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On the Relation between the Variation of the Degree-Variances and the Variation of the Anomalous Potential

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Summary. — The coefficients of a Legendre series representing a rotational invariant covariance function are called degree-variances. The covariance function related to the anomalous potential of the Earth may be interpreted as the reproducing kernel of a Hilbert space of harmonic functions. The interpretation is based upon the isomorphy between the Hilbert space and the probability space, the structure of which is defined by the covariance function.

Different rules for the variation of the degree-variance have been used, with the purpose of solving the problem, that the degree-variances from a certain step are unknown. We point out, that the use of a specific rule or parametrization of the degree-variances corresponds to the assumption of the bounded variation of the derivatives of the anomalous potential up to a particular order. This again is equivalent to assuming that the anomalous potential (T) is an element of a Hilbert space having a specific norm.

In a reproducing kernel Hilbert space we can, by means of the method of collocation, determine approximations to T which have the least norm (e.g. the least variation of the first derivatives). Hence it is of interest to know the norms of Hilbert spaces corresponding to different covariance functions.

For three simple parametrizations of the degree-variances, we have derived the norms of the corresponding Hilbert spaces. The results demonstrate a simple relation between (1) some often used models for the variation of the degree-variances, (2) the norms of the corresponding Hilbert spaces and (3) assumptions of the behaviour of T or approximations to T.

1. — INTRODUCTION.

a) *The statistical or probabilistic model for the anomalous potential of the Earth* (T).

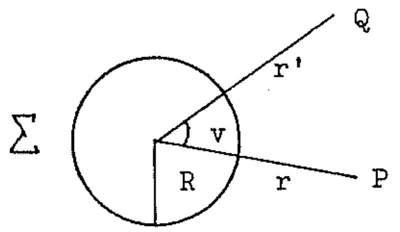
In a statistical model for the anomalous potential (called a stochastic process), the structure e.g. is defined by the one-dimensional distributions of the coefficients of T developed in a series of spherical harmonics. Supposing a Gaussian distribution of the coefficients, the covariance function related to T (i.e. the covariance function of the distributions of the values of T) defines all probabilistic properties.

With the purpose of solving the problem of having only one Earth, repetitions are introduced by regarding the harmonic functions, which by a rotation can be carried into each other, as having equal probability. (Furthermore, this means, that all values of T lying on the same sphere will have the same distribution).

This implies that the covariance function is rotational invariant, too. Supposing furthermore that T is harmonic outside a sphere with radius R, the covariance function may be represented by

$$C(P,Q) = \sum_{n=0}^{\infty} \sigma_n^2 \cdot s^{n+1} P_n(t)\,,$$

where σ_n^2 are the mean variances of the coefficients of order n, (called degree-variances), $s = R^2 / (r \cdot r')$, $t = \cos(v)$ and P and Q are points in the space Σ outside the sphere (see figure).



b) *The mathematical model for T* :

According to Parzen [5] the stochastic process (the space of harmonic functions having positive probability) is isomorphic to a reproducing kernel Hilbert space [4]. The covariance function is equal to the reproducing kernel

$$C(P,Q) = K(P,Q)\,.$$

A Hilbert space will have an inner product noted $(\,,\,)$, and we may express the reproducing property by

$$T(P) = \big(T(Q), K(P,Q)\big)\,.$$

One Hilbert space will not contain all functions harmonic outside Σ. Nevertheless a Hilbert space always exists, containing a given harmonic function.

The Hilbert spaces differ by their inner products (or by their corresponding norms, $\| V \|^2 = (V, V)$). The functions contained in a specific Hilbert space are those which have finite norm. A knowledge about in which Hilbert space a harmonic function is contained, hence corresponds to a knowledge about e.g. the variation of the first derivatives of the function.

An inner product corresponds uniquely to a reproducing kernel. Hence, considering the above mentioned isomorphy, we see that the covariance function gives us information about the variation of the anomalous potential.

Finally we note, that the solid spherical harmonics

$$(*) \qquad \left(\frac{R}{r}\right)^{i+1} \begin{cases} \overline{R}_{ij}(P) & j = 0, 1, \ldots, i \\ \overline{S}_{ij}(P) & j = 1, \ldots, i \end{cases} \qquad i = 0, \ldots, \infty$$

form a complete base in a Hilbert space of harmonic functions. ($\overline{R}_{ij}(P)$ and $\overline{S}_{ij}(P)$ are the fully normalized surface spherical harmonics cf. [2] (1 - 73)). For any rotational invariant norm (i.e. a norm which has the property, that functions which may be mapped into each other by a rotation have the same norm), the functions (*) are orthogonal and their normalizing factors will only depend on the degree i. In the following we shall only regard Hilbert spaces with this kind of norms.

2. — The problem of unknown degree-variances.

Only the first few degree-variances have been estimated. Therefore, different rules for the behavior of the σ_n^2 have been proposed and used (see [6]). Most rules or parametrizations express σ_n^2 as a fraction of two polynomials in n or as a constant raised to a power which is a simple function of n.

The covariance function may be used to compute the mean variance of a quantity related to T — for example over the surface of the Earth. The variation of the (point) gravity anomalies are for example equal to

$$\sigma^2 (\Delta g) = \sum_{n=2}^{\infty} \sigma_n^2 (n-1)^2 / R^2 \qquad \text{(spherical approximation).}$$

If we knew that the variation of Δg were finite, it would be reasonable to use rules for the variation of σ_n^2 where $\sigma_n^2 < c/n^{3+\varepsilon}$, $c > o$ and $\varepsilon > o$. All series which proceed like powers of n^{-1} dominat series which proceed like s^n, $s < 1$ (modulo a suitable constant). Hence rules of this kind imply the finite variation of all derivatives of T (cf. section 4). T is in fact harmonic down to a sphere with radius equal to $\sqrt{s} \cdot R$.

We have according to [2] (7,28) that the degree-variances are estimated by

$$\hat{\sigma}_n^2 = \sum_{m=o}^{n} (\bar{a}_{nm}^2 + \bar{b}_{nm}^2) \,,$$

where $\bar{a}_{nm}$ and $\bar{b}_{nm}$ are the coordinates of T with regard to the base (*):

$$T\,(P) = \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+1} \sum_{m=o}^{n} (\bar{a}_{nm}\,\bar{R}_{nm}\,(P) + \bar{b}_{nm}\,\bar{S}_{nm}\,(P)) \,.$$

(Note, that in the general theory of harmonic functions, this is equivalent to the statement, that T is developed in a series of solid spherical harmonics).

A change in the normalization of the solid spherical harmonics (*) implying that they are multiplied by a factor N, means that the coefficients have to be divided by N. Hence a specific chosen rule for the variation of the degree-variances implies that T can only be an element of Hilbert spaces with specific norms. (In concluding this, we use, that an element of a Hilbert space has finite norm).

3. — Degree-variances and least squares approximation methods.

Approximations to T may be determined by least squares methods. Solutions are obtained as elements of either a finite or an infinite dimensional space. When the number of observations are bigger than the dimension of the finite dimensional space, a unique solution is obtained by requiring a weighted square sum of the differences between observed and predicted values to be a minimum. The degree-variances are here used as the preliminar weights of the $2n + 1$ dimensional subspaces spanned by the spherical harmonics of degree n. In the infinite dimensional case, we suppose that an inner product (or related norm) has been chosen, making this space to a reproducing kernel Hilbert space. Hence it is possible to determine an approximation to T by the method of least squares collocation (see [3]), i.e. so that the solution agrees with given observations. Many functions in the Hilbert space have this property. But we are getting a unique solution by requiring it to have the least norm between all these.

As in the finite dimensional case we would like to determine an approximation to T which fit the observations the best possible. The method of least squares collocation requires as many normal equations to be solved as there are observations. — Hence it is not generally possible (nor for local determinations of T) to take all observations in regard. — And all observations will not contain the same amount of information.

As mentioned above, the method of least squares collocation requires the frame of a reproducing kernel Hilbert space. The kernel defines in a sense the weigths and the covariances of the observations in the same way as the mentioned covariance function. Taking this into consideration, it has been natural to try to find reproducing kernels, which behave like the covariance function of T. Hence, it is of interest to know the norms of Hilbert spaces corresponding to different covariance functions. In the following section we shall then from three simple examples demonstrate the relation between models for the variation of the degree-variances, where these depend on powers of the degree n, and the norms of corresponding Hilbert spaces.

4. — Examples of norms and corresponding reproducing kernels.

Generally a reproducing kernel $K(P, Q)$ may be expressed by a sum containing the functions V_{ij} forming an orthonormal base in the corresponding Hilbert space (see [4]).

$$K(P, Q) = \sum_{i=0}^{\infty} \sum_{j=-i}^{i} V_{ij}(P) \cdot V_{ij}(Q),$$

where

$$(V_{ij}(P), V_{i_1 j_1}(P)) = \begin{cases} 1 & \text{for } i = i_1 \text{ and } j = j_1 \\ 0 & \text{for } i \neq i_1 \text{ or } j \neq j_1 . \end{cases}$$

As a function W in the Hilbert space may be expressed by

$$W(P) = \sum_{i=0}^{\infty} \sum_{j=-i}^{i} a_{ij} V_{ij}(P),$$

with

$$\sum_{i=0}^{\infty} \sum_{j=-i}^{i} (a_{ij})^2 < \infty ,$$

we easily see the reproducing property

$$W(P) = (W(Q), K(P, Q)) = \sum_{i=0}^{\infty} \sum_{j=-i}^{i} a_{ij} (V_{ij}(Q), \sum_{n=0}^{\infty} \sum_{m=-n}^{n} V_{nm}(P) \cdot V_{nm}(Q))$$

$$= \sum_{i=0}^{\infty} \sum_{j=-i}^{i} a_{ij} V_{ij}(P).$$

Example 1:

The usual surface spherical harmonics $\bar{S}_{ij}$, $\bar{R}_{ij}$ are orthonormalized over the surface of the unit sphere. We will determine the corresponding orthonormalized solid spherical harmonics over the space outside a sphere with radius R (Σ).

With the aim of simplifying the computations, we denote all normalized surface spherical harmonics by $\bar{S}_{ij}$, $-i \leq j \leq i$.

The inner product of two functions in this Hilbert space is

$$(U, V)_1 = \frac{1}{4\pi} \int_{\Sigma} U(P) \cdot V(P) \, d\Sigma .$$

The normalizing factor may then be found by computing the norm of the solid spherical harmonics $\bar{S}_{ij}(P) \cdot \left(\frac{R}{r}\right)^{i+1}$.

$$\left\| \bar{S}_{ij}(P) \cdot \left(\frac{R}{r}\right)^{i+1} \right\|^2 = \frac{1}{4\pi} \int_{\Sigma} \left(\left(\frac{R}{r}\right)^{i+1} \bar{S}_{ij}(P) \right)^2 d\Sigma$$

$$= \frac{1}{4\pi} \int_R^{\infty} \int_{\sigma} \left(\bar{S}_{ij}^2 \cdot \left(\frac{R}{r}\right)^{2i+2} \right) \cdot r^2 \cdot dr \cdot d\sigma =$$

$$= \int_R^{\infty} \frac{R^{2i+2}}{r^{2i}} \, dr = R^3 / (2i - 1) .$$

Hence the reproducing kernel is

$$K_1(P, Q) = \sum_{i=1}^{\infty} \frac{2i - 1}{R^3} s^{i+1} \sum_{j=-i}^{i} \bar{S}_{ij}(P) \bar{S}_{ij}(Q)$$

$$= \sum_{i=1}^{\infty} \frac{4i^2 - 1}{R^3} s^{i+1} P_i(t) ,$$

where $t = \cos(v)$ and P_i is the Legendre polynomials. The degree-variances vary here like i^2.

Example 2 :

We shall now use the Direchlet norm

$$\| W \|_2^2 = \frac{1}{4\pi} \int_{\Sigma} (\nabla W)^2 \, d\Sigma ,$$

where ∇ is the gradient differential operator, $\left(\frac{\partial}{\partial x} , \frac{\partial}{\partial y} , \frac{\partial}{\partial z} \right)$.

As we have

$$(U , W)_2 = \frac{1}{4\pi} \int_{\Sigma} \nabla U \cdot \nabla W \, d\Sigma = -\frac{1}{4\pi} \int_{\sigma} \frac{\partial W}{\partial n} \cdot U \, d\sigma ,$$

where σ is the surface of the sphere, we easily see that

$$\frac{1}{4\pi} \int_{\Sigma} \nabla \left(\left(\frac{R}{r} \right)^{i+1} \bar{S}_{ij} \right)^2 d\Sigma = R \, (i + 1) .$$

The reproducing kernel is then

$$K_2 \, (P , Q) = \sum_{i=0}^{\infty} \frac{s^{i+1}}{R} \, \frac{2i + 1}{i + 1} \cdot P_i \, (t) ,$$

and we see that the degree-variances behaves like a constant $\left(\lim_{i \to \infty} \frac{2i + 1}{i + 1} = 2 \right)$.

Example 3 :

This example shows how we by introducing a weight function depending on r get a modified rule for the variation of σ_n^2

$$(U , W)_3 = \frac{1}{4\pi} \int_{\Sigma} \frac{1}{r} \, (\nabla U \cdot \nabla W) \, d\Sigma .$$

Here we cannot use Greens theorem, but may compute the first derivatives of the orthonormal functions. Representing the spherical harmonics on complex form following [1], we have

$$V_{nm}(P) = (\bar{R}_{nm}(P) + i \cdot \bar{S}_{nm}(P)) / r^{n+1}$$

where i is the imaginary unit.

The scheme below expresses the relation between a harmonic V_{nm} multiplied by $c = ((2n+3)/(2n+1))^{1/2}$ and its derivatives:

$c \cdot V_{nm}(x,y,z)$	$V_{n+1,\,m+1}$	$V_{n+1,\,m}$	$V_{n+1,\,m-1}$ $m>0$	$m=0$
$\frac{\partial}{\partial x}$	$-\frac{1}{2}((n+m+2)\cdot(n+m+1))^{1/2}$	0	$\frac{1}{2}((n-m+1)\cdot(n-m+2))^{1/2}$	$-\frac{1}{2}((n+2)\cdot(n+1))^{1/2}$
$\frac{\partial}{\partial y}$	$\frac{i}{2}((n+m+2)\cdot(n+m+1))^{1/2}$	0	$\frac{i}{2}((n-m+1)\cdot(n-m+2))^{1/2}$	$-\frac{i}{2}((n+2)\cdot(n+1))^{1/2}$
$\frac{\partial}{\partial z}$	0	$-((n+m+1)\cdot(n-m+1))^{1/2}$	0	0

Hence

$$\frac{1}{4\pi}\int_{\sigma}(\nabla V_{nm})^2\, d\sigma = \frac{1}{4\pi}\int_{\sigma}\Big(\frac{1}{2}(n+m+2)\cdot(n+m+1)\,V^2_{n+1,m+1} +$$

$$+ (n+m+1)(n-m+1)\,V^2_{n+1,m} +$$

$$+ \frac{1}{2}(n-m+1)(n-m+2)\,V^2_{n+1,m-1}\Big)\frac{2n+1}{2n+3}\,d\sigma =$$

$$= \frac{1}{2\cdot r^{2n+4}}\Big((n+m+1)(n+m+2) +$$

$$+ 2(n-m+1)(n+m+1) + (n-m+1)(n-m+2)\Big)\frac{2n+1}{2n+3} =$$

$$= \frac{(2n+1)\cdot(2n+2)}{2\cdot r^{2n+4}} = \frac{(2n+1)(n+1)}{r^{2n+4}}.$$

And we get

$$\frac{1}{4\pi}\int_{\Sigma}\frac{1}{r}(\nabla V_{nm})^2\, d\Sigma = \int_{R}^{\infty}\frac{(n+1)\cdot(2n+1)}{r^{2n+3}}\cdot dr = \frac{1}{2}\cdot\frac{2n+1}{R^{2n+2}}$$

and

$$K_3(P, Q) = 2\cdot\sum_{i=o}^{\infty} s^{i+1}\cdot P_i(t)\,.$$

5. — Conclusions.

When we change the order of a differential operator D^{α} (α a multiindex-symbol) in the inner product

$$(W, U) = \int_{\Sigma}\sum_{\alpha}(D^{\alpha} W\cdot D^{\alpha} U)\, d\Sigma$$

by one, we can see, that the original orthonormal functions have to be multiplied by the square root of a polynomial of second order in n. Hence the corresponding degree-variances of the reproducing kernel will be divided by the polynomial. The polynomial corresponding to the second order differential operator

$$D^{\alpha} = \sum_{i,j}\frac{\partial^2}{\partial x_i\,\partial x_j}$$

is for example

$$(2i+1)\cdot(i+2)/R^2 \quad \text{(see [7])}\,.$$

In a recent paper by Rapp [6] it is proposed to use a rule

$$\sigma_n^2 = \frac{A}{(n-1)(n-2)(n^2+Bn+C)}\,.$$

This rule assures finite variance of the gravity anomalies and deflections on the surface of the Earth. A solution to a collocation problem, where the kernel has these degree-variances, will correspondingly have finite variance of e.g. the second derivatives.

LITERATURE

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[6] R.H. RAPP, *Geopotential Coefficients Behaviour to High Degree*, 1972.

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REMARKS

GRAFAREND. — You mentioned that the statistics involved in collocation is non - Gaussian. That means that collocation is nonlinear.
Is that right ?

Reply. — As we know from the adjustment of geodetic networks, Gaussianity is closely related to minimalizing a mean square error. We may get solutions to the problem of approximating the anomalous potential using a sort of non-linear collocation, but then we are minimalizing a non-square error. For me non-Gaussianity means, that the best covariance function is not estimated by the usual method of taking the mean of sums of products. Instead one could try to define the covariance function as the function which in actual cases minimalizes the mean square error.

MEISSL. — How sensitive is your collocation procedure with respect to the choice of the covariance function.

(1) If, for example, you have observations consisting of gravity measurements and vertical gradient of gravity - measurements. Which covariance function would you choose ?

(2) Do you use different covariances for local and global adjustments ?

Reply. — (1) I would try to find a representation of the covariance function, which both approximates the empirical covariance functions of the gravity anomalies and the vertical gradient of gravity. A model for the degree-variances, where these were dependent of a factor S^n (where $S < 1$), would assure, that the variation of the gradient of the gravity anomalies on the surface of the Earth is finite, too. One could also use a norm which minimalizes a weighted sum of both the second and third derivatives.

(2) Local covariance functions have I in practical work tried to approximate with kernels, where the first N degree-variances was equal to zero. The number N depending on the size of the area in consideration.

KRARUP. — I should like to mention that every choice of a norm for the potential outside the Earth or, what is the same, a choice of a covariance function, induces a norm for the mass distribution and a covariance function for this mass distribution (this « covariance function » is in some cases a generalized function (distribution)). E.g. the choice of the norm

$$\| \varphi \|^2 = \int_{\Sigma} \frac{1}{r} (D^2 \varphi)^2 \, d x$$

is equivalent with choosing the norm

$$\| \rho \|^2 = \int f(v) \, \rho^2 \, d x$$

over the mass distribution. (f (v) is some simple function of v the actual form of which I do not remember to-day).