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EFFECTS OF THE LACK OF ADEQUATE HEIGHT AND GRAVITY DATA ON THE USE OF POSITIONS DETERMINED BY SPACE TECHNIQUES IN DEVELOPING COUNTRIES

C.C. TSCHERNING

GEODAETISK INSTITUT
GAMLEHAVE ALLE 22
DK-2920 CHARLOTTENLUND, DENMARK

ABSTRACT. Positions determined by space techniques give the height above the reference ellipsoid (h) and not the height above mean sea level (H). This makes these positions difficult to use in developing countries, where frequently sufficient height and gravity data lack, which otherwise makes it possible to determine H as the difference between h and the height anomaly (geoid undulation), ζ.

The error in determining ζ from height and gravity data depends on the variation of the gravity field, the spacing of the available gravity data and the distance to the nearest levelling point. A formula is given expressing the error in terms of parameters describing this dependency.

1. Introduction

In this paper we will deal with a problem, which typically may arize in a developing country, but which also may be quite important for an undeveloped area associated with or being a part of a developed country. Therefore, in this paper a developing country will be any state which territory include an area with scarce or lacking horizontal and vertical geodetic control points. A typical area could be parts of Greenland, which up to the mid-seventies had large areas without any geodetic control. Still, in Greenland, there are no areas, where precise levelling has been made, and it will probably never be done. Here trigonometric and barometric levelling are still acceptable methods for hegiht determination.

The advent of satellite positioning systems (NNSS or NAVSTAR/GPS) has made it possible to provide the geodetic control in a fast and uncomplicated manner. Commercial firms or even the Development Agency of a developed country providing assistance to a developing country may want to take advantage of such systems in order to establish e.g. the geodetic control necessary for a mapping project.

However, the use of this new technology must be done in such a way that the developing country gets a product which viewed scientifically and professionally is as good as possible: It is not professionally correct to use very large amounts of money (mainly on logistical activities) establishing a geodetic control network using satellite technique, when a much more useful product could have been obtained with a little extra effort. How and why such a situation may occur and how it may be avoided will be explained in the following.

It is probably now obvious to everybody, that the use of the satellite technology gives rize to a new kind of problems: The position systems determine the geometric position of a point in space, i.e. a set of Cartesian coordinates (X,Y,Z). From these quantities, geodetic latitude and longitude, and height (h) above an adopted reference ellipsoid may be computed.

On maps, and for a number of engineering purposes, the height above mean sea level is needed. Any height system, which can be used to show in which direction water will flow is satisfactory for practical purposes. It would be most correct to use potential differences, but orthometric (H) or normal heights (H^*) will both fulfil the need. Let us here suppose, that it is the normal heights system, which has been adopted.

The difference $\zeta = h - H^*$ is then the height anomaly, which we must determine in order to make satellite determined positions useful for height determination ζ is related to the anomalous gravity potential T through Bruns' equation

$$\zeta = T/\gamma , \qquad (1)$$

where γ is the normal gravity. (The equation is simply the first terms in a Taylor expansion, where the choice of height system assures, that the zero order term is zero).

The determination of ζ is therefore a part of the more general problem of gravity potential determination. In fact ζ is determined by solving the basic geodetic boundary value problem.

In order to solve this problem for an area, height and gravity data are needed. The result will not so much depend on which method we use in order to solve the boundary value problem, as on which data are available. (However, the use of some methods are restricted to specific data types).

The purpose of this paper is therefore to describe the effect of the lack of adequate height and gravity data, by giving approximative rules for which data spacing is needed in order to determine the height anomaly with a given mean error.

The spacing needed will depend on the local gravity field variation, and in section 2 we will discuss how this variation can be expressed through the so-called empirical covariance function. We will also describe how this function can be used to determine the estimates of the mean errors.

In section 3 some numerical examples will be given, which illustrate the use of the estimates, and in section 4 the most useful estimates are expressed by a few parameters, which characterize the gravity variation.

2. Expressions of the mean square error as a function of the covariance function

Different, but mathematically "correct" methods for gravity field approximation ("determination") can be expected to give nearly identical results if the same data are used, see e.g. Tscherning (1981). This means that we may use the error estimation procedure associated with least squares collocation in order to estimate the magnitude of the errors quite independently of the method, which subsequently will be selected as the most convenient.

Recall (cf. e.g. Moritz (1980)) that in least squares collocation, the empirical covariance function cov(T(P), T(Q)) associated with the anomalous potential, T, can be used to derive covariances between other quantities. (P, Q are two points in space). This is done by applying the linear(ized) functionals associated with these quantities on the covariance function, e.g.

$$\mathrm{cov}(\Delta g(P),\ T(Q)) = -\frac{\partial}{\partial r}\Big|_P (\mathrm{cov}(T(P),\ T(Q))) - \frac{2}{r}\mathrm{cov}(T(P),\ T(Q)), \quad (2)$$

where $\Delta g(P)$ is the gravity anomaly in the point P and r is the distance of P from the origin.

Let us suppose, that we have a set of observations x_i, $i=1,\ldots,N$, with associated functionals L_i and errors (of observations) n_i. We also suppose that the "noise" n_i has a (diagonal) variance-covariance matrix $\{\sigma_{ij}\}$. Then

$$x_i = L_i(T) + n_i. \quad (3)$$

Let us regard a quantity L(T) and its estimate $L(\tilde{T})$ computed based on the N observations. The mean square error of the difference is then

$$\sigma^2(L(\tilde{T}) - L(T)) = \sigma^2(L) - \{ cov(L,L_i) \}^T \{ cov(L_i,L_j) + \sigma_{ij} \}^{-1} \{cov(L_j,L)\}, \quad (4)$$

where we have put $\sigma^2(L) = cov(L(T), L(T))$, $cov(L,L_i) = cov(L(T), L_i(T))$ etc.

The same expression is valid if we from T have subtracted the contribution of a set of potential coefficients or of the topography. This only means, that we from our data must subtract the same effects, and add the effects back on our computed quantities. Also the covariance function must be estimated based on these "reduced" quantities.

For the type of area we have in mind, where no or little gravity data is available, the potential coefficient solutions complete to degree and order 180 will probably not give much better results than a solution complete to degree and order 36 like GEM10B (Lerch et al., 1981). This is because the 180 degree and order solutions have not received any short wavelength contribution from this area. We will therefore in the following use a quite pessimistic "synthetic" covariance function based on the global covariance model described in (Tscherning and Rapp, 1974).

$$cov(T(P), T(Q)) = \sum_{i=2}^{\infty} \sigma_i \left(\frac{R_E^2}{r\,r'}\right)^{i+1} P_i(\cos\psi_{PQ}) \;, \quad (5)$$

where r' is the distance of Q from the origin, P_i are the Legendre polynomials, ψ_{PQ} is the spherical distance between P and Q, R_E is the mean earth radius and σ_i are constants, the so-called degree-variances.

The values σ_i, $i = 2\text{-}36$, represents the variation in T due to the error in the potential coefficients and

$$\sigma_i = \frac{425.28 \text{ mgal}^2 \cdot R_E^2}{(i-1)(i-2)(i+24)} \left(\frac{R_B}{R_E}\right)^{2i+2} \;, \qquad i > 36 \quad (6)$$

where R_B is the radius of the so-called Bjerhammar sphere, $R_E - R_B = 1220$ m. The values of covariance functions $cov(\zeta(P),\zeta(Q))$, $cov(\zeta(P), \Delta g(Q))$ and $cov(\Delta g(P), \Delta g(Q))$ for $r = r' = R_E$ and varying spherical distance are given in Table 1.

This covariance function may be too rough for an area with a smoothly varying gravity field, but will probably not be too smooth for a mountainous area, if the topographic effects have been taken properly into account.

Table 1. Values of covariance functions for varying spherical distance ψ computed using eq. (5) and (6).

ψ	Height anomaly m^2	Covariance between height and gravity anomalies m x mgal	Gravity anomaly $mgal^2$
0° 0'	5.61	56.62	1389.72
0 5	5.58	54.61	1013.61
0 10	5.51	51.64	784.54
0 15	5.41	48.60	641.88
0 20	5.29	45.63	540.09
0 25	5.15	42.74	461.97
0 30	5.00	39.97	399.25
0 35	4.83	37.30	347.34
0 40	4.66	34.73	303.45
0 45	4.48	32.28	265.71
0 50	4.30	29.92	232.86
0 55	4.11	27.67	203.99
1 0	3.92	25.52	178.42
1 10	3.54	21.50	135.18
1 20	3.16	17.83	100.21
1 30	2.79	14.51	71.58
1 40	2.43	11.51	47.96
1 50	2.09	8.82	28.43
2 0	1.76	6.41	12.28
2 10	1.46	4.27	-1.04
2 20	1.18	2.39	-11.93
2 30	0.92	0.75	-20.76
2 40	0.68	-0.67	-27.78
2 50	0.47	-1.88	-33.25
3 0	0.29	-2.89	-37.35
3 10	0.12	-3.72	-40.26
3 20	-0.02	-4.38	-42.13

We are then able to express the relationship between a certain data spacing (d) and the mean error using eq. (4). However, it would be more convenient, if we could describe this relationship through a few parameters, which characterize the covariance functions as done in (Forsberg and Tscherning, 1981). In order to see how this could be done, we will first give some examples, which will show how eq.(4) can be applied.

3. Numerical examples

In order to illustrate the use of eq. (4), a number of numerical experiments have been made using a configuration of observation points, which will be quite typical for an area of the type we discuss here.

Let us suppose, that the distance to the closest levelling point (P_o) is less than $5^{\circ} \approx 550$ km. This point could also be a datum point, where the height (H_o^*) have been fixed by convention. If we by satellite technique determine the ellipsoidal height (h_o) in this point, then we can find the height anomaly in the point. $\zeta_o = H_o^* - h_o$.

Then we can use this value to find a correction to the height anomalies obtained in other points based on potential coefficients and topographic heights used to calculate topographic effects. In Figure 1 we see the error of these extrapolated values as a function of the error σ_o of ζ_o and of the spherical distance ψ.

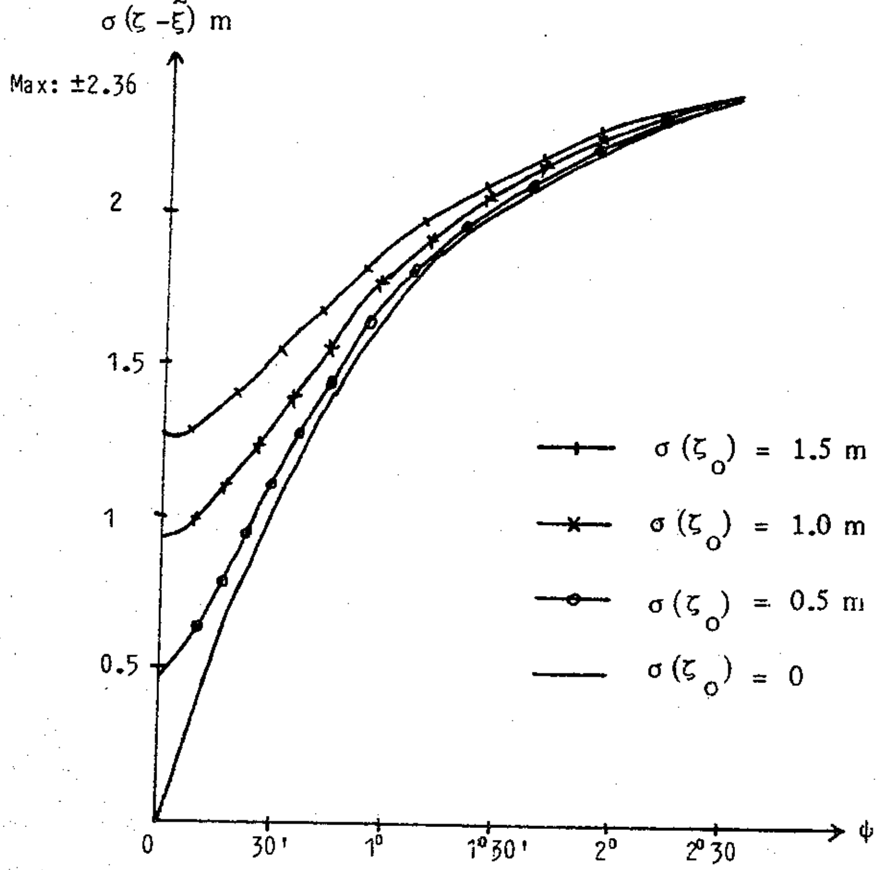

Figure 1. Root mean square error of a predicted height anomaly as a function of the distance ψ from the closest levelling point for four different values of the error $\sigma(\zeta_o)$ of the height anomaly given in the point.

Note, that the maximal error is attained in the distance equal to the distance where cov(ζ(P),ζ(Q)) first time becomes equal to zero, cf. Table 1.

The addition of gravity data will make the error decrease. This dependence is illustrated in Figure 2 for varying spacing of the gravity data. The values are supposed to have errors below 2 mgal, (mainly due to errors in the (preliminary) height determination). In all cases, the data were spaced equidistantly in an area of extend 5° in longitude and 2° in latitude. The point P_o with an observed value ζ_o was supposed to be located in the middle of the eastern boundary of the area, see Figure 3. The values shown in Figure 2 are for a line running east-west through the middle of the area. The error will naturally increase as we arrive to the western boundary, as also seen from Figure 2. The increase of the error as we move away from the middle of the area is shown in Figure 3 for a gravity data spacing d = 20'. Some examples were also computed, where no observed height anomaly were used. The result was similar to what is shown in Figure 2 for the middle and the western part of the area.

What is most significant is the decrease in the magnitude of the error which is obtained using just very few gravity values. Height anomalies are determined better than ±0.75 m using gravity data spaced 15' ≈ 27 km apart. – Or by having one gravity value per 700 km².

This result correspond very well to what has been achieved in Greenland, see Forsberg and Madsen (1981, Table 2 and 3).

4. The error estimates expressed by a few parameters

The error estimates shown in Figure 1-3 depends on the adopted covariance function, the data spacing d and the error $\sigma(\zeta_o)$. If the covariance function values given in Table 1 are shown graphically as curves, it is seen, that they may easily be approximated by simple polynomials. cov(ζ(P),ζ(Q)) and cov(ζ(P), Δg(Q)) may be approximated using a straight line for ψ less than the distance to the first zero point. cov(Δg(P), Δg(Q)) may be approximated using two second order polynomials. One valid for ψ less than ψ_1, the distance in which the covariance function attains half of its value for $\psi = 0$ (This distance is called the correlation distance). The other polynomial is then valid for $\psi_1 < \psi < \psi_o$, where ψ_o is the distance out to the first zero point. In (Forsberg and Tscherning, 1981) the following expression was used for $\psi < \psi_1$.

$$\mathrm{cov}(\Delta g(P),\ \Delta g(Q)) = C(\psi) \simeq C_o(1-\tfrac{1}{2}(\psi/\psi_1)^2)\ , \qquad (7)$$

for $r = r' = R_E$ and $C_o = C(0)$.

Let us for $K(\psi) = \mathrm{cov}(\zeta(P), \zeta(Q))$ introduce corresponding quantities: $K(0) = K_o$ and ζ_1 is the correlation distance. Then

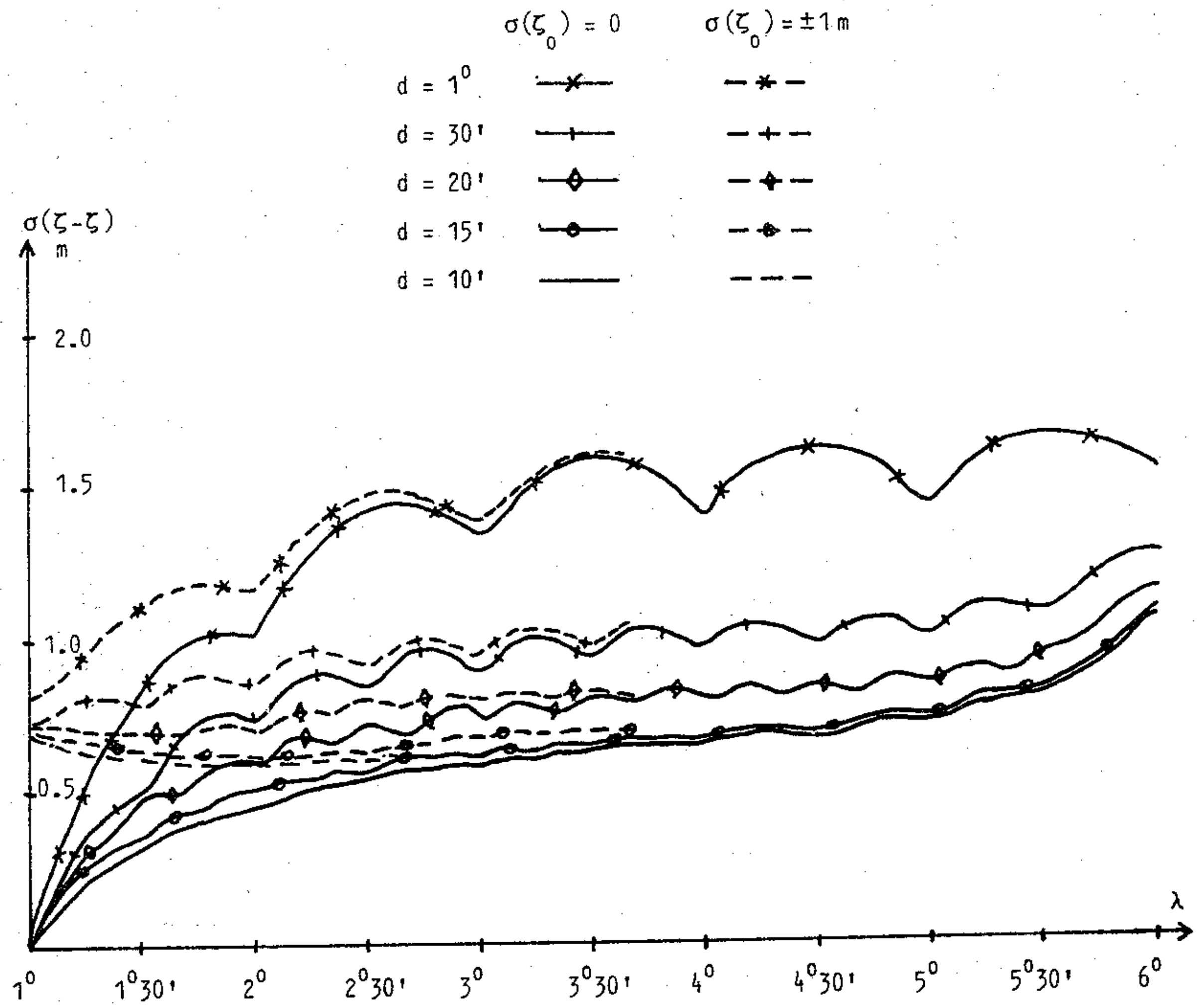

Figure 2. Root mean square error of predicted height anomalies located on the line $1° < \lambda < 6°$, $\varphi = 2°$ computed based on (1) regularity distributed gravity data in the area $1° < \lambda < 6°$, $1° < \varphi < 3°$ for varying spacing (d) between the points and on (2) one height anomaly ζ_0 located in P_0 with $\varphi = 2°$, $\lambda = 1°$ having $\sigma(\zeta_0) = 0$ or ±1 m. We see, that the error of prediction for points with $\lambda > 5°$ could have been improved, if gravity data outside the area ($\lambda > 6°$) had been available. Also note, that from a distance of 1° from P_0 the "observed" height anomaly does not help in improving the result. Outside this area, it is the gravity anomalies which contribute most.

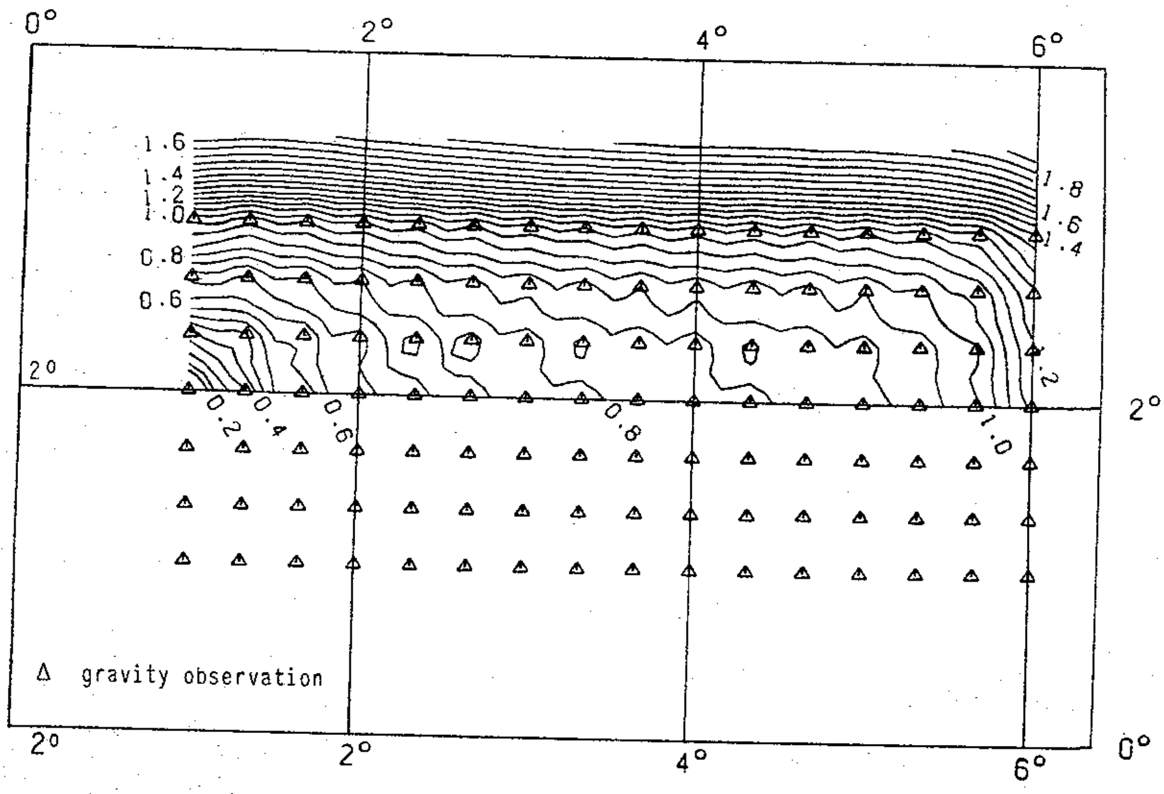

Figure 3. Height anomaly root mean square prediction error corresponding to a gravity data spacing of $d = 20'$. One height anomaly held fixed in the point P_0 with $\varphi = 2^0$ and $\lambda = 1^0$. (Contour interval 0.05 m).

$$K(\psi) \approx K_0(1 - \frac{1}{2}\psi/\xi_1) \qquad (8)$$

In the first numerical example, we only used one observation. Here eq. (4) becomes for ζ located the distance ψ from the known point P_0 with height anomaly ζ_0, $\sigma_0 = \sigma(\zeta_0)$,

$$\sigma(\zeta-\tilde{\zeta})^2 = K_o - K(\psi)^2/(K_o + \sigma_o^2)$$

$$= K_o\left(1 - \frac{K_o}{K_o+\sigma_o^2}\left(1 - \frac{1}{2}\,\psi/\xi_1\right)^2\right)$$

$$= \frac{K_o}{K_o+\sigma_o^2}\left(\sigma_o^2 + K_o\left(\psi/\xi_1 - \frac{1}{4}(\psi/\xi_1)^2\right)\right). \qquad (9)$$

Using ξ_1 1°30' and $K_o = 5.61\ m^2$ it is easily seen that (the square-root of) this expression corresponds very well to the values shown in Figure 1.

The error curves in Figure 2 are more difficult to express as simple functions of the parameters which characterize the covariance functions. In (Forsberg and Tscherning, 1981) is given an expression for the error $\sigma(\varepsilon)$ for predicted deflections of the vertical in terms of the data spacing (d), valid for $d < \psi_1$. Here

$$\sigma(\varepsilon) \approx \frac{C_o^{\frac{1}{2}}}{6.6}\, 0.3\ d/\psi_1\ , \qquad (10)$$

where C_o is given in mgal and d and ψ_1 in the same units.

As a rough estimate for the error in the height anomaly difference $(\sigma(\zeta_o) = 0)$, we then have

$$\sigma(\tilde{\zeta}_P - \zeta_o) \approx \sigma(\varepsilon) \times 0.005 \times D, \qquad (11)$$

where D is the distance from P_o to P given in km and $\sigma(\varepsilon)$ is given in arcsec. For the covariance function of Table 1 we have $\psi_1 = 13'$ and C_o = 1390 mgal². Hence

$$\sigma(\varepsilon) \approx 5''\!.6\ (0''\!.3\ \frac{d}{13'}) \qquad (12)$$

and then for d = 15'

$$\sigma(\tilde{\zeta}_P - \zeta_o) \approx 0.001\ m \times D(km).$$

This equation can only be used for $D < \psi_1$, but here it agreed quite well the result shown in Figure 2.

From Figure 2 we see, that for points in the middle of the area, the error does not go linearity to zero as a function of the spacing d. In Figure 4 is this error shown as a function of d, and we see that the error decreases linearity, but towards a constant. This also holds for points at the boundary most far away from P_o. The slope of the line is probably related to the slope and maximal value of the function cov($\zeta(P),\Delta g(Q)$), but further investigations are needed in order to find the correct relationship. The magnitude of the constant (and the fact that it is non-zero) depends on the local area covered by the gravity observations, and the distance of the point of prediction from the boundary of the area. Also here further investigations are needed in order to relate these quantities to each other.

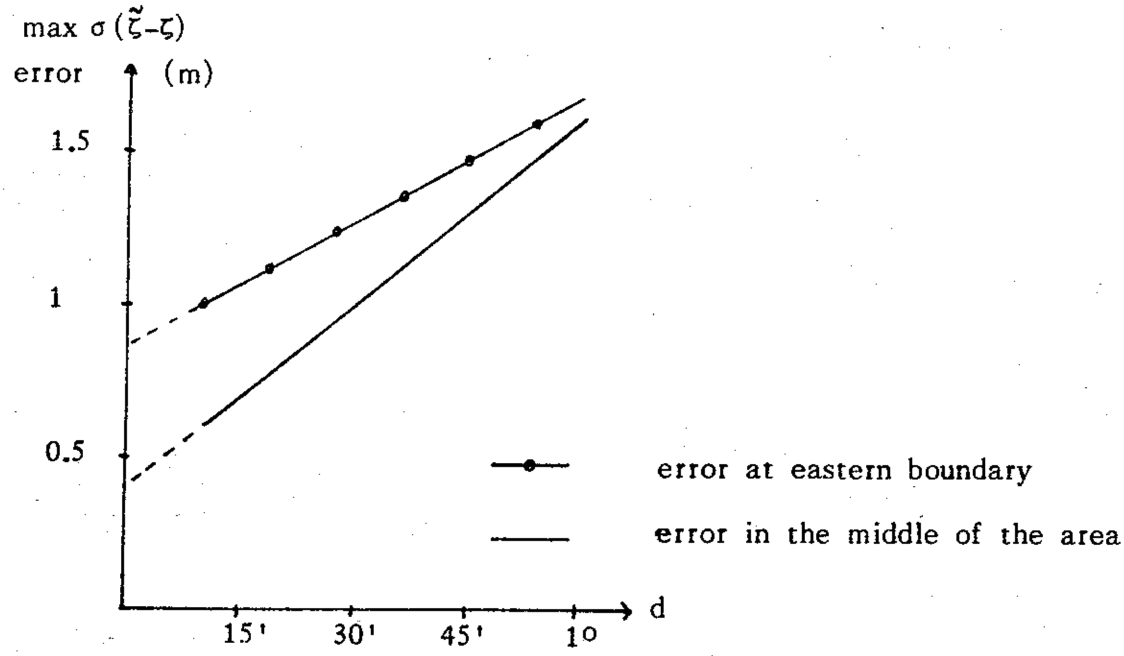

Figure 4. Relationship between the gravity spacing d and the maximal height anomaly root mean square error of prediction in the middle of the area and at the middle of the eastern boundary, cf. Figure 3. Values from Figure 2 are used.

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