cdot X_Q \, d\Phi, \text{ where } \Phi \text{ is the probability measure.}$$

From this covariance functions, covariances between other random variables can be derived, if the random variables are related to the fundamental random variables X_P by a linear or limit of linear operations on these fundamental random variables (for details see e.g. [18]). This will define covariances or covariance functions of random variables which involve differentiation or integration. The covariances can be obtained by performing the corresponding linear operations on $C(T_P, T_Q)$. The covariance function of the gravity anomalies becomes:

$$(8) \qquad C(\Delta g_P, \Delta g_Q) = -D_r\,(C(\Delta g_P, T_Q)) - \frac{2}{r}\, C(\Delta g_P, T_Q)$$

$$= D^2_{r,r'}\, C(T_P, T_Q) + \frac{2}{r}\, D_{r'}\, C(T_P, T_Q) + \frac{2}{r'}\, D_r\, C(T_P, T_Q) + \frac{4}{r \cdot r'}\, C(T_P, T_Q)$$

where r and Δg_P are the spherical distance and the gravity anomaly in P respectively, r' and Δg_Q the same quantities in Q and D_r, $D_{r'}$ the partial derivatives with respect to r and r'.

The prediction or approximation formula becomes the one mentioned in the introduction. The very complicated problem is the estimation of the covariance functions (7) or (8).

Both theoretically and practically, the estimation problem is difficult. Since we have only one Earth, repetitions must be introduced by regarding a rotated Earth as a new body. The covariances are then estimated by taking the mean of the product sums of all points lying some fixed spherical distance apart. Such a procedure is possible, when the probabilistic structure fulfills an ergodic property.

The most simple probability structure is the one discussed by P. Meissl [12], i.e. a Gaussian distribution of the random variables with mean value zero. The probabilistic problems have been intensively studied by S. L. Lauritzen [9]. Unfortunately, he proves that the Gaussian probability distribution will imply that, as a result of the above mentioned estimation procedure, variances of the estimated covariances will not go to zero, when the number of products tends to infinity.

In the model used by T. Krarup [7], the anomalous potential T is supposed to be an element of a Hilbert space of functions, which are all harmonic outside a surface (generally a sphere) enclosed by the Earth and regular at infinity.

(A Hilbert space is a linear vector space (elements f, g) with an inner product $\{f, g\}$ and corresponding norm $\|f\|$. The space must be complete, i.e., every Cauchy-sequence converges to an element of the space, see [3] or [13]). For all reasonable norms, this kind of Hilbert space will have a reproducing kernel, K(P, Q). The reproducing kernel is an element of the Hilbert space for either of its variables, P or Q (i.e., harmonic as a function of P or Q) and symmetric K(P, Q) = K(Q, P). The kernel makes it possible to represent the identity operator I(f) = f by means of the inner product, and is therfore called the reproducing kernel (referring to the kernel of an integral operator):

(9) $$I(f)(P) = \{K(P, Q), f(Q)\} = f(P)$$

The problem of "exact" collocation is solved in the same straight forward way in this model as in the statistical model:

(10) $$\tilde{T}(P) = \sum_{i=1}^{n} c_i \ell_i K(P_i, P),$$

where the coefficients c_i are determined by:

(11) $$\{\ell_i \ell_j K(P_i, P_j)\} \{c_j\} = \{x_i\} .$$

x_i are the observed quantities and ℓ_i, ℓ_j are the linear functionals, which relate the measurements to the anomalous potential, i.e.

(12) $$\ell_i(T) = x_i .$$

The notation $\ell_i \ell_j K(P_i, P_j)$ indicates that ℓ_i is applied with respect to the first variable and ℓ_j with respect to the second. (Note that the theory of sample functions [4] is a special case of this model. The Hilbert space contains only as many harmonics as there are observations and the inner product is chosen so that the coefficient matrix of (11) is an identity matrix).

The linear functionals are the same mapping, which relate the covariance function (7) to the covariance functions of the stochastic variable associated to the measured quantity. It is possible to interpret the reproducing kernel K(P, Q) as the fundamental covariance function (7), see e.g. [18] or [9].

Hence, a practical solution to the estimation problem is the representation of the covariance function by a reproducing kernel having as many of the known properties of the covariance as possible. For further details see [21] and [23]. We will just mention what this implies for the elements of the Hilbert space.

Because we only have gravity data available, the covariance function (8) is estimated instead of the fundamental covariance function (7). The above mentioned estimation procedure gives us a covariance function which is rotational invariant, i.e. only dependent on the spherical distance ψ between P and Q and the distance r and r' of the points from the origin. The fundamental covariance function (7) will get the same property, and can hence be expressed by:

$$C(T_P, T_Q) = \sum_{i=0}^{\infty} \tau_i^2 \left(\frac{R^2}{r \cdot r'}\right)^{i+1} P_i(\cos\psi), \tag{13}$$

where τ_i^2 are constants ≥ 0, called the degree-variances of the covariance function and R less than or equal to a mean Earth radius.

The reproducing kernel must then have the same property. This implies that the norm of the Hilbert space must be rotational invariant and hence the set of harmonicity equal to an open set outside a sphere with center at the origin and totally enclosed by the Earth (a so called Bjerhammar-sphere). The use of this kind of norm has the advantage that the usual solid spherical harmonics are orthogonal, (but not necessarily orthonormal) and that all harmonics of the same degree will have the same norm:

$$\langle V_{ij}, V_{pq} \rangle \begin{cases} = 0 & \text{for } i \neq p \text{ or } j \neq q \\ = w_i & \text{for } i = p \text{ and } j = q, \; w_i \geq 0. \end{cases} \tag{14a}$$

$$V_{ij}(P) = \left(\frac{R}{r}\right)^{i+1} \cdot \begin{cases} S_{ij}(\theta, \lambda) & 0 \leq j \leq i \\ R_{ij}(\theta, \lambda) & -i \leq j < 0 \end{cases} \tag{14b}$$

where r, θ, λ are the spherical coordinates of P and S_{ij} and R_{ij} the surface harmonics, cf. [5] (1-67).

This implies that the approximation (10) or (6) will always be harmonic down to the Bjerhammar sphere and by using the collocation method we obtain a solution to the so-called Bjerhammar problem [5] p. 321. The actual potential of the Earth is only assured harmonicity down to the surface of the Earth. One could then question whether the collocation solution $\widetilde{T}$, obtained using the mentioned rotational invariant norm, would converge towards the true T. T. Krarup in [7] and later in [8] proved a variant of the so called Runge theorem, which assures the existence of arbitrary good approximations to T being harmonic down to the Bjerhammar-sphere.

It is necessary for the computation of the coefficients $\{c_j\}$ in (11) and later predicition using the approximation formula (10) that the linear functionals corresponding to the measured quantities are represented by simple expressions.

The point gravity anomaly Δg, the deflections of the vertical ξ, η and the height anomaly can all be represented by linear expressions in spherical approximation:

$$\ell_{\Delta g}(T) = -D_r(T) - \frac{2}{r} T(P) = \Delta g_P \tag{15}$$

$$\ell_{\xi}(T) = -\frac{1}{\gamma} \cdot D_{\varphi} T = \xi_P \tag{16}$$

$$\ell_{\eta}(T) = -\frac{1}{\gamma \cdot \cos\varphi} D_{\lambda} T = \eta_P \tag{17}$$

(18) $\ell_\zeta(T) = \frac{T}{\gamma} = \zeta_P$ where φ, λ are the latitude and logitude of P and γ the ference gravity.

The formula for the mean gravity anomaly over an area A becomes

(19) $$\ell_{\overline{\Delta g}}(T) = \frac{1}{A} \int_A \ell_{\overline{\Delta g}}(T)\, dA = \frac{1}{A} \int_A (-D_r T - \frac{2}{r} T)\, dA .$$

Let us by $\overline{V}_{ij}$ denote the functions V_{ij} of e.g. (14b) normalized by a suitable rotational invariant norm. For the coefficients v_{ij} of T, developed in a series with respect to the harmonics $\overline{V}_{ij}$ we have (by definition):

(20) $$\ell_{V_{ij}}(T) = \langle \overline{V}_{ij}(P),\ T(P) \rangle = v_{ij}$$

Unfortunately, not every geophysical quantity can fit into this model. Only quantities related to functionals, which are elements of the space dual to the selected space of harmonic functions can be represented in this way. (In the statistical model this will correspond to exclusing random-variables with infinite variance). This excludes e.g. mass-density anomalies and seismic information.

The application of the functionals (15) - (18) to the reproducing kernel (giving the appropriate covariances) presents no problems. The use of (20) is more simple than it looks. The reproducing kernel has (cf. [13], p.42) a simple representation in terms of the orthonormalized functions, $\overline{V}_{ij}$:

(21) $$K(P,Q) = \sum_{i=0}^{\infty} \sum_{j=-i}^{i} \overline{V}_{ij}(P) \cdot \overline{V}_{ij}(Q)$$

(Note, that by using (21) and [5] (1-82) we can get (13)).

This means that:

(22) $$\ell_{V_{kl}} K(P,Q) = \langle \overline{V}_{kl}(P), \sum_{i=0}^{\infty} \sum_{j=-1}^{i} \overline{V}_{ij}(P) \cdot \overline{V}_{ij}(Q) \rangle$$
$$= \sum_{i=0}^{\infty} \sum_{j=-1}^{i} \overline{V}_{ij}(Q) \cdot \langle \overline{V}_{kl}(P),\ \overline{V}_{ij}(P) \rangle = \overline{V}_{kl}(Q) ,$$

where we have used the linearity of the inner product and equation (14a). This means that the covariance between the random variable associate to the k,l coefficient of T, v_{kl} and the fundamental random variable X_q is equal to the value of the normalized solid spherical harmonic $\overline{V}_{kl}$ evaluated in Q. In fact this is a simple covariance model.

Different rules of thumb are used for the computation of the functions $\ell_{\overline{\Delta g}}$ applied on K(P, Q). The starting point is the covariance function of the point gravity anomalies (8) developed in a Legendre series similar to (13) but with other coefficients, σ_i^2. The most simple rule just breaks off the series for degree 180/v where v is the diameter of the area in degrees. For a spherical cap it is possible to compute the smoothing coefficients β_i given by P. Meissl [12]. Neither one of these two are completely satisfactory. A practical solution is to multiply the coefficients σ_i^2 by a damping factor ρ^{i+1}, $\rho<1$. This kind of damping has a simple interpretation, which can be seen by multiplying the term $\frac{R^2}{r \cdot r'}$ in (13) by ρ. The mean anomaly functional is represented by a point anomaly functional, where the point of evaluation has the distance from the origin equal to $r \cdot \rho^{-1}$. Because of the mentioned representation problem, mean gravity anomalies of the equal angular or equal area type are not (in my opinion) a very useful type of geodetic data. Mean gravity anomalies over spherical caps would be somewhat more useful.

Note, finally, closed expressions for the covariance function can be obtained if the quantities τ_i^2 in (13) are simple rational functions of i, i.e., e.g.:

(23) $$\tau_i^2 = \frac{1}{i \cdot (i-1)}, \quad \frac{1}{(i-1) \cdot (i-2)} \quad \text{or} \quad \frac{1}{(i-1)(i-2)(i+5)},$$

cf. [21] or [22].

Some implementation problems

The approximation (10) allows us in principle to compute an approximation $\tilde{T}$ from all existing gravity data and deflections. It would be both impossible and ridiculous to try to do this. Generally, we would be more interested in a set of local solutions, which agree well in common areas. The later use of the collocation solution for prediction purposes will then only involve the computation of the product sum of a limited number of covariance values and coefficients c_i in (10). Thus, we must in some way try to exclude the influence of the gravity anomalies in Denmark, when we want to predict a gravity anomaly in Ohio.

In prediction theory, this is done by estimating a local covariance function and subtracting the local mean value of the data from the observed values. The local covariance function will give a prediction model in which it is reasonable to exclude data which have spherical distance, ψ, from the point of prediction, greater than ψ_0, where ψ_0 is the value of the first zero-point of the local covariance function.

Now, the behavior of a local covariance function can often be predicted from the size of the area over which the considered gravity anomalies are distributed. By considering a local sample and subtracting the mean value, we are in fact trying to exclude the effect of the harmonics of wavelengths greater than the diameter, d°, of the area. As a rule of thumb, the harmonics of degree i, $i < 180/d^\circ$ are excluded.

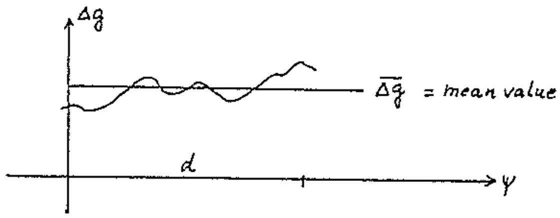


Figure 1

For an area of diameter 3°, this means that the quantities τ_i^2 in (13) will be zero for $i < 60$. It is possible to give this heuristic consideration a more precise interpretation.

Let us consider an example, where we had determined n potential coefficients $\bar{S}_{ij}$ and $\bar{C}_{ij}^*$ (cf. [19], p. 2). We will denote the corresponding coefficients, normalized with respect to the norm of the Hilbert space, by $\{v_i\}$, using a single subscripting by i or j, i, j = 1. . . , n. We will use the same subscripting for corresponding functionals ℓ_i and normalized solid spherical harmonics, i. e., so that the reproducing kernel can be expressed by:

$$K(P,Q) = \sum_{i=1}^{\infty} V_i(P) \cdot V_i(Q).$$

Furthermore, let there be given m other observations $\{x_p\}$, subscripted by p or q, p, q = 1, . . . , m and corresponding linear functionals ℓ_p, ℓ_q.

Using (14a) and (20), it is easy to see that the submatrix of the coefficient matrix of the normal equations (11), formed by the covariances between the potential coefficients, becomes the n × n identity matrix (I). The covariances between the coefficients and the data $\{x_p\}$ become:

$$\ell_p(\ell_i K(P,Q)) = \ell_p(V_i(Q)) = c_{pi} \quad , \tag{24}$$

where the constant c_{pi} is the contribution of the i-th harmonic to the measured quantity x_p.

We get the normal equations:

$$\begin{Bmatrix} I & \{c_{pj}\} \\ \{c_{qi}\} & \{\ell_p \ell_q K(P,Q)\} \end{Bmatrix} \cdot \begin{Bmatrix} \{a_i\} \\ \{b_p\} \end{Bmatrix} = \begin{Bmatrix} \{v_j\} \\ \{x_q\} \end{Bmatrix} \tag{25}$$

and the approximation:

(26) $$\widetilde{T}(Q) = \sum_{j=1}^{n} a_j V_j(Q) + \sum_{p=1}^{m} b_p \cdot \ell_p K(P,Q)$$

From equation (25) we get:

$$\{a_j\} = \{v_j\} - \{c_{pj}\}\{b_p\} \quad \text{and}$$

$$\{c_{qi}\}\{v_i\} - \{c_{qi}\}\{c_{ip}\}\{b_p\} + \{\ell_p \ell_q K(P,Q)\}\{b_p\} = \{x_q\} \quad \text{or}$$

(27) $$(\{\ell_p \ell_q K(P,Q) - \{c_{qi}\}\{c_{ip}\}) \cdot \{b_p\} = \{x_q\} - \{c_{qi}\}\{v_i\}$$

The coefficients in the m × m matrix at the left-hand side in (27) are:

$$\ell_p \ell_q K(P,Q) - \{c_{qi}\}\{c_{ip}\} = \ell_p \ell_q K(P,Q) - \{\ell_p V_i(P) \cdot \ell_q V_i(Q)\}$$

$$= \ell_p \ell_q \left(\sum_{i=1}^{\infty} V_i(P) \cdot V_i(Q) - \sum_{j=1}^{n} V_j(P) \cdot V_j(Q) \right)$$

$$= \ell_p \ell_q \left(\sum_{i=n+1}^{\infty} V_i(P) \cdot V_i(Q) \right) = \ell_p \ell_q K_1(P,Q)$$

The right hand side of (27) is the data $\{x_p\}$ minus the influence of the reference field

$$T_0(Q) = \sum_{i=1}^{n} v_i V_i(Q).$$

Hence, the use of observed potential coefficients makes it possible to regard the approximation $\widetilde{T}(Q)$ as the sum of a reference field T_0 and an approximation T_1. The latter is determined using anomalies now referring to T_0 and a covariance function (reproducing kernel) $K_1(P, Q)$ which is equal to the original covariance function having the terms corresponding to the degree and order of T_0 removed.

The new covariance function will generally make the remaining observations less dependent. For example, using the potential coefficients of [19] up to and inclusive of degree 20 and a model where $\tau_i^2 = 1/(i-1) \cdot (i-2))$ will give a representation of the covariance function of the gravity anomalies, which has its first zero-value for $\psi \approx 5°$. The corresponding empirical covariance function has its first zero value for $\psi \approx 35°$.

Even when a reference field of order 20 is used, the covariances will be too large. Since a field of this order is about the best we have at present, another way must be found to construct an improved reference field. The 1° x 1° mean gravity anomalies contain a kind of damped information, which we can use. The equations corresponding to a set of n mean gravity anomalies $\{\overline{\Delta g}_i\}$. i = 1, ... n and a set of e.g. m point anomalies and deflections $\{x_p\}$, p = 1, ... m are:

$$\text{(29)} \qquad \begin{Bmatrix} \{C(\overline{\Delta g}_i, \overline{\Delta g}_j)\} & \{C(\overline{\Delta g}_j, x_p)\} \\ \{C(\overline{\Delta g}_i, x_q)\} & \{C(x_q, x_p)\} \end{Bmatrix} \cdot \begin{Bmatrix} \{a_i\} \\ \{b_p\} \end{Bmatrix} = \begin{Bmatrix} \{\overline{\Delta g}_j\} \\ \{x_q\} \end{Bmatrix}$$

or in more compact form:

$$\begin{Bmatrix} C_{ij} & C_{ip} \\ C_{qi} & C_{pq} \end{Bmatrix} \cdot \begin{Bmatrix} a_i \\ b_p \end{Bmatrix} = \begin{Bmatrix} \overline{\Delta g}_j \\ x_q \end{Bmatrix}$$

We then have, as above,

$$a_i = C_{ij}^{-1} \overline{\Delta g}_j - C_{ij}^{-1} C_{ip} b_p \quad \text{and}$$

$$C_{iq}^{T} \cdot C_{ij}^{-1} \overline{\Delta g}_j + (C_{pq} - C_{iq}^{T} C_{ij}^{-1} C_{ip}) b_p = x_q \qquad \text{or}$$

$$\text{(30)} \qquad (C_{pq} - C_{iq}^{T} C_{ij}^{-1} C_{jp}) b_p = x_q - C_{iq}^{T} C_{ij}^{-1} \overline{\Delta g}_j .$$

Introducing the function $T_1(P)$,

$$\text{(31)} \qquad T_1(P) = \sum_{i=1}^{n} d_i \cdot C(T_p, \overline{\Delta g}_i), \{d_i\} = \{C_{ij}\}^{-1} \{\overline{\Delta g}_j\} \quad ,$$

We see that the right hand side of (30) is the predicted value of $\{x_q\}$ using mean gravity anomalies.

The change in the covariance function is not so obvious, but it is reasonable to expect that the harmonics up to degree ≈ 90 have been approximately removed.

A possible method for the estimation of the maximal order of removed harmonics is to compute gravity anomalies using the reference field $T_0(P) + T_1(P)$. Then, the empirical covariance function of the gravity anomalies is computed and approximated by covariance functions derived from (13) with different number of coefficients equal to zero. The representation which gives the best approximation of the empirical covariance function, will then furnish us with an estimate of the numbers of removed coefficients.

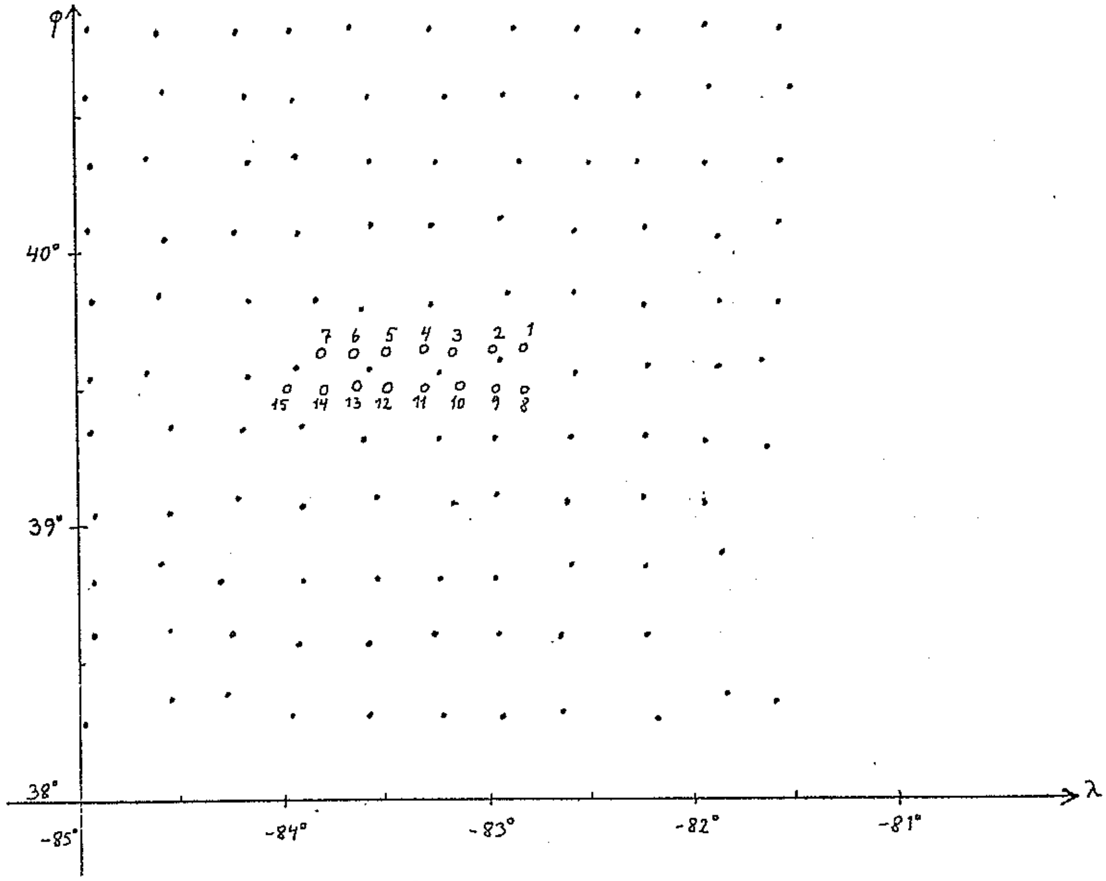
For a more general discussions of some of these problems, see Moritz [16].

Some results

The described collocation procedure has been first tested in Denmark cf. [23], using the algol-program described in [20], and later in Ohio, U.S.A., using a further refined computer program, now written in the IBM 360/370 version of FORTRAN IV.

A 2°30'× 3°40' square in Ohio was selected (see figure 2) because it contained good gravity coverage and a consistent set of 15 pairs of deflections of the vertical, cf. [17]. In the middle of the selected area, Badekas and Mueller [17] have computed a local approximation to the geoid. The deflections were transformed into an approximate geocentric reference system by using the datum-shift components for NAD 1927 given in [11].

A reference field T_0 was defined by the 20 × 20 solution given in [19]. An improved reference field T_1 were derived from (13) with τ_i^2 = a constant/(i· (i-1)), i > 20 and zero for i ≤ 20. The mean gravity anomalies were represented by point anomalies at 10 km's height, because this height gave a satisfying damping of the covariance function. The ratio between the radius of the Bjerhammar sphere and the mean radius of the Earth was 0.998. Then approximately 21 × 21 point gravity anomalies were selected (spaced as equally as possible with 7'30" distance in latitude and 10' distance in longitude). 11 × 11 of these points ($\Delta\varphi$ = 15', $\Delta\lambda$ = 20') were used as measured values. (See figure 2).



• Gravity anomaly
o_5 Deflection of the vertical, station no. 5, cf. Table 2.

Fig. 2. Distribution of the 117 gravity anomalies used as observations in the computation of the approximations T_2 and T_3 and the 15 pairs of deflections of the vertical used as test values for T_2 and as observations in the approximation T_3.

The remaining 299 points were used as test values, i.e. the gravity anomalies were predicted and compared with the observed values. The empirical covariance function of the anomalies with respect to T_0 and T_1 was computed. It was estimated that the first 110 harmonics, τ_i^2, should be removed from (7) to give a proper representation of the local covariance function.

Using this covariance function and the 11 × 11 (actually 117) point anomalies an approximation T_2 was computed. Then the 299 point values and 2 × 15 deflection components were predicted and compared with the observed values, c.f. Table 1, 2 and Figure 3.

Test Data						
	Gravity anomalies		Deflections of the vertical: Latitude component ξ		Longitude component η	
Number of test values	299		15		15	
Reference field	Mean mgal	Variance $mgal^2$	Mean arc sec	Variance $arc\ sec^2$	Mean arc sec	Variance $arc\ sec^2$
NAD 1927 and int. gravity formula	-7.0	331.7	1.4	2.1	1.7	25.2
Datum shift [11], $T_0 + T_1$	1.2	200.0	0.2	1.7	0.8	15.0
Datum shift [11], $T_0 + T_1 + T_2$	0.4	58.5	1.2	1.2	-0.6	2.4
Datum shift [11], $T_0 + T_1 + T_3$	0.3	50.1				

Table 1. Comparison of predicted and measured gravity anomalies and deflections of the vertical. The table shows the mean and the variance of the difference between observed and predicted values using different approximations of T as reference fields.

Table 2. Comparison of observed and predicted deflections of the vertical, (NAD).

Station no.	Latitude ° ′ ″	Longitude ° ′ ″		Observation	Differ. obs-pred	Datum shift	Contribution from T_0	T_1	T_2	$T_0+T_1+T_2$	Predicted value
1	39 38 57.47	-82 49 14.00	ξ	1.75	1.18	-0.25	0.45	-0.24	0.11	0.32	0.57
			η	-1.45	-2.52	-0.25	0.03	2.00	1.68	3.71	3.97
2	39 37 58.36	-82 59 10.12	ξ	1.50	1.68	-0.25	0.48	0.15	-1.06	-0.44	-0.18
			η	5.40	-3.35	-0.24	0.04	2.19	6.29	8.51	8.75
3	39 39 37.67	-83 10 21.68	ξ	0.20	0.96	-0.25	0.50	0.67	-2.18	-1.01	-0.76
			η	10.90	1.15	-0.22	0.04	1.94	7.55	9.53	9.75
4	39 39 18.42	-83 20 33.72	ξ	2.70	2.84	-0.25	0.52	1.04	-1.96	-0.39	-0.14
			η	4.80	0.36	-0.20	0.05	1.34	2.85	4.24	4.44
5	39 39 39.65	-83 32 39.13	ξ	3.10	1.02	-0.25	0.55	1.19	0.10	1.83	2.08
			η	-3.30	-1.58	-0.19	0.06	0.02	-1.98	-1.91	-1.72
6	39 40 30.56	-83 40 55.98	ξ	2.40	-0.22	-0.25	0.56	1.03	0.78	2.37	2.62
			η	-2.90	0.05	-0.17	0.07	-0.76	-2.43	-3.12	-2.95
7	39 41 8.40	-83 50 46.65	ξ	1.90	0.35	-0.25	0.57	0.64	0.09	1.31	1.55
			η	-1.10	1.25	-0.16	0.07	-1.24	-1.35	-2.51	-2.35
8	39 31 37.45	-82 49 38.34	ξ	0.70	-0.12	-0.26	0.46	-0.08	0.18	0.56	0.82
			η	0.40	-2.99	-0.25	0.05	2.28	0.81	3.13	3.39
9	39 30 53.79	-82 57 54.47	ξ	0.90	1.26	-0.26	0.48	0.02	-1.12	-0.61	-0.36
			η	5.40	-1.72	-0.24	0.06	2.41	4.42	6.88	7.12
10	39 32 4.04	-83 10 54.17	ξ	-0.40	1.00	-0.26	0.51	0.21	-2.38	-1.66	-1.40
			η	9.40	-0.07	-0.22	0.06	2.20	6.98	9.24	9.47
11	39 32 14.94	-83 20 16.15	ξ	-1.30	-0.19	-0.25	0.53	0.33	-2.22	-1.36	-1.11
			η	6.70	1.05	-0.21	0.07	1.56	3.82	5.45	5.65
12	39 32 10.48	-83 29 39.62	ξ	1.10	1.39	-0.25	0.55	0.34	-1.44	-0.55	-0.29
			η	-0.60	-1.32	-0.19	0.08	0.33	0.12	0.53	0.72
13	39 32 23.63	-83 42 5.84	ξ	4.00	3.57	-0.25	0.58	0.19	-0.59	0.18	0.43
			η	-4.30	-0.91	-0.17	0.08	-1.19	-2.45	-3.56	-3.39
14	39 31 43.32	-83 51 10.77	ξ	2.03	2.31	-0.25	0.59	-0.08	-1.04	-0.53	-0.28
			η	-2.99	1.05	-0.16	0.09	-1.68	-2.62	-4.20	-4.04
15	39 32 49.12	-83 59 24.33	ξ	-0.30	0.86	-0.25	0.61	-0.18	-1.84	-1.41	-1.16
			η	-3.60	-0.27	-0.15	0.10	-1.75	-1.83	-3.48	-3.33

Note in Table 1 and 2 the relatively big systematic differences between the measured and predicted deflection components. For a more well distributed set of deflections, such differences can be used to determine a correction to the used datum-shift components, see [23].

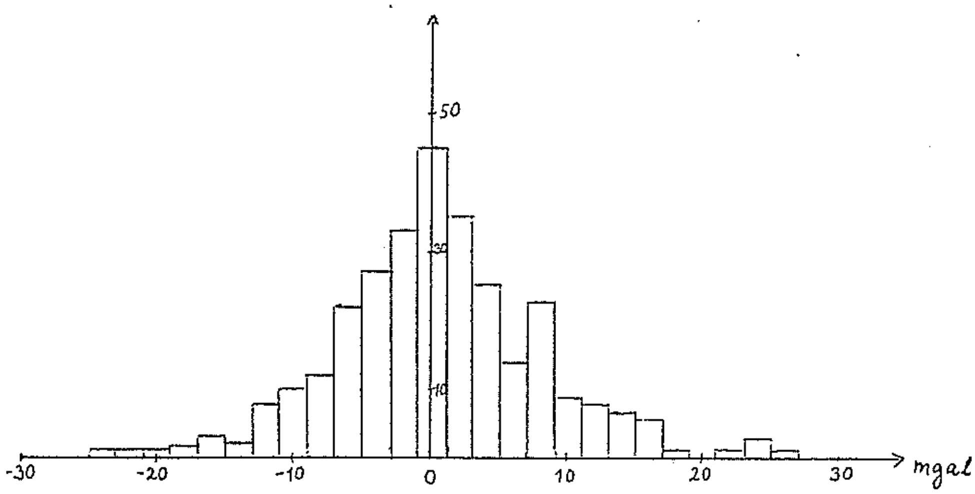


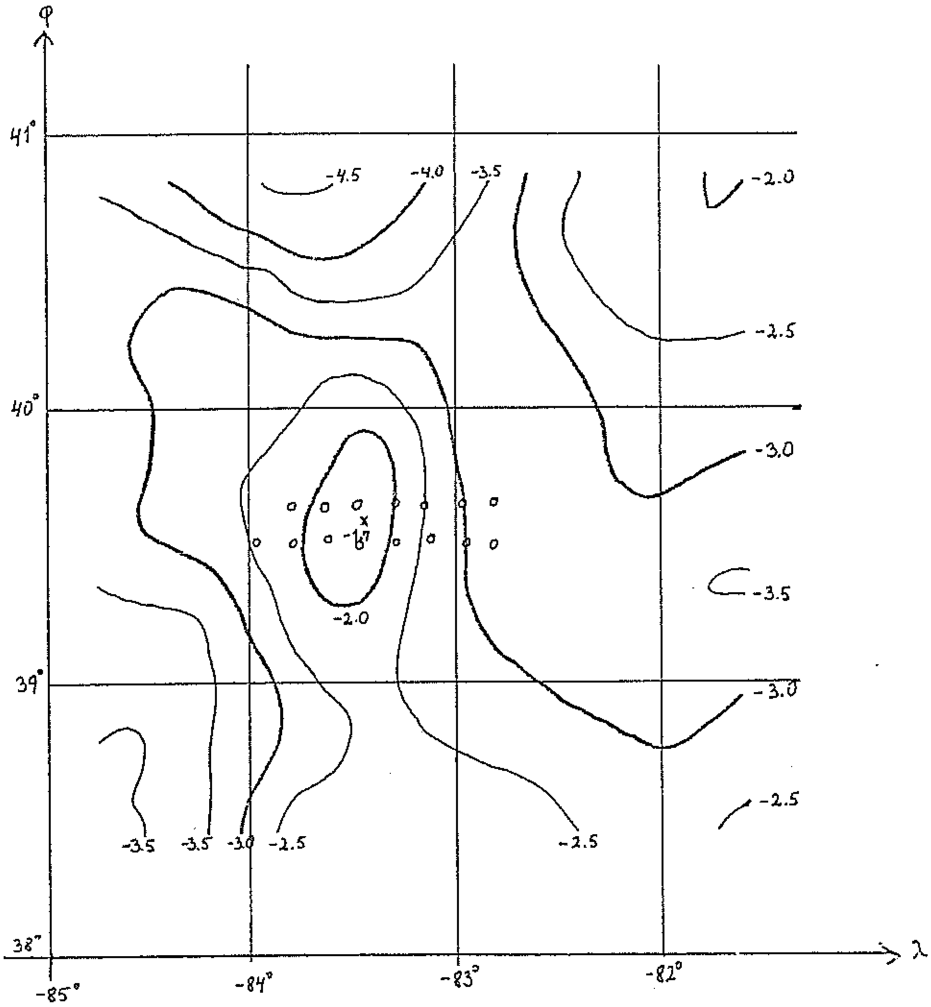
Fig. 3. Histogram of differences between predicted and measured gravity anomalies using the approximation $T_0 + T_1 + T_2$ for the computation of the predicted values. Total number of anomalies 299.

The gravity data used for the computation of T_2 was then used together with the 2 x 15 deflection components. This gave singularities in the normal-equations. The covariance functions were then changed so that only the first 90 τ_i^2 were set to zero and the ratio between the radius of the Bjerhammar sphere and the mean radius was changed to 0.999. This removed the singularities and an approximation T_3 was computed. The gravity anomalies could then be computed at the test points and the prediction results were somewhat improved, cf. Table 1.

An approximation to the geoid was computed using $T_0 + T_1 + T_2$ for the same area as used in [1], see figure 4.

Conclusion

Computation tests show that the anomalous potential can be approximated very well in areas with good gravity coverage or with a reasonable number of deflections of the vertical. A gravity coverage with station equally spaced 25 km apart will



° indicates astronomical station
Equidistance 0.5m.

Fig. 4. Geoid of South-west Ohio computed from the approximation $T_0 + T_1 + T_2$ and transformed into NAD 1927 by the datum shift given by [11]. Zero level not fixed.

represent about 75% of the variation of the anomalous potential.

Tests described in [6] indicates that collocation gives predictions of at least the same quality as other least square prediction methods.

Good results have also been obtained using collocation as a global approximation method, cf. Lelgemann [10]. Computation experiments by the author show that the method, using a selection of $5^\circ \times 5^\circ$ mean gravity anomalies so that maxima and minima are represented, gives a better representation of the mean gravity anomaly field, with less parameters, than a representation by spherical harmonics.

A problem not discussed here is the many inconsistencies in the available data. The first order triangulations need to be readjusted before we e.g. can get consistent deflections of the vertical. The next step must be to combine the approximation method described above with the adjustment of first order triangulations as described by Eeg and Krarup in [2].

Acknowledgement:

This work was supported in part by Air Force contract no. F19628-72-C-0120, The Ohio State University Research Foundation project no. 3368B1, sponsored by Air Force Cambridge Research Laboratories and by the Danish Geodetic Institute.

References

[1] Badekas, J. and I. I. Mueller: Interpolation of the Vertical Deflections from Horizontal Gravity Gradients, J. G. R. Vol. 73, No. 22, 1968.

[2] Eeg, J. and T. Krarup: Integrated Geodesy, (Internal Report No. 7, Danish Geodetic Institute), 1973.

[3] Davis, P. F.: Interpolation and Approximation, 1963.

[4] Giacaglia, G. E. O. and C. A. Lundquist: Sampling Functions for Geophysics, SAO Special Report No. 344, 1972.

[5] Heiskanen, W. and H. Moritz: Physical Geodesy, 1967.

[6] Heitz, S. and C. C. Tscherning: Comparison of two Methods of Astrogeodetic Geoid-Determination Based on Least-Squares Prediction and Collocation, Tellus XXIV (1972), 3.

[7] Krarup, T.: A Contribution to the Mathematical Foundation of Physical Geodesy, (Report No. 44, Danish Geodetic Institute), 1969.

[8] Krarup, T: On Potential Theory, Vol. I, (Internal Report No. 6, Danish Geodetic Institute), 1973.

[9] Lauritzen, S. L.: The Probabilistic Background of some Statistical Methods in Physical Geodesy, (Report No. 48, Danish Geodetic Institute), 1973.

[10] Lelgemann, D.: A Mathematical Model of the Geoid for Altimetry, (Int. Symp. Use of Artif. Sat.), Athens, 1973.

[11] Marsh, J. G., Douglas, B. C., and Klosko, S. M.: A Global Station Coordinate Solution Based Upon Camera and Laser Data, GSFC 1973, Goddard Space Flight Center, 1973.

[12] Meissl, P.: A Study of Covariance Functions Related to the Earth's Disturbing Potential, (Rep. No. 151, Dep. of Geod. Science., OSU), 1973.

[13] Meschkowski, H.: Hilbertsche Raume mit Kernfunktion, 1962.

[14] Moritz, H.: Schwerevorhersage und Ausgleichungsrechnung, Z. F. V., Vol. 90, 1965.

[15] Moritz, H.: Advanced Least-Squares Methods, (Rep. No. 175, Dept. Geod. Science, OSU), 1972.

[16] Moritz, H.: Stepwise and Sequential Collocation, (Rep. No. 203, Dep. Geod. Science, OSU), 1973.

[17] Mueller, I. I., and Preuss, H. D.: Astro-Geodetic Deflections in Southwest Ohio, (Rep. No. 61, Dep. Geod. Science, OSU), 1965.

[18] Parzen, E.: Statistical Inference on Time Series by Hilbert Space Methods. I., Tech. Rep. No. 23, Stanford, 1959 (Also in: "Time Series Analysis Papers", 1967).

[19] Rapp, R.H.: Numerical Results from the Combination of Gravimetric and Satellite Data Using the Principles of Least Squares Collocation, (Rep. No. 200, Dep. Geod. Science, OSU), 1973.

[20] Tscherning, C.C.: An Algol Program for Prediction of Height Anomalies, Gravity Anomalies, and Deflections of the Vertical, (Internal Report No. 2, Danish Geodetic Institute), 1972.

[21] Tscherning, C.C.: Representation of Covariance Functions Related to the Anomalous Potential of the Earth Using Reproducing Kernels, (Internal Rep. No. 3, Danish Geodetic Institute), 1972.

[22] Tscherning, C.C.: On the Relation between the Variation of the Degree-Variances and the Variation of the Anomalous Potential, Boll. Geod. Sc. Aff., XXXII, No. 3, 1973.

[23] Tscherning, C.C.: Determination of a Local Approximation to the Anomalous Potential of the Earth using "Exact" Astro-Gravimetric Collocation, Lecture Notes, Ramsau, 1973.