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Collocation Methods in Harmonic Spaces *)

Summary

It is suggested to use collocation when solving free boundary value problems, where the boundary values only are given in discrete points. Collocation is here used to label approximation methods by which not only values of the function but also for example of its partial derivates are maintained.

Collocation problems may be solved in a very straightforward way when the function is an element of a reproducing kernel Hilbert space. It is noted that subspaces of the harmonic spaces of Heinz Bauer are of this kind, when suitable scalar products are introduced.

For harmonic spaces theorems of the Runge type may be proved. This makes it valid to determine a solution being harmonic in a bigger area, which furthermore may have a boundary making the corresponding reproducing kernels computable.

To the reproducing kernel there corresponds a stochastic process with this as covariance kernel. It is therefore proposed to select the Hilbert space by chosing the reproducing kernel so that it approximates an empirically determined covariance function.

Finally the concept of ε-entropy and its heuristic use is mentioned.

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1. Introduction

A free boundary value problem or a usual boundary value problem may in some cases be solved by transforming the problem to an approximation problem. We determine a function which satisfy the partial differential equation and which agrees with the boundary values in a sufficient dense grid of points.

We use the term collocation on approximation methods which maintains boundary values (or other sorts of data) exactly. A collocation method is applied in geodesy with the purpose of determining an approximation to the (harmonic part of) the potential of the Earth. The method is described in section 2.

Approximate solutions to boundary value problems, which also may be related to partial differential equations different from the Laplace equation, may be solved using the same type of collocation. Especially the method will be well suited to solve problems in the general harmonic spaces of Heinz Bauer [1] . These spaces are described in section 3.

The surface of the Earth has (as known) an irregular form. However, a theorem of the Runge type shows that arbitrarily good approximations exist within the space of functions harmonic outside e.g. a sphere enclosed in the Earth. In section 4 the validity of theorems of the Runge type in general harmonic spaces is discussed.

We may have additional information about our solution. This determines a subspace in the space of functions satisfying the considered partial differential equation. When this information is of statistical character we may consider the approximation problem as a prediction problem in a stochastic process. This is considered in section 5. In the final section we mention the concept of ε-entropy, which may be used to measure the interpolation properties of a subspace.

2. Collocation in Geodesy

The determination of the anomalous potential of the Earth (T) may be regarded as a free boundary value problem. The boundary values are observations of gravity anomalies and deflections of the vertical in discrete points.

Corresponding to these types of observations there exist linear functionals, which are linear combinations of the evaluation functional and some of the first partial derivatives. E.g., the gravity anomaly in a point P with distance r from the origin, is expressed by means of the anomalous potential T:

$$L_{\Delta g}(T) = -\frac{\partial}{\partial r} T|_P - \frac{2}{r} \cdot T(P) .$$

The function T is harmonic in the open space outside the Earth, V. It is furthermore regular at infinity. By introducing the topology of uniform convergence on compact subsets, the space of functions harmonic in V becomes a nuclear Frechet space, (see e.g. [4], lemma 4). Scalar products - noted as (,) - may be introduced, making the set of functions for which the corresponding norm $\| \|$ is finite, a Hilbert space. This space possesses a so-called reproducing kernel, a function K(P,Q), $K: V \times V \rightarrow \mathbb{R}$, which for P or Q fixed is a regular harmonic function. It has the reproducing property, i.e. the scalar product of T(Q) with the kernel K(P,Q) equals T(P):

$$T(P) = (K(P,Q), T(Q)) .$$

Conversely, corresponding to a given kernel with these properties, there exist a unique scalar product and norm.

Between many possible scalar products we may chose

$$(T,U) = \int_{V^*} T(P) \cdot U(P) \, dV^* , \qquad (1)$$

where V* is the boundary of V.

If V* is a sphere with radius R and with center at the origin, we will get the "Poisson"-kernel: (cf. [3] p. 43).

$K(P,Q) = \frac{s(1-s)}{l}$, $l = (1-2st+s^2)^{\frac{1}{2}}$, $s = \frac{R}{rr'}$, $t = \cos(v)$. (See Figure 1).

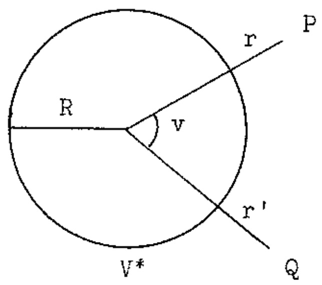

Fig. 1

Our harmonic function T will be an element of either this or another reproducing kernel Hilbert space, H(V). This kind of spaces have the property, that collocation problems may be solved in a very simple way. It is required that the linear functionals related to the observations, L_i are members of the dual Hilbert space H'(V), i.e. their norm must be finite.

If this requirement is fulfilled, we shall get a unique solution to the collocation problem, by requiring the solution to have the least norm, [5] p. 115, [8]).

The solution will be represented on the following form

$$T(Q) = \sum_{i=1}^{n} a_i L_i (K(P_i, Q)).$$

n is the number of observations, P_i the points of observations and a_i are certain constants determined as solutions to a system of normal equations:

$$\begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix} = \begin{Bmatrix} L_1 L_1 K(P_1,P_1), & \cdots & L_n L_1 K(P_n,P_1) \\ \vdots & \ddots & \vdots \\ L_1 L_n K(P_1,P_n), & \cdots & L_n L_n K(P_n,P_n) \end{Bmatrix}^{-1} \cdot \begin{Bmatrix} m_1 \\ \vdots \\ m_n \end{Bmatrix}$$

m_i are the observations.

The linear functional related to the boundary values will not always be members of the dual space $H'(V)$. This depends on the scalar product selected. But the functionals will generally be members of a dual space $H'(U)$, where $U \supset V$. Hence allowing our approximation $\tilde{T}$ to be harmonic in a set U which encloses V, the problem is solved. By choosing U^* equal to a sphere we will furthermcre get reproducing kernels which are computable.

3. Harmonic Spaces

In the geodetic collocation method, two properties (among others) have been used: (a) The functions harmonic in U are harmonic in $V \subset U$ too, (b) corresponding to a scalar product there exists a reproducing kernel.

The general harmonic spaces of Heinz Bauer have these two properties. In the following short description we follow [4] .

A harmonic space is a pair (X,H), where X is Hausdorff (e.g. three dimensional space R^3) and H is a sheaf of functions $X \rightarrow R \cup \{-\infty, \infty\}$. A sheaf is a mapping from a system of open subsets $\{U\}$ of X to the functions harmonic on these subsets, $H(U)$, which have the following properties. For $V \subset U$ the restrictions of the functions in $H(U)$ to V shall be elements of $H(V)$,(cf. property (a) above).

When the restrictions of a function g to open sets U_i, (where i runs through an index set I) all are elements of $H(U_i)$, then g must be an element of $H(U)$, where $U = \bigcup_{i \in I} U_i$.

The sheaf must satisfy three axioms:

I. $H(U)$ is a linear subspace of the functions continuous on U.

II. Each $P \in U$ has a basis $\{V(P)\}$ of open relatively compact neighbourhoods V, called the regular sets.

1. V^*, the boundary is not empty.
2. A function f continuous on V^* can be extended onto $\bar{V}$, so that its restriction to V is a harmonic function noted H_f^V. The mapping $f \rightarrow H_f^V(x)$, $x \in V$ is a positive linear functional.
3. If f is equals the restriction of a function h, harmonic in a set $U \supset \bar{V}$, then $H_f^V = h$ in V.

III. For $P, Q \in X$ there exist hyperharmonic functions f, g so that $f(P) \cdot g(Q) \neq f(Q) \cdot g(P)$.
(A hyperharmonic function g is (1) downward bounded, (2) downward continuous and (3) f continuous on the boundary V^* and $f < g$ on V^* implies $H_f^V \leq g$ in V).

It can be proved by these axioms, that the Dirichlet problem has a unique solution for each regular set.

Solutions to both elliptic and to some classes of parabolic partial differential equations as well form harmonic spaces (see [1]). If we introduce scalar products like (1) K. Maurin [4] shows, that there will exist a reproducing kernel. From this is realized that collocation may be used to solve other types of boundary value problems than these arising from the Laplace equation.

It is important to note, that the boundary values may be combined with other data when using collocation. The data can be the values of some linear functionals not related to the boundary, as for example the coefficient to the i'th term of a Fourier expansion of the solution to the boundary value problem.

4. Theorems of the Runge type

We would like to know, if functions harmonic in $V \subset U$ can be approximated arbitrarily well by functions harmonic in U. T. Krarup [3] has proved this for usual Laplace-harmonic functions, (U^* a sphere). The problem is, as far as I know, still unsolved in general harmonic spaces. In [9] it has been shown, that the union of all functions harmonic in sets $U_i \supset V$ is dense in H(V). It would be of great interest if a theorem of the Runge type could be proved for general harmonic spaces.

The validity of the Runge's theorem ([3] p. 74) has made it perfectly natural to determine approximations $\tilde{T}$, being harmonic in $U \supset V$. Earlier, much time has been spent discussing the convergence near the surface of the Earth of some canonical harmonic expansion.

5. Reproducing Kernels and Covariance Functions

In section 2 it was noted, that requiring the norm of $\tilde{T}$ to be minimum would produce a unique solution to the collocation problem. Now, there will be many Hilbert spaces in which T can be expected to be a member. The question can therefore be raised if one space might be better for approximation purposes than another.

Such a subspace is given in cases, where we have more information than this given by the boundary values. A statistical treatment of the boundary values, e.g. by a computation of an empirical covariance function of the boundary values, may also give us information about an optimal space for approximation.

An additional information of statistical character may lead us to consider the approximation problem as a prediction problem in a stochastic process.

In geodesy the potential T is in this context regarded as the realization of a gaussian stochastic process, i.e. all the properties of the process are determined by a mean value and a covariance function. Practical computations has shown, that reproducing kernels which approximate empirically determined covariance functions gave good approximations. By choosing reproducing kernels in this way we have (as mentioned in section 2) determined a scalar product and a norm and hence a subspace for approximation.

There is a near connection between reproducing kernels and covariance functions. Parzen [6] has shown, that a stochastic process is isomorphic to a Hilbert space with reproducing kernel, which will equal the covariance function.

Noting again, that Hilbert subspaces of harmonic spaces do possess a reproducing kernel, it should be feasible, as in geodesy, to introduce statistical information, when solving boundary value problems for elliptic and parabolic equations.

6. ε-entropy

When we approximate functions harmonic in V by functions in U, it would be of interest to know, if it was easier to "catch" the functions in V if U was near to V than otherwise.

The ε-entropy is a measure for how many open sets of diameter 2ε we shall need for covering a compact set; the ε-entropy is the logarithm with base 2 of this number, $H(\varepsilon, A) = \log_2(N(\varepsilon, A))$, where A is the compact set.

Let the compact set A be the functions which are bounded by a constant c and Laplace-harmonic in U (the open space outside a sphere with radius R). The topology is defined by putting $\|T\| = \sup(T(P))$, $|P| > r > R$. The variation of the ε-entropy $H(\varepsilon, A)$ gives us an impression of the interpolation properties for varying R, (cf. [7]).

$$H(\varepsilon, A) = \frac{2}{3}(\log_2 \ \frac{r}{R}(\log_2 \frac{1}{\varepsilon})^2 + O((\log_2 \frac{1}{\varepsilon})^2 \log_2(\log_2 \frac{1}{\varepsilon})) .$$

Finally a result from [2] (p.93, theorem 9) shall be mentioned:

For a separable Hilbert space H, the necessary and sufficient condition to be nuclear is, that

$$\lim_{\varepsilon \to 0} \frac{\ln(H(\varepsilon, A, U))}{\ln \frac{1}{\varepsilon}} = 0 ,$$

for every compact set $A \subset H$ and for every neighbourhood U of the origin. Thus this property is valid for the general harmonic spaces.

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