GEOPHYSICAL RESEARCH LETTERS, VOL. 1, NO. 9, PAGES 821-824, SEPTEMBER 1984

THE PRECISE COMPUTATION OF GEOID UNDULATION DIFFERENCES WITH COMPARISON TO RESULTS OBTAINED FROM THE GLOBAL POSITIONING SYSTEM

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Abstract. Ellipsoidal height differences have been determined for 13 station pairs in the central Ohio region using measurements made with the Global Positioning System. This information was used to compute geoid undulation differences based on known orthometric heights. These differences were compared to gravimetrically computed undulations (using a Stokes integration procedure and least squares collocation having an internal r.m.s. agreement of ±1 cm in undulation differences). The two sets of undulation differences have an r.m.s. discrepancy of ±5 cm while the average station separation is of the order of 14 km. This good agreement suggests that gravimetric data can be used to compute accurate geoid undulation differences that can be used to convert ellipsoidal height differences obtained from GPS to orthometric height differences.

Introduction

The use of the satellites of the Global Positioning System (GPS) has enabled the very accurate determination of the relative position of points that are separated by distance on the order of 100 km (Collins, 1984). The coordinate differences can be given in a rectangular coordinate system, or in a local system such that latitude, longitude, and ellipsoidal height differences (Δh) are given.

Ellipsoidal heights are heights measured from a defined reference ellipsoid. Orthometric heights are measured from a reference equipotential surface, the geoid. The difference between the two heights is dependent on the separation between the geoid and the reference ellipsoid, that is the geoid undulation. For most mapping applications, orthometric height differences (ΔH), not ellipsoidal height differences are needed. Such differences can be obtained if geoid undulation differences (ΔN) are computed using equation (1):

$$(H_2-H_1) = (h_2-h_1) - (N_2-N_1) \qquad (1)$$

The computation of N_2-N_1 requires fairly dense gravity coverage in the area of the stations.

In order to test this technique we have used a set of data taken by Geo/Hydro with the MACROMETER Model V-1000 with the results (Geo/Hydro 1983) provided to us by the Franklin County (Ohio) Engineers Office for two small networks in Central Ohio. The data supplied included the ellipsoidal height differences, and the orthometric heights of the points in the nets. We have computed the undulation differences that were used in equation (1) to check the accuracy of the observations and computations.

Networks

The first network consisted of thirteen stations while the second network was of nine stations. Of these stations, the orthometric elevations were known for nine stations in the first network and four in the second. The vertical control was of varied accuracy (first order, etc.) although the final results did not show any dependence on this accuracy. Based on the stations that were occupied by the Macrometers, and for which orthometric heights were available, it was possible to construct thirteen lines for which the ellipsoidal and orthometric height differences were available for the calculation of the geoid undulation differences. These station pairs and the Macrometer implied undulation differences are given in Table 1. Additional Macrometer measurements over selected lines in this area were also made by the National Geodetic Survey. The undulation differences implied by these computations (Goad, private communication, 1984) are also shown in Table 1. For two of the lines the agreement is quite good, but for the other two the discrepancies reach 25 cm.

Gravimetric Undulation Differences Through the Stokes Equation

We will compute the geoid undulation at the points of the network using gravity anomalies, Δg, in a cap (σ_c) surrounding the area, and a set of potential coefficients. In a spherical approximation this computation can be written in the form (Rummel and Rapp, 1976):

$$N = \frac{R}{4\pi\gamma} \iint_{\sigma_c} (\Delta g + \delta g_A)\, S(\psi)\, d\sigma + \frac{R}{2\gamma} \sum_{n=2}^{n_{MAX}} Q_n(\psi_0)\, \Delta g_n(\bar{\phi}, \lambda) \qquad (2)$$

where
- R is the mean earth radius;
- γ is the average value of gravity;
- Δg is the free-air anomaly;
- δg_A is the atmospheric correction;
- $S(\psi)$ is the Stokes' function;
- Q_n is the Molodensky truncation function;
- ψ_0 is the radius of the cap in which the anomalies are given;

Paper number 4L6206.
0094-8276/84/004L-6206$03.00

TABLE 1. Geoid Undulation Differences (cm)

Line	To	From	Dist.(km)	ΔN(MACR)	ΔN(NGS)	ΔN(STK)	ΔN(Coll.)
1	Clark	Rhodes	10	-7		-3 ±4	-4
2	Clark	18-83	11	-19	-19	-14 ±5	-15
3	Rhodes	18-83	4	-13		-11 ±2	-11
4	18-83	Britton	13	-19		-21 ±6	-21
5	Clark	Hoover	10	19		12 ±4	15
6	RNA	Glenrest	4	2		3 ±2	2
7	Glenrest	Reynoldsburg	2	0		0 ±1	1
8	Reynoldsburg	Livingston	1	1		0 ±10	0
9	Shannahan	18-83	22	-25	-14	-11 ±9	-12
10	Neil	18-83	24	11		17 ±10	18
11	Britton	Jackson	24	1	-2	0 ±10	-1
12	Jackson	Smith	14	32	57	50 ±7	53
13	Hoover	Smith	35	-13		3 ±14	1

$g_n(\bar{\phi}, \lambda)$ is the n^{th} degree harmonic of the gravity anomalies, at latitude $(\bar{\phi})$ and longitude (λ). This is computed from the given potential coefficients;

n_{MAX} is the maximum degree of the potential coefficients being used.

Equation (2) neglects the zero order undulation of the geoid which is needed to refer the undulation to a specific ellipsoid. In taking the undulation difference this term cancels out. The error in the spherical approximation of the integral in (2) can be removed using the equations of Rapp (1981a). However, this effect is nearly constant in a small area and therefore cancels in taking undulation differences in these applications. In writing the integral we have also neglected the effect of the terrain (Rapp and Wichiencharoen, 1984). In Ohio this is small and would cancel for the undulation differences.

The errors in undulations (and undulation differences) computed from equation (2) will depend on the errors in the gravity anomalies, the cap size, the errors in the potential coefficients, and the degree (n_{MAX}) of truncation of the potential coefficient series. Error estimates can be made with reasonable approximation using the equations given by Christodoulidis (1976). Tests indicated that the use of gravity data on a 2' grid, in a cap of about 2°, with the potential coefficients of Rapp (1981) given complete to degree and order 180, should give undulation differences to an accuracy of about ±8 cm for the average spacing (14 km) of the points in this test. The accuracy for each undulation difference is dependent on the distance between relevant staions.

In the area in which the free-air gravity anomalies were needed, there were available 13190 point values referred to the gravity formula of the Geodetic Reference System 1967. For all subsequent computations, these anomalies were converted to the Geodetic Reference System (GRS80) and all constants used in the computations were those of GRS80. A preliminary regularization of this data was obtained by selecting one point (closest to the center) in each 2'x 2' (about a 3½ x 3½ km block) block in the area. The resultant data consisted of 8862 point free-air anomalies that had an average accuracy (s.d.) of ±1.5 mgal and signal variance of 268 $mgal^2$. (The Bouguer anomaly signal variance was 227 $mgal^2$).

For the integration of the Stokes integral in (2) we created a uniform 2'x 2' grid of anomalies (an array of 192 x 150 elements) which were estimated at the geographic center of each cell. The predictions were first made from the Bouguer anomalies using the five closest (to the center) known anomalies. The predicted Bouguer anomalies were then converted to free-air anomalies using the elevation at the prediction point obtained from a digital terrain model given at a 30"x 30" intervals. This procedure was used to reduce prediction errors caused by the correlation of free-air anomalies with elevation. The anomaly estimation was carried out using least squares prediction techniques with the covariances computed from the Tscherning-Rapp (1974) degree variance model with the first 36 degrees removed and a variance scaled to be that of the Bouguer anomalies.

For the Stokes' integration, the Stokes' function was first tabulated at a spherical interval of 1" out to a spherical distance of 3°. The evaluation of $S(\psi)$ for each block was done by numerical integration. To reduce integration errors, each block was subdivided into elements depending on the separation (ψ) of the point and the cell. For ψ less than 4', the number of subdivisions was 64; for $4'<\psi\leq 8'$ there were 16 subdivisions; for $8'<\psi\leq 12'$, the subdivisions were 4 and for $12'<\psi\leq 2°$, the evaluation was made at the center point.

Using the potential coefficients to degree 180 of Rapp (1981b) and the above gravity data, the geoid undulations were computed on a 2'x 2' grid and contoured (see Figure 1), and also computed at each point of the Macrometer network. The undulations refer to a geocentric ellipsoid whose flattening is 1/298.257 and whose equatorial radius is 6378136 ±1 m. The undulation differences were computed for each available line and are given in Table 1. The standard deviations of the differences were estimated using the procedures of Christodoulidis (1976) con-

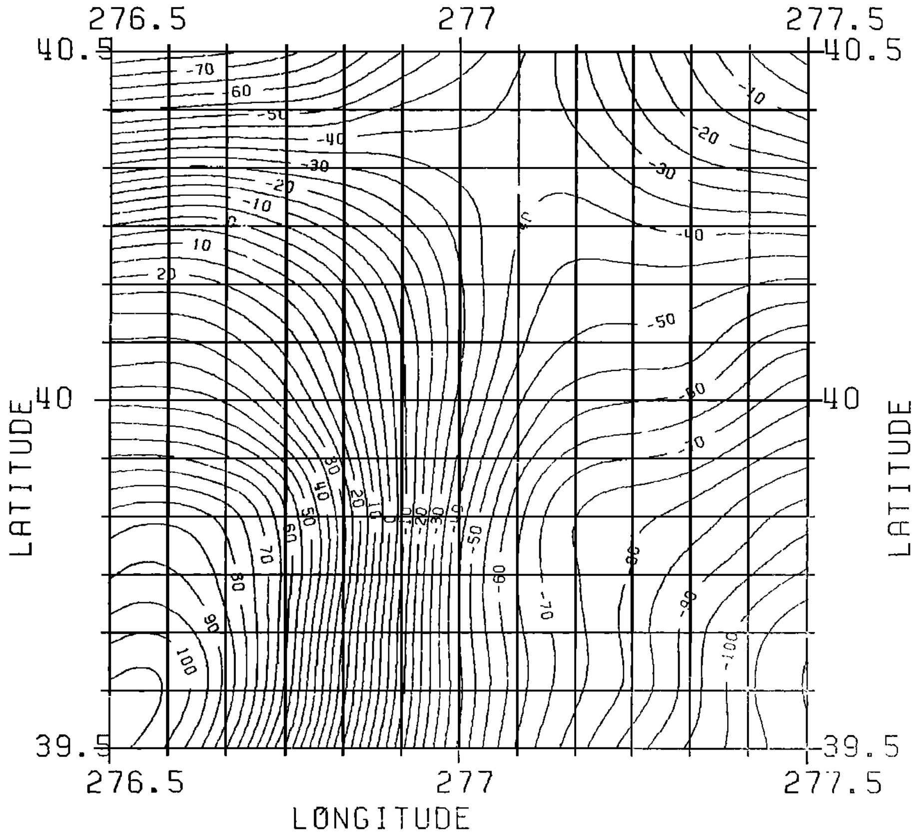

Fig. 1. Geoid in Central Ohio (add -34 m to obtain full value) (contour interval is 5 cm)

sidering the errors in the potential coefficients, the effect of neglected higher degree coefficients, and the effect of the discretization of the Stokes' integral. The effect of anomaly errors was not specifically computed due to the error correlation of the individual values. Additional study is needed for this to provide a reliable accuracy estimate for the undulation differences. These values are shown in Table 1.

Gravimetric Undulation Differences Through Least Squares Collocation

Using least squares collocation, geoid undulations have been predicted directly from gravity anomalies Δg_i, i=1, ..., n using

$$N = \sum_{i=1}^{n} a_i \, \mathrm{cov}(N, \Delta g_i) \qquad (3)$$

$$\{a_i\} = \{\mathrm{cov}(\Delta g_i, \Delta g_j) + e_{ij}\}^{-1}\{\Delta g_j\} \qquad (4)$$

where $\{\mathrm{cov}(\Delta g_i, \Delta g_j)\}$ is the n x n matrix of covariances between the observed gravity anomalies, e_{ij} the error variances and covariances of the observations (e_{ii} = 1 mgal, e_{ij} = 0) and $\mathrm{cov}(N, \Delta g_i)$ is the covariance between the geoid undulations (in a point P) and the observed gravity anomaly Δg_i.

The anomalies used were given with respect to a reference field complete to degree and order 180 (Rapp, 1978). Hence, the contribution from the reference field had to be added subsequently to the predicted value of N.

Based on 414 anomalies in the area bounded by $39.25 < \phi° < 40.75$; $276.25 < \lambda° < 277.75$ empirical covariance function values were estimated. The following model was used to represent the estimated covariances:

$$\mathrm{cov}(\Delta g_P, \Delta g_Q) = \sum_{i=2}^{\infty} \sigma_i \left(\frac{R_B^2}{r_P r_Q}\right)^{i+2} P_i(\cos \psi_{PQ}) \qquad (5)$$

σ_i, $i \leq 170$ representing the error in the used potential coefficients

$\sigma_i = \frac{225 \text{ mgal}^2 \, (i-1)}{(i-2)(i+24)}$, $i > 170$

R_B = 6370 km, the radius of the so-called Bjerhammar-sphere

P_i the Legendre polynomials, and

ψ_{PQ} the spherical distance between P and Q.

The corresponding cross-covariance function for geoid undulations and gravity anomalies is then simply obtained by multiplying the degree-variances σ_i by $r/((i-1)/\gamma)$, where γ is the normal gravity.

The 414 anomalies were then used for the prediction of the geoid undulations in the two Macrometer networks. The relevant differences between the values are found in Table 1.

Error estimates were computed for the prediction of the geoid undulations. They were all close to ±50 cm. The estimates for the error of the undulation differences are approximately 0.5 cm multiplied by the distance between the points in km. This gives values very close to those listed for the Stokes' equation in Table 1.

An Algol version (Tscherning, 1978) of the FORTRAN IV program (Tscherning, 1974) was used for all computations.

Conclusions

We first note the excellent agreement of the undulation differences computed through a Stokes' integration and least squares collocation. Although computed in an independent way, the results generally agree to a cm. The agreement with the Macrometer results is not as good but it is still excellent for most lines. The root mean square difference between the Macrometer ΔN values from Geo-Hydro is ±5 cm. Two lines (number 9 and number 12) show much larger differences being 14 cm and 25 cm respectively. However, when the results from NGS are considered, the gravimetric and Macrometer (NGS) ΔN values agree to several cm.

The results of this study can be viewed in several ways. First we see that ellipsoidal elevation differences appear to be determined (for the most part) at the ±5 cm level from the Macrometer results for stations that are separated up to 35 km. Second we have shown that gravimetric data can be used to accurately convert ellipsoidal height differences to orthometric height differences to an accuracy of ±5 cm over lines of the length considered here provided sufficient gravity information exists in an area. We therefore have shown that it is possible to compute geoid undulation differences to an accuracy of ±5 cm or better for lines tested in this paper. It is then possible to directly use the Macrometer results to get orthometric heights needed for vertical control. With this procedure it may not be necessary to occupy several sites with known orthometric heights with the Macrometer to determine an undulation difference surface to be used for interpolation purposes as suggested by Collins (1984). The obvious benefit is the cost and time reduction in the field operations of the Macrometer.

Acknowledgements. The original report of Geo-Hydro's work, and additional information, was provided to us by Dean Ringle, of the Franklin County Engineer's Office. The research described in this paper was supported in part, by NASA Grant NGR 36-008-161, administered by the Goddard Space Flight Center. This is also a contribution to a project supported by NATO Grant 320/82.

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(Received May 30, 1984;
revised June 27, 1984;
accepted June 28, 1984.)