Determination of the geoid by collocation

by

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Summary: The purpose of physical geodesy is to determine a potential field compatible with given discrete observations (values of linear functionals operating on the potential). Collocation or generalized interpolation is well suited for such a determination. This is due to the lucky fact, that the disturbing potential T is a function in a reproducing kernel hilbert space. Formulas in spherical approximation are used when determining the astrogeodetic quasigeoid of the south-west scandinavian area.

Introduction.

There will (almost) ever be an infinite number of potentials compatible with a finite set of measurements. So our problem will be to select one between these many: As we expect the equipotential surfaces to be "nice" smooth surfaces, it is natural to look for the potential, which has the least norm. But there can be introduced (infinitely) many norms, i.e. corresponding to, taking higher derivates in regard. We shall look at

Example 1.

The set of potentials regular in the open set outside the surface of the Earth (Ω, boundary ω),(and regular at ∞), H_Ω, equiped with the Dirichlet norm:

$$\|T\|^2 = \iiint_\Omega (\nabla T)^2 \, d\Omega$$

and the corresponding scalar product:

$$(T,U) = \iiint_\Omega \nabla T \cdot \nabla U d\Omega = \iint_\omega T \frac{\partial U}{\partial n} d\omega = \iint_\omega \frac{\partial T}{\partial n} U d\omega .$$

Equiped with this norm, (the completion of) H_Ω becomes a Hilbert space with reproducing kernel [2] , [6] :

$$K(P,Q) = G(P,Q) - N(P,Q),$$

the difference between the Greens and Neumanns functions of Ω , P and Q points in Ω.

$$(T(P),K(P,Q)) = \iiint_{\Omega} \Delta T(P) \cdot \Delta (K(P,Q)) d\Omega$$

$$= \iint_{\omega} T \frac{\partial G(P,Q)}{\partial n} d\omega = \iint_{\omega} \frac{\partial T(P)}{\partial n} N(P,Q) d\omega = T(Q).$$

2. Formal determination of the anomaleous potential.

The value of the anomaleous potential T in a point P (coordinates φ, λ, r_P), can be regarded as the value of the functional

$$L_P: H \rightarrow \dot{R}, \text{ given by } L_P(T) = T(P).$$

Correspondingly gravity anomalies, deflections of the vertical and height anomalies can be viewed as the picture of T by the functionals (in spherical approximation):

(2.1) $L_{\Delta g(P)} : H_{\Omega} \rightarrow \dot{R}$ given by $L_{\Delta g(P)}(T) = -\left.\frac{\partial T}{\partial r}\right|_P - \frac{2}{r_P} T(P)$,

(2.2) $L_{\xi(P)} : H_{\Omega} \rightarrow \dot{R}$ given by $L_{\xi(P)}(T) = -\left.\frac{\partial T}{\partial \varphi}\right|_P \cdot \frac{1}{r_P \gamma_P}$

(2.3) $L_{\eta(P)} : H_{\Omega} \rightarrow \dot{R}$ given by $L_{\eta(P)}(T) = -\frac{1}{\cos(\varphi)} \left.\frac{\partial T}{\partial \lambda}\right|_P \cdot \frac{1}{r_P \gamma_P}$

(2.4) $L_{\zeta(P)} : H_{\Omega} \rightarrow \dot{R}$ given by $L_{\zeta(P)}(T) = T/\gamma_P$ (γ_P: gravity in P).

If we have chosed a norm in H_{Ω} (i.e. the Directlet-norm, cf. example 1), and accordingly a reproducing kernel K(P,Q), together with a set of measurements $\{m_i\}$, then the solution is given on the form

(2.5) $$\tilde{T}(P) = \sum_{i=1}^{n} a_i L_i K(P_i, P),$$

where L_i is the functional corresponding to the measurement m_i.

The constants a_i are determined by

$$(2.6)\quad \begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix} = \begin{Bmatrix} L_1L_1K(P_1,P_1) & \cdots & L_1L_nK(P_1,P_n) \\ & \ddots & \\ L_nL_1K(P_n,P_1) & \cdots & L_nL_nK(P_n,P_n) \end{Bmatrix}^{-1} \begin{Bmatrix} m_1 \\ \vdots \\ m_n \end{Bmatrix}$$

Note, that $L_i(\tilde{T}) = \sum_{j=1}^{n} a_j L_i L_j K(P_i,P_j) = m_i$, i.e. the measurements are maintained exactly.

The uncertainly of the value of a linear functional of this potential is

$$(2.7)\qquad L_PL_P(K(P,P)) - \{L_PL_iK(P,P)\}' \{L_iL_jK(P_i,P_j)\}^{-1} \{L_jL_PK(P_j,P)\} ,$$

(cf. [7] (7-63) and (7-64)). (2.5 - 7 are easily proved indirectly using the reproducing property of the kernel, cf. [6] p.115).

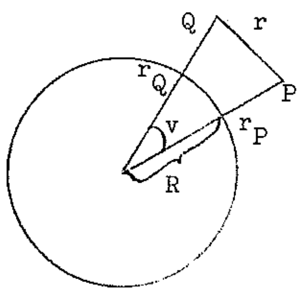
The solution (2.5) to the collocation problem is a linear combination of regular potentials. It is a condition for a unique solution, that the functionals L_i are linearly independent in $H_\Omega{}'$ (the dual space of H_Ω), - or that the potentials $L_iK(P_i,Q)$ are linearly independent in H_Ω. But besides the linear equation (2.6) will always be positively semidefinite and symmetric. This means that the method of Cholesky can be used, when solving the normal equations.

3. Selection of reproducing kernel (= selection of norm).

We will once more consider an

Example 2.

The Greens function of a sphere is $G(P,Q) = \frac{1}{r} + \frac{1}{\ell}$, where



$r = (r_P{}^2 + r_Q{}^2 - 2r_Pr_Q\cos(v))^{\frac{1}{2}}$ and

$\ell = (((r_Pr_Q)/R)^2 - 2r_Pr_Q\cos(v)+R^2)^{\frac{1}{2}}$, (see figure)

It is more difficult to determine the Neumanns function; but we can fortunately easily compute the reproducing kernel. If φ_{nm}, where $|m| \leq n$, $n=0, \ldots, \infty$ is an orthonormaleous system in H_Ω , then

$$K(P,Q) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \varphi_{nm}(P)\cdot\varphi_{nm}(Q).$$

When H_Ω is equiped with the Dirichletnorm, it is easily seen, that the orthonormal system becomes

$$\varphi_{nm} = \frac{1}{(n+1)^{\frac{1}{2}}} \frac{R^{n+\frac{1}{2}}}{r_P^{\,n+1}} \begin{cases} \bar{R}_{nm} \\ \bar{S}_{nm} \end{cases} ,$$

where R_{nm} and S_{nm} are the fully normalized spherical harmonics, cf. [7], (1-73). Accordingly we have

$$K(P,Q) = \frac{1}{4\pi} \sum_{o}^{\infty} \frac{2n+1}{n+1} \frac{R^{2n+1}}{(r_P r_Q)^{n+1}} P_n(\cos(v))$$

$$= \frac{1}{4\pi} \left(\frac{2}{\ell} + \frac{1}{R}\ln\left(\frac{r_P r_Q(1+\cos(v))}{R\ell+R^2-r_P r_Q\cos(v)}\right)\right) .$$

As well-known, the Greens and Neumanns functions are strongly dependent of the border ω, and this is also true for the reproducing kernel, independent of the norm we has selected.

In the case of the Earth we do not know the boundary, the surface, - our job is to determine the surface; in the same way it is not a good idea to look for a solution in H_Ω, as the kernel might be singular on the boundary to which the measurements often will refer. (cf. the formula of Vening Meinesz, which are wrong in their usual form, as the kernel of Stokes is ∞ at the boundary [12]).

We can, as stressed in [5] , find an arbitrary good approximation to the potential in the space $H_{\Omega'}$, where $\Omega' \supset\supset \Omega$. Therefore it is possible to use a so-called "Bjerhammar sphere" totally enclosed in the Earth. The reproducing kernel will then get the form shown in example 2 (on the condition that we use the Dirichlet-norm), where R = the radius of the Bjerhammar sphere.

It is also possible to select the kernel according to our <u>statistical</u> knowledge of the properties of the potential field of the Earth. This presuppose that we can regard the potential field as a <u>stationary</u> stochastic process - which it only is to a very rough approximation.

Do we suppose, that the potential might be represented as a stationary stochastic process, then the measurements will correspond to the outcome of a set of stochastic variables (linear functionals) $X(t)$, $t \in \omega'$, where ω' is a index set. We furthermore suppose $E(X(t)) = X(t)(T) = m_t$.

The covariance function $K(s,t) = E(X(t) \cdot X(s))$ can be estimated either by means of the degree variances or by computing the product sums, parametrisized by the spherical distance and the distance from the origin.

Kaula has estimated the covariance function for $X(t) = L_{\Delta g(t)}$, $\omega' \subset \omega$ [4].

Let us regard the set of all linear combinations of the (bounded) stochastic variables, $L_1(X(t), t \in \omega')$. This linear space can be extended to a hilbert space $L_2(X(t), t \in \omega')$, equiped with norm

$$\|X(t)\|^2 = E|X(t)|^2 .$$

This space is isomorphic to a hilbert space with reproducing kernel $K(s,t)$, [10].

We will then get the best possible prediction of the outcome of a stochastic variabl $Z \in L_2(X(t), t \in \omega')$ given certain measurements, using expressions (2.5) and (2.6). cf. [7] Chapt. 7.

As the stochastic variables, which correspond to the functionals (2.2) - (2.4), only will be elements in $L_2(X(t), t \in \omega')$, when $\omega' = \omega$, $X(t) = L_{\Delta g}$, is it not possible (contrary to [3]) to use a locally determined covariance function when determining an estimate of T.

We therefore choose to determine $K(P,Q)$ so that $L_{\Delta g}\, L_{\Delta g}\, K(P,Q)$ agrees with the empirically determined covariance function of Kaula. This might be done in the following way.

As we presuppose the stochastic process (in spherical approximation) to be stationar (i.e. isotropic on spheres with center in the gravity center of the Earth), the general form of the covariance function must be

$$K(P,Q) = \sum_{i=0}^{\infty} A_i \left(\frac{R^2}{r_P r_Q}\right)^{i+1} P_i(\cos(v)), \quad A_i \geq 0.$$

Kaula has determined 32 so called degree variances of which only 25 can be used. These degree variances of the gravity anomalies σ_i^2 are related to the A_i's by

$$\sigma_i^2 = \frac{(i-1)^2}{R_e^2} A_i \, , \qquad R_e \text{ the mean radius of the Earth.}$$

Principally we do not know anything about the degree variances of higher order, except that $\sum_0^\infty A_i < \infty$, i.e., that the coefficients must go faster than $\frac{1}{i}$ against zero for $i \to \infty$.

Of computational reasons it would be nice to have a model of the covariance function, which is as simple as possible.

If we require $A_i := \text{constant}$, $i > N$, then the kernel can be expressed as

$$K(P,Q) = \sum_{i=0}^{N} A_i' \left(\frac{R^2}{r_P r_Q}\right)^{i+1} P_i(\cos(v)) + \frac{c \cdot R}{\ell} \, , \text{ where } c = \text{a constant number,}$$

and now

$$\ell = \left(1 + \left(\frac{R^2}{r_P r_Q}\right)^2 - 2\,\frac{R^2}{r_P r_Q}\cos(v)\right)^{\frac{1}{2}}$$

(use (2.1) and [7] , (1-81)). Unfortunately we will not get a simple expression for $A_i := \frac{c}{i^2}$, $i > N$. But for $A_i := \frac{c}{(i-1)i}$ we will get reasonable expressions:

$$K(P,Q) = \sum_{i=2}^{N} A_i' \left(\frac{R^2}{r_P r_Q}\right)^{i+1} P_n(\cos(v)) + c r \cdot \left(1 - \ell + (\cos(v) \cdot r - 1) \cdot \ln \frac{2}{1 - \frac{R^2}{r_P r_Q}\cos(v) + \ell}\right)$$

When we has chosed a parametrisizing of the degree variances, we must at last determine R, the radius of the Bjerhammar sphere. This might be done by requiring K(P,Q) to fit the best possible (in a least squares sence) to the values of a empirically determined covariance function. (For example the values of Kaulas [4] table 7). Dependent on different weighting or fitting one will find values of R/R_e between 0.98 - 0.999. Whereas different kernels gives nearly the same prediction results, we will here recommand the so called Kaula's rule of thumb for the degree variances $A_i = c/i^2 \approx c/(i(i-1))$.

4. Forming the normal equations (2.6).

The forming of the normal equations requires the computation of $L_i L_j K(P_i, Q_j)$. As an example we will go through the computation, when we has selected a parametrization $A_i := \frac{c}{i\times(i-1)}$.

We will use the usual trick: $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$ and find closed expressions for

$$\sum_2^\infty \frac{r^{n+1}}{n} P_n(\cos(v)) \text{ and } \sum_2^\infty \frac{r^{n+1}}{n-1} P_n(\cos(v)$$

by means of

$$\frac{1}{\ell} = \sum_{n=0}^\infty r^n P_n(\cos(v)), \text{ where } r = \frac{R^2}{r_P r_Q} \text{ and } \ell = (1-2\cdot r\cdot \cos(v) + r^2)^{\frac{1}{2}}$$

$$\int \frac{dr}{r\ell} = \int (\sum_{n=0}^\infty r^{n-1} P_n \cos(v) dr = \ln \frac{2}{1-\cos(v) r+\ell} + \ln(r) + c_1$$

$$\int \frac{dr}{r^2 \ell} = \int (\sum_{n=0}^\infty r^{n-2} P_n \cos(v)) dr = -\frac{\ell}{r} + \cos(v)(\ln \frac{2}{1-\cos(v)\cdot r+\ell} + \ln(r) + c_2) + c_3$$

If we require the expression to be 0 at infinity (i.e. for r = 0), then

$$r\cdot \sum_2^\infty \frac{r^n}{n} P_n(\cos(v)) = -r^2 \cos(v) + r\cdot \ln \frac{2}{1-\cos(v)\ r+\ell}$$

$$r \sum_2^\infty \frac{r^n}{n-1} P_n(\cos(v)) = r + r\cos(v)\cdot (\ln \frac{2}{1-\cos(v)\cdot r+\ell} - 1) - \ell\cdot r$$

and

$$\sum_2^\infty \frac{r^{n+1}}{n(n-1)} P_n(\cos(v)) = r\cdot (1-\ell + (r\ \cos(v)-1) \ln \frac{2}{1-r\ \cos(v)+\ell})$$

By means of these expressions, we can easily compute the coefficients $L_i L_j K(P_i, Q_j)$:

$$L_{\Delta g} K(P,Q) = \sum_2^N \frac{A_i'\,(i-1)}{r_P}(r)^{i+1} P_i \cos(v) + \frac{c}{r_P} r\cdot(-\cos(v)\cdot r + \ln \frac{2}{1-r\cos(v)+\ell})$$

$$L_{\Delta g} L_{\Delta g} K(P,Q) = \sum_2^N \frac{A_i'(i-1)^2}{r_P r_Q} r^{i+1} P_i \cos(v) + \frac{c\cdot r}{r_P r_Q}(\frac{1}{\ell} - 1 - \ln \frac{2}{1-r\cos(v)+\ell})$$

As

$$L_{\xi(P)} K(P,Q) = \frac{\partial \cos(v)}{\partial \varphi_P} \frac{\partial K(P,Q)}{\partial \cos(v)} \frac{1}{r_P \gamma_P}$$

$$L_{\eta(P)} K(P,Q) = \frac{1}{\cos \varphi_P} \frac{\partial \cos(v)}{\partial \lambda_P} \frac{\partial K(P,Q)}{\partial \cos(v)} \frac{1}{r_P \gamma_P}$$

(4.1) $$L_{\xi(P)} L_{\xi(Q)} K(P,Q) = (\frac{\partial \cos(v)}{\partial \varphi_P} \frac{\partial \cos(v)}{\partial \varphi_Q} \frac{\partial^2 K(P,Q)}{\partial \cos(v)^2} + \frac{\partial^2 \cos(v)}{\partial \varphi_P\, \partial \varphi_Q} \frac{\partial K(P,Q)}{\partial \cos(v)}) \frac{1}{r_P r_Q \gamma_P \gamma_Q} \quad \text{etc.}$$

we will first write the expressions:

$$\frac{\partial K(P,Q)}{\partial \cos(v)} = \sum_{i=2}^N A_i' \, r^{i+1} P_n' \cos(v) + c\cdot r^2 (\frac{r\cdot \cos(v)}{1-r\cos(v)+\ell} + \ln \frac{2}{1-r\cos(v)+\ell})$$

$$\frac{\partial^2 K(P,Q)}{\partial \cos(v)^2} = \sum_{i=2}^N A_i' \, r^{i+1} P_n'' \cos(v) + c\cdot r^3 (1 + \frac{(1+\ell)^2}{\ell(1-r\cos(v)+\ell}) \frac{1}{1-r\cos(v)+\ell}$$

$$\frac{\partial}{\partial \cos(v)}(L_{\Delta g} L(P,Q)) = \sum_{i=2}^N \frac{A_i'(i-1)}{r_p} P_n' \cos(v) + c\cdot \frac{r^2}{r_p}((1+\frac{1}{\ell})(1-r\cos(v)+\ell)^{-1} - 1)$$

Computation of the sums

$$\sum_{i=2}^N B_i P_i \cos(v), \quad \sum_{i=2}^N C_i P_i' \cos(v) \text{ and } \sum_{i=2}^N D_i P_i'' \cos(v)$$

are easily done by "backward" recursion by a generalized hornerian scheme [1], see appendix 1. When computing the higher derivates of the legendre polynomials we will as a by-product get the lower derivates, which simplifies the computation of expressions like (4.1).

5. Conclusive remarks.

It appears from the preceding, that the anomaleous potential might be determined to a certain approximation (in principle as good as wanted), as well from gravity anomalies, deflections of the vertical, satellite observations and so on.

As deflections of the vertical are connected to a datum for example ED 1950, a combination of observations of gravity anomalies and of deflections are difficult, if an absolute position of the quasigeoid is wanted.

This problem is not severe, as we now knows the gravity center of the Earth very much better than before. Thus a formulation of the collocation problem as a least squares problem with elements which determines a datum shift, will possibly give good results.

It is a condition for a (unique) solution, that the determinant of the equations system (2.6) is different from zero. If the observations consists of many and dense observations (i.e. 50 gravity stations within 100 km^2) one could expect a number of (numerical) singularities, when working with 9 decimal digits.

Working with a local sample of observations means that one do not take in regard the lower frequencies of the covariance function. And if one takes away the first 20-30 of the coefficients of the parametrisized global model one will get covariance functions, which looks very much alike locally computed functions.

A change in the radius of the Bjerhammar sphere will also change the covariance function on the surface of the Earth. And it looks like the ratio r has to be very near to 1, if the observations only are a short distance from each other.

The values of the height anomalies must be fixed at a point, if on is to compare solutions computed by means of different samples of observations. When computing the Sout-West Scandinavian quasigeoid, the height anomaly at Røst, Germany has been fixed as to connect it to the of Heitz determined West-German geoid.

In this way it is possible to compute the quasigeoid in parts, and to use different covariance functions in low-lands and in mountainous areas.

A map showing the height anomalies in ED 1950 has been computed from 136 deflections of the vertical in Denmark, Germany and Sweden. The estimated error on the difference between the height anomalies of Røst (Germany) and Landskrona (Sweden) is less than 10 cm.

Note [8] and [9] where Moritz gives an exposition of the collocation method.

Literature.

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appendix 1.

```
real procedure ddp(f,degree,angle);
integer degree; real angle; array f;
begin comment The procedure computes the value of the second
   derivate of a function expanded in a (finite) sum of legendre-
   polynomials. The value is assigned to the name of the procedure.

   The degree of the polynomial of highest order is stored in the
   variable degree, the coefficients of the polynomials in the
   array f, and the argument of the polynomial in the variable angle.

   If the value of the first derivate or of the function itself are
   wanted, the variables da0 and a0 must be declared as exterior
   variables.

   Reference: Clenshaw, Math. Tables and other aids to computation,
   Vol. 9, 1955, p. 118 - 120;

   integer i;
   real a0,a1,a2,da0,da1,dd0,dd1,s1,s2,sc,cosangle;
   a0:= a1:= da0:= da1:= dd0:= dd1:= 0;
   cosangle:= cos(angle); s1:= 2-1/(degree+2);

   for i:= degree step -1 until 0 do
   begin
      s2:= s1-1; s1:= 2-1/(i+1);
      sc:= s1×cosangle;

      a2:=  a1; a1:=  a0; a0:=  a1×sc -a2×s2+f[i];
      a2:= da1; da1:= da0; da0:= da1×sc-a2×s2+a1×s1;
      a2:= dd1; dd1:= dd0; dd0:= dd1×sc-a2×s2+2×da1×s1;
   end i - loop;

   ddp:= dd0;
end of procedure ddp;
```