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EVALUATION OF ISOTROPIC COVARIANCE FUNCTIONS OF TORSION BALANCE OBSERVATIONS

Abstract

Torsion balance observations in spherical approximation may be expressed as second–order partial derivatives of the anomalous (gravity) potential, T ,

$$T_{13} = \frac{\partial^2 T}{\partial x_1 \, \partial x_3}, \; T_{23} = \frac{\partial^2 T}{\partial x_2 \, \partial x_3}, \; T_{12} = \frac{\partial^2 T}{\partial x_1 \, \partial x_2}, \; T_\Delta = \frac{\partial^2 T}{\partial x_1^2} - \frac{\partial^2 T}{\partial x_2^2},$$

where x_1 , x_2 *and* x_3 *are local coordinates with* x_1 *"east",* x_2 *"north" and* x_3 *"up." Auto– and cross–covariances for these quantities derived from an isotropic covariance function for the anomalous potential will depend on the directions between the observation points. However, the expressions for the covariances may be derived in a simple manner from isotropic covariance functions of torsion balance measurements. These functions are obtained by transforming the torsion balance observations in the points to local (orthogonal) horizontal coordinate systems with first axes in the direction to the other observation point. If the azimuth of the direction from one point to the other point is* a , *then the result of this transformation may be obtained by rotating the vectors*

$$\begin{Bmatrix} T_{13} \\ T_{23} \end{Bmatrix} \text{ and } \begin{Bmatrix} T_\Delta \\ 2\,T_{12} \end{Bmatrix}$$

the angles $a - 90°$ *and* $2\,(a - 90°)$ *respectively.*

The reverse rotations applied on the 2×2 *matrices of covariances of these quantities will produce all the direction dependent covariances of the original quantities.*

1. Introduction

Let $K(P, Q)$ be a rotational invariant reproducing kernel of a Hilbert space of harmonic functions, or equivalently a so–called empirical covariance function of the anomalous gravitational potential, T . It is a function of two variables, P , Q , both points in R^3 outside some sphere with radius R_B , the Bjerhammar sphere, bounding the set of harmonicity for the functions in the Hilbert space. Then

$$K(P,Q) = \sum_{i=2}^{\infty} \sigma_i \left(\frac{R_B^2}{r r'}\right)^{i+1} P_i(\cos\psi), \tag{1}$$

where ψ is the spherical distance between P and Q , r and r′ the radial distances of P , Q from the origin respectively, and P_i are Legendre polynomials. σ_i are positive constants, the so–called degree–variances.

The inner product of two linear functionals L_1 and L_2 or equivalently their covariance, is obtained by applying these functionals on K ,

$$L_1(L_2(K(P,Q))) := K(L_1, L_2) \tag{2}$$

Expressions for such quantities, where L_1 and L_2 are linear functionals associated with zero, first– or second–order derivatives of T are given, e.g., in (Tscherning, 1976). Similar expressions have been derived in (Keller and Meier, 1980) for harmonic functions given in a half space $\{x_1, x_2, x_3) \mid x_3 > 0\}$.

If we want to derive the quantities $K(L_1, L_2)$ for cases where L_1 or L_2 are linear functionals associated with torsion balance (or gravity gradiometer) observations, then the equations given in (Tscherning, 1976) cannot be used directly. Certain linear combinations of the equations must be used.

The covariance functions will depend on the directions between P and Q . This makes the direct derivation of expressions for these functions rather involved. We will show here that the expressions may be derived in a very straightforward manner from *isotropic* covariance functions of torsion balance observations. Such functions are also useful in cases where empirical covariances have to be estimated in a situation where few observations are available. (This is because sampling of products of pairs of observations will have to be done only with respect to the distance between the observation points).

Isotropic covariances are derived in section 4 following some mathematical preparations in sections 2 and 3. Finally in section 5 the isotropic covariances are used to derive not only the general expressions for the covariances of torsion balance observations , but also expressions for covariances of several other gravity field quantities.

2. Basic equations

Suppose we have a usual Cartesian coordinate system with axes x , y , z so that the last one coincides with the rotation axis of the Earth. All points not located on the z–axis will then have the usual spherical coordinates φ , latitude, λ , longitude, and r , the distance from the origin. For these points we may also define a local coordinate system with coordinates (x_1, x_2, x_3) so that the first axis points east, the second north and the third in the direction of the radius vector. We may keep the origin or move it to the point, as we like.

In spherical approximation we may then express the torsion balance observations as follows.

$$T_{13} = \frac{\partial^2 T}{\partial x_1 \partial x_3} = \frac{\partial}{\partial r}\left(\frac{1}{r\cos\varphi}\frac{\partial T}{\partial \lambda}\right) \tag{3}$$

$$T_{23} = \frac{\partial^2 T}{\partial x_2 \, \partial x_3} = \frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial T}{\partial \varphi}\right) \tag{4}$$

$$T_\Delta = \frac{\partial^2 T}{\partial x_1^2} - \frac{\partial^2 T}{\partial x_2^2} = \frac{-1}{r^2}\left(\frac{1}{\cos\varphi}\,\frac{\partial}{\partial\varphi}\,(\cos\varphi\,\frac{\partial}{\partial\varphi}\,T) - \frac{1}{\cos^2\varphi}\,\frac{\partial^2 T}{\partial\lambda^2}\right) \tag{5}$$

$$2 \cdot T_{12} = 2\,\frac{\partial^2 T}{\partial x_1 \, \partial x_2} = \frac{2}{r}\,\frac{\partial}{\partial\varphi}\left(\frac{1}{r\cos\varphi}\,\frac{\partial T}{\partial\lambda}\right) \tag{6}$$

see (Mueller, 1963) and (Tscherning, 1976a, eq. (58)).

For the sake of completeness, we write the equations associated with some other important types of observations, the height anomaly ζ, the gravity anomaly Δg, and the deflections of the vertical (η , ξ). Then, with γ equal to the normal gravity, and still using spherical approximation, we have

$$\zeta = T/\gamma \;,$$

$$\Delta g = -\frac{\partial T}{\partial x_3} - \frac{2}{r}T = -\frac{\partial T}{\partial r} - \frac{2}{r}T \;,$$

$$\eta = -\frac{1}{\gamma}\,\frac{\partial T}{\partial x_1} = -\frac{1}{\gamma\, r\cos\varphi}\,\frac{\partial T}{\partial\lambda} \;,$$

and

$$\xi = -\frac{1}{\gamma}\,\frac{\partial T}{\partial x_2} = -\frac{1}{\gamma\cdot r}\,\frac{\partial T}{\partial\varphi} \;.$$

It is obvious that the evaluation functional or the functional $\frac{\partial}{\partial x_3}\, l_P$ will result in quantities which are still isotropic and of the form given in eq. (7) ; except for factors r^k or $(r')^k$, $k = -1$ or -2 . T_{13} and T_{23} will in this respect be treated as if they were deflections of the vertical (η , ξ) . In order to see this,

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial\varphi}\,T\right) = \frac{1}{r}\,\frac{\partial}{\partial\varphi}\left(\frac{\partial T}{\partial r} - \frac{1}{r}T\right)$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r\cos\varphi}\,\frac{\partial}{\partial\lambda}\,T\right) = \frac{1}{r\cos\varphi}\,\frac{\partial}{\partial\lambda}\left(\frac{\partial T}{\partial r} - \frac{1}{r}T\right)$$

Then with $i, j = 1$ or 2

$$K\left(\frac{\partial^2}{\partial x_j \partial x_3}\, l_P \,,\, Q\right) = \frac{\partial}{\partial x_j}\, l_P\left(\frac{\partial K}{\partial r} - \frac{1}{r}K\right)$$

$$K\left(P \,,\, \frac{\partial^2}{\partial x_j \partial x_3}\, l_Q\right) = \frac{\partial}{\partial x_j}\, l_Q\left(\frac{\partial K}{\partial r'} - \frac{1}{r'}K\right)$$

$$K\left(\frac{\partial^2}{\partial x_j \partial x_3}\,|_P\,,\ \frac{\partial^2}{\partial x_i \partial x_3}\,|_Q\right) = \frac{\partial}{\partial x_j}\,|_P\,\frac{\partial}{\partial x_i}\,|_Q\left(\frac{\partial^2 K}{\partial r\,\partial r'} - \frac{1}{r}\frac{\partial K}{\partial r'} - \frac{1}{r'}\frac{\partial K}{\partial r} + \frac{1}{rr'}K\right)$$

In order to facilitate the use of these quantities in the subroutine COVAX (Tscherning, 1976), we introduce the variable $s = R_B^2/(r\,r')$ and compute the derivatives using s. (Note, that we later use the term "s" for another quantity).

Because

$$\frac{\partial}{\partial r} = \frac{\partial s}{\partial r}\frac{\partial}{\partial s} = -\frac{s}{r}\frac{\partial}{\partial s}\ ,\quad \frac{\partial}{\partial r'} = \frac{\partial s}{\partial r'}\frac{\partial}{\partial s} = -\frac{s}{r'}\frac{\partial}{\partial s}$$

we get

$$\frac{\partial T}{\partial r} - \frac{1}{r}T = -\frac{1}{r}\left(s\frac{\partial K}{\partial s} + K\right),$$

$$\frac{\partial T}{\partial r'} - \frac{1}{r'}\,T = -\frac{1}{r'}\left(s\frac{\partial K}{\partial s} + K\right),$$

$$\frac{\partial^2 K}{\partial r\,\partial r'} - \frac{1}{r}\frac{\partial K}{\partial r'} - \frac{1}{r'}\frac{\partial K}{\partial r} + \frac{1}{rr'}K = \frac{s}{rr'}\frac{\partial}{\partial s}\left(s\frac{\partial K}{\partial s} + K\right) + \frac{1}{rr'}\left(s\frac{\partial K}{\partial s} + K\right)$$

$$= \frac{1}{rr'}\left(s^2\frac{\partial^2 K}{\partial s^2} + 3s\frac{\partial K}{\partial s} + K\right).$$

With

$$D = s\frac{\partial K}{\partial s} + K\ ,$$

$$E = \left(s^2\frac{\partial^2 K}{\partial s^2} + 3s\frac{\partial K}{\partial s} + K\right)s/R_B^2$$

we have

$$K\left(\frac{\partial^2}{\partial x_j \partial x_3}\,|_P\,,\ Q\right) = \frac{\partial}{\partial x_j}\,|_P\,(D)/(-r)\ , \tag{7}$$

$$K\left(P\,,\ \frac{\partial^2}{\partial x_i \partial x_3}\,|_Q\right) = \frac{\partial}{\partial x_i}\,|_Q\,(D)/(-r')\ , \tag{8}$$

$$K\left(\frac{\partial^2}{\partial x_i \partial x_3}\,|_P\,,\ \frac{\partial^2}{\partial x_j \partial x_3}\,|_Q\right) = \frac{\partial}{\partial x_i}\,|_P\,\frac{\partial}{\partial x_j}\,|_Q\,(E) \tag{9}$$

Note that

$$D = \sum_{i=2}^{\infty} \sigma_i\,(i+2)\,s^{i+1}\,P_i\,(\cos\psi)$$

and

$$E = \sum_{i=2}^{\infty} \frac{\sigma_i}{R_B^2} (i+2)^2 s^{i+2} P_i(\cos\psi) .$$

3. Change of covariances caused by a rotation of the local coordinate system

It is well known that isotropic covariance functions for deflections of the vertical are obtained by using the longitudinal and transversal components of a pair of deflections in a point P relative to a point Q, see (Jordan, 1972), (Moritz, 1972), (Nash and Jordan, 1978) and (Tscherning, 1972). These components arise as first–order partial derivatives with respect to the coordinates of a local coordinate system obtained by rotating the local coordinate system the angle $\alpha - 90°$ with axes pointing east and north, where α is the azimuth from the point P to the point Q. Another angle α', will obviously have to be used in the point Q, see *Figure 1.*

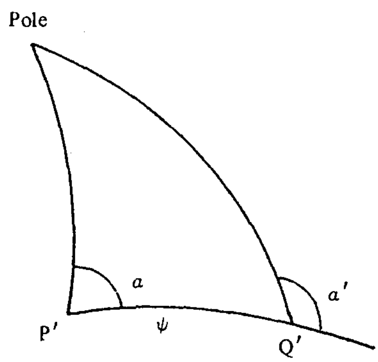

Fig. 1 – P′ *and* Q′ *are the projections of* P, Q *respectively on the unitsphere.*

Let us then compute the change in the second–order horizontal derivatives following a rotation β, and let us denote the new coordinates z_1 and z_2. Then

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = R(\beta) \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{10}$$

with

$$R(\beta) = \begin{Bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{Bmatrix} \tag{11}$$

For the first–order derivatives we have

$$\begin{Bmatrix} \dfrac{\partial}{\partial z_1} \\ \\ \dfrac{\partial}{\partial z_2} \end{Bmatrix} = R(\beta) \begin{Bmatrix} \dfrac{\partial}{\partial x_1} \\ \\ \dfrac{\partial}{\partial x_2} \end{Bmatrix} \qquad (12)$$

and for the second–order derivatives

$$\begin{Bmatrix} \dfrac{\partial^2}{\partial z_1^2} & \dfrac{\partial^2}{\partial z_1 \partial z_2} \\ \\ \dfrac{\partial^2}{\partial z_1 \partial z_2} & \dfrac{\partial}{\partial z_2^2} \end{Bmatrix} = R(\beta) \begin{Bmatrix} \dfrac{\partial^2}{\partial x_1^2} & \dfrac{\partial^2}{\partial x_1 \partial x_2} \\ \\ \dfrac{\partial^2}{\partial x_1 \partial x_2} & \dfrac{\partial^2}{\partial x_2^2} \end{Bmatrix} R(-\beta) \qquad (13)$$

The functionals related to the torsion balance observations $2 \cdot T_{12}$ and T_Δ are then transformed as follows.

$$\frac{\partial^2}{\partial z_1^2} - \frac{\partial^2}{\partial z_2^2} = (cos^2\beta - sin^2\beta)\frac{\partial^2}{\partial x_1^2} - (cos^2\beta - sin^2\beta)\frac{\partial^2}{\partial x_2^2}$$

$$- 2\, sin\,\beta\, cos\,\beta \cdot 2\, \frac{\partial^2}{\partial x_1 \partial x_2},$$

$$2 \cdot \frac{\partial^2}{\partial z_1 \partial z_2} = 2\, cos\,\beta\, sin\,\beta \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right) + 2\,(cos^2\beta - sin^2\beta) \cdot \frac{\partial^2}{\partial x_1 \partial x_2}.$$

Hence

$$\begin{Bmatrix} \dfrac{\partial^2}{\partial z_1^2} - \dfrac{\partial^2}{\partial z_2^2} \\ \\ 2\dfrac{\partial^2}{\partial z_1 \partial z_2} \end{Bmatrix} = \begin{Bmatrix} cos\, 2\beta & - sin\, 2\beta \\ \\ sin\, 2\beta & cos\, 2\beta \end{Bmatrix} \begin{Bmatrix} \dfrac{\partial^2}{\partial x_1^2} - \dfrac{\partial^2}{\partial x_2^2} \\ \\ 2\dfrac{\partial^2}{\partial x_1 \partial x_2} \end{Bmatrix} \qquad (14)$$

The vector containing $(T_\Delta, 2T_{12})$ is then transformed by a rotation the angle 2β, while the vector (T_{13}, T_{23}) is transformed by a rotation the angle β, just as if it were a vector containing the negative deflection components $(-\eta, -\xi)$.

Because of the rotational invariance of $K(P, Q)$ we may now choose the local coordinate systems in a convenient way, so that the derivative with respect to the first coordinate variable corresponds to the directional derivatives between the points. We will then work with two systems, so that the first makes the evaluation of the linear functionals associated with P easy and so that the second makes the evaluations in Q easy. The original derivatives in northern and eastern directions may then be obtained by executing the rotations $90° - \alpha$ in P and $270° - \alpha'$ in Q.

4. Evaluation of isotropic covariances

We now introduce the two coordinate systems. One is associated with P having coordinates (x_1, x_2, x_3) and the other with Q having coordinates (y_1, y_2, y_3). They are obtained from the above described local coordinate systems by rotations $\alpha - 90°$ and $\alpha' - 270°$ respectively.

The third axis will in both systems coincide with the radius vector passing through the point, and the first axis will be in the plane spanned by the two radius vectors. In both cases the direction of the first axis is selected so that the new azimuth is 0 . The second axis is selected so that the three axes span a right–handed Cartesian coordinate system. Here $P = (0, 0, r)$ in the x–system and $Q = (0, 0, r')$ in the y–system. Let the angle between the radius vectors be ψ, (= the spherical distance) and denote the coordinates of a point x in the y–system by $\tilde{x}$ and vice versa. Then we have the following coordinate transformation :

$$\left.\begin{aligned} \tilde{x}_1 &= -x_1 \cos\psi + x_3 \sin\psi = -x_1 \cdot t + x_3 \cdot s \\ \tilde{x}_2 &= -x_2 \\ \tilde{x}_3 &= x_1 \sin\psi + x_3 \cos\psi = x_1 \cdot s + x_3 \cdot t \end{aligned}\right\} \qquad (15)$$

Note that this transformation is its own inverse, i.e., $\tilde{\tilde{x}} = x$, and that it is also valid for $\tilde{\tilde{y}}$. (We now put $t = \cos\psi$ and $s = \sin\psi$) .

We will now express the covariance function by some more convenient parameters

$$u = (x, \tilde{y}) = (\tilde{x}, y) = \sum_{i=1}^{3} x_i \tilde{y}_i = \sum_{j=1}^{3} \tilde{x}_j y_j \,, \qquad (16)$$

$$v = \frac{1}{2} |x|^2 \cdot |y|^2 \,, \qquad (17)$$

so that

$$K(x, y) := f(u, v)$$

(However, we may here use D or E instead of K , see section 1) .

Because of our choice of coordinate system, we may now compute the derivatives in these two new systems, and subsequently evaluate the result in P and Q . The derivatives may then be transformed back to the old coordinate system using eq. (12) and (14).

We denote the derivatives of f with respect to u and v using subscripts, so that

$$\frac{\partial^m f}{\partial u^i \partial v^j} := f_{ij} \,, \quad m = i + j \,.$$

We will need the derivatives of u and v with respect to the coordinates,

$$\frac{\partial u}{\partial x_i} = \widetilde{y}_i \ , \ \frac{\partial v}{\partial x_i} = x_i \, |\widetilde{y}|^2 = x_i \, |y|^2$$

and also

$$\frac{\partial \widetilde{y}_1}{\partial y_1} = -t \ , \frac{\partial \widetilde{y}_1}{\partial y_3} = s \ , \frac{\partial \widetilde{y}_2}{\partial y_2} = -1 \ .$$

Then

$$\frac{\partial}{\partial x_i} K = f_{10} \, \widetilde{y}_i + f_{01} \, x_i \, |y|^2 \ ,$$

and

$$\frac{\partial^2}{\partial x_i \partial x_j} K = f_{20} \, \widetilde{y}_i \widetilde{y}_j + f_{11} \, (\widetilde{y}_i x_j + \widetilde{y}_j x_i) \, |y|^2$$

$$+ f_{02} \, x_i x_j |y|^4 + \frac{\partial x_i}{\partial x_j} \, f_{01} \, |y|^2 \ .$$

When evaluating the derivatives in $\mathbf{P}$, and then subsequently in Q following (=), we may after each *completed* differentiation insert the actual values of the coordinates. This gives us the following simple equations.

$$\frac{\partial}{\partial x_1} K = f_{10} \, \widetilde{y}_1 \ (=) \ f_{10} \, r' s \ , \tag{18}$$

$$\frac{\partial}{\partial x_2} K = f_{10} \, \widetilde{y}_2 (=) \ 0 \ , \tag{19}$$

$$F := \frac{\partial^2 K}{\partial x_1^2} - \frac{\partial^2 K}{\partial x_2^2} = f_{20} \ (\widetilde{y}_1^2 - \widetilde{y}_2^2)(=) \ f_{20} \, (r' s)^2 \ , \tag{20}$$

$$G := \frac{\partial^2 K}{\partial x_1 \, \partial x_2} = f_{20} \, \widetilde{y}_1 \, \widetilde{y}_2 \ (=) \ 0 \tag{21}$$

We must then differentiate these quantities with respect to coordinates y_i , using $\mathbf{P} = (0, 0, r)$ in x–coordinates.

$$\frac{\partial^2 K}{\partial y_j \partial x_i} = \widetilde{y}_i \, (f_{20} \, \widetilde{x}_j + f_{11} \, y_j \cdot |x|^2) + \frac{\partial \widetilde{y}_i}{\partial y_j} \, f_{10}$$

$$+ x_i (|y|^2 \, (f_{11} \, \widetilde{x}_j + f_{02} \, y_j |x|^2) + 2 y_j \, |y| \, f_{01}) \ ,$$

and then

$$\frac{\partial^2 K}{\partial y_1 \, \partial x_1} = \widetilde{y}_1 \, (f_{20} \, \widetilde{x}_1 + f_{11} \, y_1 \, |x|^2) - t \, f_{10} \; (=) \; s^2 \, r r' f_{20} - t \, f_{10} \tag{22}$$

$$\frac{\partial^2 K}{\partial y_2 \, \partial x_1} = \widetilde{y}_1 (f_{20} \widetilde{x}_2 + f_{11} \, y_2 \, |x|^2) \; (=) \; \widetilde{y}_1 \widetilde{x}_2 \, f_{20} = 0 \tag{23}$$

$$\frac{\partial^2 K}{\partial y_1 \, \partial x_2} = \widetilde{y}_2 \, (f_{20} \, \widetilde{x}_1 + f_{11} \, y_1 \, |x|^2) \, (=) \; 0 \tag{24}$$

$$\frac{\partial \;\; K}{\partial y_2 \, \partial x_2} = \widetilde{y}_2 (f_{20} \widetilde{x}_2 + f_{11} \, y_2 \, |x|^2) - f_{10} (=) - f_{10} \tag{25}$$

$$\frac{\partial F}{\partial y_1} = \frac{\partial}{\partial y_1} \, (\widetilde{y}_1^2 \,) \, f_{20} \; + (\widetilde{y}_1^{\,2} - \widetilde{y}_2^{\,2}) (f_{30} \; \widetilde{x}_1 \; + f_{21} \, y_1 \, |x|^2)$$

$$(=) - 2 \, tsr' f_{20} + (s r')^2 \, r \, s \, f_{30} \tag{26}$$

$$\frac{\partial F}{\partial y_2} = - 2 \, y_2 \, f_{20} + (\widetilde{y}_1^2 - \widetilde{y}_2^2)(f_{30} \, \widetilde{x}_2 + f_{21} \, y_2 \, |x|^2) \; (=) \; 0 \tag{27}$$

$$\frac{\partial \;\; F}{\partial y_1 \, \partial y_2} = - 2 \, y_2 \, \frac{\partial}{\partial y_1} \, (f_{20}) + y_2 \, |x|^2 \frac{\partial}{\partial y_1} \, ((y_1^2 - y_2^2) f_{21}) \; (=) \; 0 \tag{28}$$

$$\frac{\partial^2 F}{\partial y_1^2} = + \, 2 t^2 \, f_{20} \; - 2 t \widetilde{y}_1 \, (f_{30} \widetilde{x}_1 + \; f_{21} \, y_1 \, |x|^2) - 2 \, t \widetilde{y}_1 \, (f_{30} \widetilde{x}_1 + f_{21} \, y_1 \, |x|^2)$$

$$+ (\widetilde{y}_1^2 - \widetilde{y}_2^2) \, (\widetilde{x}_1 \, (f_{40} \widetilde{x}_1 + f_{31} \, y_1 \, |x|^2 + |x|^2 \, (f_{21} + y_1 \, \frac{\partial}{\partial y_1} \, f_{21}))$$

$$(=) + 2 \, t^2 \, f_{20} - 4 \, t \, s^2 \, r r' f_{30} + (r' s)^2 \, ((r s)^2 \, f_{40} + r^2 \, f_{21}) \tag{29}$$

$$\frac{\partial^2 F}{\partial y_2^2} = - \, 2 \, f_{20} - 2 \, y_2 \, \frac{\partial}{\partial y_2} (f_{20}) + (\widetilde{y}_1^2 - \widetilde{y}_2^2) \, |x|^2 \, (f_{21} + y_2 \, \frac{\partial}{\partial y_2} (f_{21}))$$

$$+ \, 2 \widetilde{y}_2 \, \frac{\partial}{\partial y_2} \, (f_{21} \, y_2 \, |x|^2) \, (=) - 2 \, f_{20} + (r r' s)^2 \, f_{21} \tag{30}$$

Hence

$$(\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2}) F = 2 \, (t^2 + 1) \, f_{20} - 4 \, t \, s^2 \, r r' f_{30} + (r r')^2 \, s^4 \, f_{40} \tag{31}$$

$$\frac{\partial G}{\partial y_1} = \widetilde{y}_2 \, (\frac{\partial \widetilde{y}_1}{\partial y_1} \, f_{20} + \widetilde{y}_1 \, (f_{30} \widetilde{x}_1 + f_{21} \, y_1 \, |x|^2)) \; (=) \; 0 \tag{32}$$

$$\frac{\partial G}{\partial y_2} = \tilde{y}_1 \left(-f_{20} + \tilde{y}_2 \left(f_{30}\tilde{x}_2 + f_{21} y_2 |x|^2\right)\right) (=) - r' s f_{20} \tag{33}$$

$$\frac{\partial^2 G}{\partial y_2 \partial y_1} = t f_{20} - \tilde{y}_1 \left(f_{30}\tilde{x}_1 + f_{21} y_1 |x|^2\right) + \tilde{y}_2 \frac{\partial}{\partial y_2} (\ldots\ldots)$$

$$(=) t f_{20} - r r' s^2 f_{30} \tag{34}$$

These expressions may be slightly simplified by introducing the derivatives of K, (D and E) with respect to $t = \cos\psi$.

Because

$$\frac{\partial}{\partial u} = \frac{\partial t}{\partial u}\frac{\partial}{\partial t} = \frac{1}{r r'} \frac{\partial}{\partial t},$$

then with

$$K_n = \frac{\partial^n}{\partial t^n} K \qquad \text{(and similarly for D and E)}$$

we have the following non–zero expressions

$$\frac{\partial K}{\partial x_1} = \frac{s}{r} K_1 \tag{35}$$

$$\frac{\partial^2 K}{\partial x_1^2} - \frac{\partial^2 K}{\partial x_2^2} = \frac{s^2}{r^2} K_2 , \tag{36}$$

$$\frac{\partial^2 K}{\partial y_1 \partial x_1} = (s^2 K_2 - t K_1)/(r r'), \tag{37}$$

$$\frac{\partial^2 K}{\partial y_2 \partial x_2} = - K_1/(r r'), \tag{38}$$

$$\frac{\partial}{\partial y_2}\left(\frac{\partial^2 K}{\partial x_1 \partial x_2}\right) = - s K_2 /(r^2 r'), \tag{39}$$

$$\frac{\partial}{\partial y_1}\left(\frac{\partial^2 K}{\partial x_1^2} - \frac{\partial^2 K}{\partial x_2^2}\right) = (s^3 K_3 - 2 t s K_2)/(r^2 r'), \tag{40}$$

$$\frac{\partial^2}{\partial y_2 \partial y_1} \frac{\partial^2 K}{\partial x_2 \partial x_1} = (t K_2 - s^2 K_3)/(r r')^2 \tag{41}$$

$$\left(\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2}\right)\left(\frac{\partial^2 K}{\partial x_1^2} - \frac{\partial^2 K}{\partial x_2^2}\right) = (2(t^2+1)K_2 - 4 t s^2 K_3) + s^4 K_4)/(r r')^2 \tag{42}$$

When evaluating the non–zero quantities related to T_{12} and T_{23}, we substitute D for K in eq. (35), (37) – (40) and E for K in eq. (37) and (38), and use eq. (7) – (9).

Then

$$\frac{\partial^2 K}{\partial x_3 \, \partial x_1} = -\frac{s}{r^2} D_1 \, , \tag{43}$$

$$\frac{\partial^3 K}{\partial y_1 \, \partial x_3 \, \partial x_1} = -(s^2 D_2 - t D_1)/(r^2 r') , \tag{44}$$

$$\frac{\partial^3 K}{\partial y_3 \, \partial y_1 \, \partial x_1} = -(s^2 D_2 - t D_1)/(r(r')^2) , \tag{45}$$

$$\frac{\partial^3 K}{\partial y_2 \, \partial x_3 \, \partial x_2} = D_1/(r^2 r') , \tag{46}$$

$$\frac{\partial^2 K}{\partial y_3 \, \partial y_2 \, \partial x_2} = D_1/(r(r')^2) , \tag{47}$$

$$\frac{\partial^2}{\partial y_3 \, \partial y_2}\left(\frac{\partial^2 K}{\partial x_1 \, \partial x_2}\right) = -s D_2/(r r')^2 , \tag{48}$$

$$\frac{\partial^2}{\partial y_3 \, \partial y_1}\left(\frac{\partial^2 K}{\partial x_1^2} - \frac{\partial^2 K}{\partial x_2^2}\right) = -(s^3 D_3 - 2 t s D_2)/(r r')^2 , \tag{49}$$

$$\frac{\partial^4 K}{\partial y_3 \, \partial y_1 \, \partial x_3 \, \partial x_1} = (s^2 E_2 - t E_1)/(r r') , \tag{50}$$

$$\frac{\partial^4 K}{\partial y_3 \, \partial y_2 \, \partial x_3 \, \partial x_2} = -E_1/(r r') . \tag{51}$$

The quantities (35) – (51) are then the basic isotropic quantities computed in the two local coordinate systems. Quantities involving gravity anomalies could have been obtained from functions similar to D and E , but where the degree–variances σ_i are multiplied with $(i-1)$ and $(i-1)^2$ respectively.

5. General covariance expressions

The general covariance expressions $cov(L_1(T), L_2(T)) = K(L_1, L_2)$ are then obtained using eq. (35) – (51). We introduce of few more "shorthand" expressions,

$$K_{ijkn} = \frac{\partial^4 K}{\partial x_i \, \partial x_j \, \partial y_k \, \partial y_n} \, , \quad i, j, k, n = 1, 2 \text{ or } 3 ,$$

$$K_{\Delta k n} = \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)\left(\frac{\partial^2 K}{\partial y_k \, \partial y_n}\right),$$

and similarly $K_{ij\Delta}$ and $K_{\Delta\Delta}$.

Then

$$\begin{Bmatrix} cov(T_\Delta, T_\Delta) & cov(T_\Delta, 2T_{12}) \\ cov(2T_{12}, T_\Delta) & cov(2T_{12}, 2T_{12}) \end{Bmatrix} = R(180^\circ - 2a) \begin{Bmatrix} K_{\Delta\Delta} & 0 \\ 0 & 4K_{1212} \end{Bmatrix} R(2a' + 180^\circ),$$

$$\begin{Bmatrix} cov(T_{13}, T_\Delta) & cov(T_{13}, 2T_{12}) \\ cov(T_{23}, T_\Delta) & cov(T_{23}, 2T_{12}) \end{Bmatrix} = R(90^\circ - a) \begin{Bmatrix} K_{13\Delta} & 0 \\ 0 & 2K_{2312} \end{Bmatrix} R(2a' + 180^\circ),$$

$$\begin{Bmatrix} cov(T_{13}, T_{13}) & cov(T_{13}, T_{23}) \\ cov(T_{23}, T_{13}) & cov(T_{23}, T_{23}) \end{Bmatrix} = R(90^\circ - a) \begin{Bmatrix} K_{1313} & 0 \\ 0 & K_{2323} \end{Bmatrix} R(a' + 90^\circ).$$

Quantities such as $K_{\Delta\Delta}$ or K_{1212} are easily obtained using the subroutine COVAX, (for the degree–variance models described in (Tscherning, 1976)). However, the computation of $K_{13\Delta}$, K_{1313}, K_{2323} and K_{2312} requires some small modifica– tions. A modified version of COVAX is available on request from the authors.

6. Conclusion

We have here derived covariance expressions for torsion balance observations based on an isotropic covariance function (or reproducing kernel). It is planned to evaluate these functions using the degree–variance models discussed in (Tscherning, 1975) and compare these values with values obtained from empirical data.

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