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On the Use and Abuse of Molodensky's Mountain

by

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ABSTRACT

Methods for the approximation of the anomalous gravitational potential may be tested using a potential generated by point masses and with the topographic effects simulated by a conical mountain with a rounded top, the Molodensky Mountain Model (MMM).

It is shown, that this model has a degree-variance spectrum σ_n, which tend to zero like n^{-1} for $n \rightarrow \infty$. This is then essentially different from that of the Earth's anomalous gravitational potential, but similar to the spectrum implicitly used in Bjerhammar's so-called Dirac approach. This explains some of the results obtained by (Katsambalos, 1981) using the MMM. Here collocation with covariances resembling these of the anomalous potential of the Earth gave results at high altitudes which compared unfavourably to results obtained using the Dirac approach.

A numerical example is then given using collocation with covariance functions having a degree-variance spectrum similar to that of the model, which at high altitudes gave better results than these obtained by Katsambalos using the Dirac approach. This shows that conclusions obtained from such model studies may not be valid for the Earth's gravity field.

1. INTRODUCTION

Let us suppose, that the anomalous gravitational potential T is a harmonic function outside the solid earth surface and that it fulfil certain regularity conditions at infinity. T may then be determined (approximated) using classical methods for the solution of boundary value problems or more general modern approximation techniques. A survey of the today used or proposed methods have been given in (Tscherning, 1981), and many details can be found in (Moritz, 1980).

It is naturally of interest to find the "best" method (or to show that a method which has been claimed to be best is not so good), and to discover problems, which may occur when using a method. Here theoretical investigations have been a main tool, but the advent of electronic computers have made numerical studies using simulated data feasible.

However, we have to be very careful drawing conclusions from numerical studies. We may use the studies for a preliminary rejection of a hypothesis, but even that is not always possible. Furthermore, if we get numerical results, which are not in agreement with a theoretical result, a programming error has probably been made.

The purpose of this paper is to discuss where model studies can be used, and where and how they should not be used. This discussion has been inspired by numerical studies of two approximation techniques in (Katsambalos, 1981) and remarks in (Bjerhammar, 1982, p. 323). The main features of the two techniques are explained briefly in section 2.

The specific properties of the Molodensky Mountain Model (MMM) and its related gravity potential are then derived in section 3, and we discuss also the use of such models here. In section 4, I will give an example of the use of MMM, which give results which in some way contradict earlier results obtained using the model. I hope the reader will not regard this use of the model as an abuse.

In the conclusion, section 5, I try to describe a number of problems, which could be investigated using numerical studies.

2. REMARKS ON TWO TYPES OF METHODS FOR APPROXIMATING THE ANOMALOUS POTENTIAL

Two types of methods (A), (B) are presently used for constructing approximations $\tilde{T}$ to the anomalous potential. Suppose, that we have given a set of (errorfree) observations x_i, $i=1,\dots,N$, related to T through linear functionals

$$L_i(T) = x_i\,, \qquad i=1,\dots,N\,. \tag{1}$$

Then we may describe the two methods as follows

(A) Select base functions f_j, $j=1,\dots,k$, $k \leq N$, so that

$$\tilde{T} = \sum_{j=1}^{k} a_j f_j \tag{2}$$

and determine the coefficients $\{a_j\}$ so that

$$\{L_j(\tilde{T})-x_j\}^T \{w_{ij}\}\{L_i(\tilde{T})-x_i\} \tag{3}$$

attains its minimum value. ($\{w_{ij}\}$ is a weight matrix). A special case occurs when k=N; here the coefficients are determined simply by

$$L_i(\tilde{T}) = \sum_{j=1}^{k} a_j L_i(f_j) = x_i, \qquad i=1,\ldots,N . \tag{4}$$

In this case we have an example of a collocation technique, because we have an exact agreement between the model and the data.

(B) Select a normed linear vector space with dimension $\geq$ N, preferably a reproducing kernel Hilbert Space (RKHS), so that the linear functionals are elements of the dual space. (The RKHS may be spanned by a stochastic process, in which case the linear functionals are stochastic variables). $\tilde{T}$ is then selected as the element of the space, which has the minimum norm, and which agrees with the observations, i.e. $\tilde{T}$ fulfils eq. (1). In a RKHS with kernel K(P,Q) the solution is then given by eq. (2) and (4) with $f_j = L_j K(P,Q) = K(L_j,Q)$ and k=N. (We follow here the notation used in (Krarup, 1978)). Note that the system of equations (4) in this case becomes symmetric

$$\{K(L_i,L_j)\}\{a_j\} = \{x_i\} \ , \tag{5}$$

and positive definite if the linear functionals are linearily independent.

It is then obvious, that the method (A) and (B) becomes identical if K=N and $f_j = K(L_j,Q)$. This also shows, that a solution to method (A) in this case implicitly will be the one, which has a minimum norm in the RKHS, which has K(P,Q) as its reproducing kernel.

Both methods have advantages and disadvantages. (Lelgemann, 1981) seems to prefer methods of type (A) but with base functions associated with reproducing kernels. This permit the control of stability properties by selecting the kernels depending on the data spacing in cases where the data are spaced regularily. (Bjerhammar, 1979) also seems to prefer methods of type (A) because of the favourable numerical properties (condition number for the matrix $\{L_i(f_j)\}$), which are obtained for certain types of base functions.

Several other authors, including myself, prefer methods of type (B) because these methods have a very fine theoretical base, but also because it is possible to choose a RKHS so that $\tilde{T}$ gets a predefined smoothness, even resembling that of the true anomalous potential. This is the well known method of least squares collocation as developed originally by (Moritz, 1965) and further extended by (Krarup, 1969).

The numerical problems which arize (size of and conditioning of the matrix $\{K(L_i,L_j)\}$) are solved by using RKHS with subspaces removed, which represent long-wavelength information, and by only using data from a local area.

The RKHS which is used in least squares collocation is the one in which the empirical covariance function of T is the reproducing kernel. Suppose we work in spherical approximation, and that T may be expressed by a series in solid spherical harmonics,

$$T(\varphi,\lambda,r) = \frac{GM}{r} \sum_{i=2}^{\infty} \left(\frac{R}{r}\right)^i \sum_{j=0}^{i} \bar{P}_{ij}(\sin\varphi) * (\bar{C}_{ij}\cos(j\lambda)+\bar{S}_{ij}\sin(j\lambda)) \qquad (6)$$

Here φ is the latitude, λ the longitude, r the distance from the origin, GM the product of the gravitational constant and the mass of the Earth, R the mean earth radius (= 6371 km), $\bar{P}_{ij}$ are the fully normalized associated Legendre functions and $\bar{C}_{ij}$, $\bar{S}_{ij}$ the coefficients of the series. Then

$$K(P,Q) = \sum_{i=2}^{\infty} \sigma_i \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\psi), \qquad (7)$$

$$\sigma_i = \sum_{j=0}^{i} (\bar{C}^2_{ij} + \bar{S}^2_{ij}) \left(\frac{GM}{R}\right)^2, \qquad (8)$$

where r' is the distance of Q from the origin and ψ is the spherical distance from P to Q. Estimates for the degree-variances σ_i can be found in (Tscherning and Rapp, 1974) or in (Moritz, 1977). What is essential is that there seems to be a general agreement that σ_i tend to zero at least like i^{-3} for $i \to \infty$. This means that $\|\tilde{T}\|$ will be minimum in a norm which minimalizes the surface integral of a linear combination of up to the second order derivatives of $\tilde{T}$, see (Tscherning, 1972) and Table 1.

The users of methods of type (A) have mainly used base functions which in the limiting case k=N would correspond to the use of much weaker

Table 1. Relationship between the variation of σ_i for $i \to \infty$, and the norm which is minimalized.

Variation of σ_i	Norm expressed as	
	Surface integral of	Volume integral of
i^2		T^2
i	T^2	
const.		$(\nabla T)^2$
i^{-1}	$(\nabla T)^2$	
i^{-2}		$\sum_{i,j} (\frac{\partial^2 T}{\partial x_i \partial x_j})^2$
i^{-3}	$\sum_{i,j} (\frac{\partial^2 T}{\partial x_i \partial x_j})^2$	

norms, see Table 1. Let us regard the so-called Dirac approach (Bjerhammar, 1976) and let us suppose, that we have k=N gravity observations, $L_i(T) = \Delta g_i$, related to points P_i, which all have the same height, r_o. Then the method requires that

$$r_o \, \Delta g_i = \sum_{j=1}^{N} a_j \cdot L_i^*(f_j) , \tag{9}$$

where $L_i^* = r \cdot L_i$ and in this case with $R_B < r_o$

$$L_i^*(f_j) = K(L_i^*, L_j^*)$$

$$= R_B/r_o(1 - (R_B/r_o)^2)/(1-2R_B/r_o \cos\psi_{ij} + (R_B/r_o)^2)^{\frac{3}{2}}$$

$$= R_B(r_o^2 - R_B^2) / (r_o^2 - 2R_B r_o \cos\psi_{ij} + R_B^2)^{\frac{3}{2}} \tag{10}$$

In fact, what we have here is nothing but the Poisson-kernel used for the upward continuation from a sphere with radius R_B of the function $r \cdot \Delta g$, which in spherical approximation is a harmonic function.

In order to find K(P,Q), we use the expansion of $L_i^*(f_j)$ in Legendre polynomials, cf. (Heiskanen and Moritz, 1967, p. 35), and the fact that $L_i^* = -r\frac{\partial}{\partial r}|_{P_i} - 2\,ev_i$, where ev_i is the evaluation functional in the point P_i.

Then using eq. (7) we must have

$$K(L_j^*, L_k^*) = \sum_{i=2}^{\infty} \sigma_i (i-1)^2 \left(\frac{R^2}{r_o^2}\right)^{i+1} P_i(\cos\psi_{jk})$$

$$(=) \sum_{i=0}^{\infty}(2i+1) \left(\frac{R_B}{r_o}\right)^{i+1} P_i(\cos\psi_{jk}) \quad . \qquad (11)$$

Hence

$$\sigma_i = \frac{2i+1}{(i-1)^2} \left(\frac{R_B \cdot r_o}{R^2}\right)^{i+1} \quad . \qquad (12)$$

Note, that the use of the Poisson kernel is inconsistent with our basic requirement that $\tilde{T}$ fulfils the usual regularity conditions at infinity. Their correct use requires, that the two first terms are removed.

This show, that the use of Bjerhammar's Dirac approach in this limiting case will give us solutions, which minimalize the first order derivatives of $\tilde{T}$ at the surface of the sphere with radius $(R_B \cdot r_o)^{\frac{1}{2}}$, see Table 1.

Let me finally remind the reader, that methods of type (A) have been used for the determination of sets of harmonic coefficients of T. However, it has always been useful to introduce minimum norm conditions in order to get a realistic and stable solution, see e.g. (Moritz and Schwarz, 1973),(Reigber and Ilk, 1976) or (Lerch et al., 1977).

3. THE MOLODENSKY MOUNTAIN MODEL

The MMM was introduced in the English language literature in (Molodensky et al., 1962). It has probably, because of its simplicity, been used frequently since then, e.g. by (Moritz, 1965a).

The model may easily be changed, so it has been very useful when testing different approximation methods under varying conditions. However, we should keep in mind, that a method like least squares collocation specifically tries to take advantage of an apriori knowledge of the function, which one wants to approximate. We will therefore here derive this type of information, i.e. the empirical covariance function, for the MMM.

The anomalous potential is generated by m point masses M_i located at points P_i, $i=1,\ldots,m$, which all are on the same straight line passing through the origin. Let us suppose, that this line is taken as the Z-axis of our coordinate system. The points P_i will then have coordinates $(\varphi_i,\lambda_i,r_i) = (90^\circ,0^\circ,R_i)$ or $(-90^\circ,0^\circ,R_i)$. The potential of the i'th mass is then (with $q_i = P_j(\sin\varphi_i) = 1$ or -1, $r > R_i$),

$$V_i(\varphi,\lambda,r) = GM_i/(r^2+R_i^2-2rR_i\cos\psi)^{\frac{1}{2}}$$

$$= \frac{GM_i}{r}\sum_{j=0}^{\infty} q_i^j\left(\frac{R_i}{r}\right)^j P_j(\sin\varphi) = \frac{GM_i}{r}\sum_{j=0}^{\infty}\left(\frac{R_i}{r}\right)^j(2i+1)^{-\frac{1}{2}} q_i^j\,\bar{P}_{j0}(\sin\varphi)$$

$$= \frac{GM_i}{R}\sum_{j=0}^{\infty}\left(\frac{R}{r}\right)^{j+1}\left(\frac{R_i}{R}\right)^j(2j+1)^{-\frac{1}{2}} q_i^j\,\bar{P}_j(\sin\varphi)\ . \qquad (13)$$

We then have

$$T(P) = T(\varphi,\lambda,r) = \sum_{j=0}^{\infty}\left(\frac{R}{r}\right)^{j+1}\left(\sum_{i=1}^{m}\frac{GM_i}{R}\left(\frac{R_i}{r}\right)^j q_i^j\right)(2j+1)^{-\frac{1}{2}}\,\bar{P}_j(\sin\varphi),$$

an expression which only is valid for $r > \max_{i=1,\ldots,m}(R_i)$. We see, that in order that T fulfil the usual regularity conditions at infinity, we must to each point P_i, have another point $P_j = (-\varphi_i, 0, R_i)$ and $M_j = M_i$. Furthermore, one mass point must be located at the origin with mass equal to minus the sum of all other masses. This condition seems not to have been fulfilled by the MMM used up to now. However, the differences between the models used and a "corrected" model will generally be small, but not insignificant. The differences will be relatively larger for values of T than for values of derivatives of T.

Most users of the MMM have used a cone with height 4100 m and with Ω between 2' and 12'. Mostly only two point masses have been used with R_1 = 6367 km and R_2 = 6373 km, i.e. $R_2 > R$. The masses are then fixed by the requirement that a specific value of $\partial T/\partial r$ is produced by each mass at the vertex of the cone, e.g. 100 and 50 mgal, respectively.

The field of gravity disturbances ($-\partial T/\partial r$) produced by this will be negative everywhere outside the surface of the model, it has no random characteristics, and does not have a smoothness which resembles that of the Earth. Hence, the MMM can not at all be used to show which base functions (out of a certain sample) are most useful when applying method (A) for approximating the true anomalous potential. It may be used to compare methods of type of (A) with methods of type (B), but it can not show very much about how successful any of these methods will be when approximating the true T.

Because the empirical covariance function for $\max(R_i) < R$ may be computed explicitly, it should be possible to get useful insight into the behaviour of least squares collocation using covariance functions determined by various estimation procedures. On the other hand, it is obvious, that least squares collocation will loose in the competition with certain methods of type (A). Imagine, that we as base functions f_j select point mass potentials given by eq. (13) but with varying R_i. Hereby we will be able to find a $\tilde{T}$ which agrees very well, if not exactly, with any MMM.

4. AN EXAMPLE OF THE USE OF THE MMM

In (Katsambalos, 1981, Chapter 7) tests using the collocation method have been made using a covariance function related to the true anomalous potential. It is concluded (p. 98) that large errors may occur at altitudes larger than 10 km, and that taken for the full range of heights, the best results are obtained using the Dirac approach.

From the results of section 3, we should expect, that by using collocation with a covariance function for which σ_i tend to zero like i^{-1} for $i \rightarrow \infty$, we should be able to get results as good as these obtained using the Dirac approach. This was investigated using gravity disturbance data $-\partial T/\partial r$ on a conical mountain with Ω = 12', a data grid interval of 3' and

If we take all mass points to be below the mean Earth surface, then we see that the degree-variances are

$$\sigma_i = \left(\sum_{j=1}^{m} \frac{GM_j}{R} \left(\frac{R_j}{R}\right)^i q_j^i \right)^2 \frac{1}{2i+1} \; .$$

We see, that the degree-variances tend to zero like i^{-1}, so we should expect to get reasonably good results using the Dirac method as described in section 2. On the other hand, the "real" local field is if all $R_i < R$ decreasing more slowly, like when the degree-variances are constant. However, the most important conclusion is, that T varies in a manner significantly different from that of the Earth.

Besides a mass point gravity field, the MMM consist of a topography with the shape of a cone with a rounded top, see Fig.1 and (Katsambalos, 1981, Chapter 3). The model may be varied by changing the height of the cone and the extend of the cone in terms of the angular distance Ω on the mean earth sphere from the axis of the cone to the intersection of the cone and the sphere.

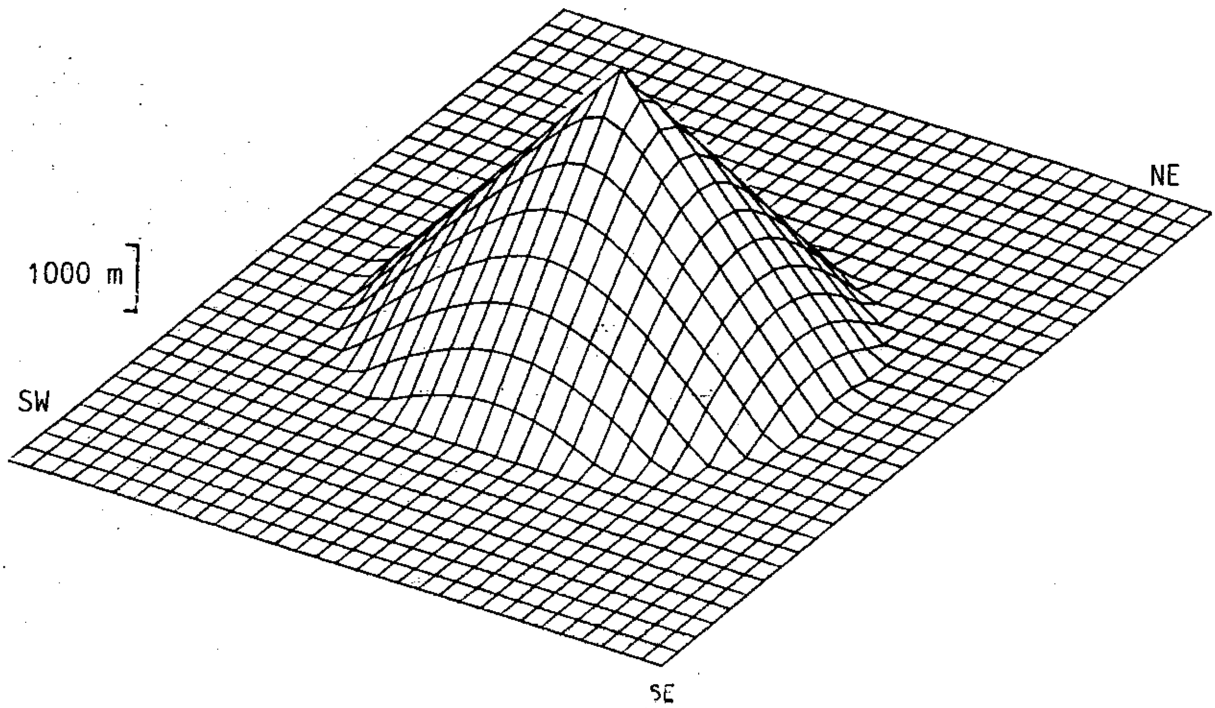

Fig. 1: Molodensky's Mountain with Ω =12'

a data region of 24' x 24', situated symmetrically around the MM . As covariance functions I used

$$K(P,Q) = \sum_{i=2}^{\infty} \frac{A(i-1)}{(i-2)(i+k)} \left(\frac{R_B^2}{rr'}\right)^{i+1} P_i(\cos\psi),$$

which could be evaluated using the subroutine COVAX, see (Tscherning, 1976), simulating that the data was of data type 5 (originally vertical gravity gradient $\partial^2 T/\partial r^2$).

It is clear, that the smoothness of T may be varied changing either the value of k or of the radius of the Bjerhammar sphere, R_B. Doing this, I found after a few iterations values of k and R, which at the altitudes 20 km and 100 km, respectively, gave results, which were better than these obtained by (Katsambalos, 1981) using the same cone, data type and region, but a data grid interval 1'. The results are given in Table 2.

It should be mentioned, that the use of other data grid intervals gave different optimal values of k and R_B. However, the test has definitively shown, that the above mentioned conclusion of (Katsambalos, 1981) must be modified. Personally, I would dare to conclude, that it seems to be worthwhile to take advantage of apriori information about the smoothness of the function, which one tries to approximate.

Table 2. Results of prediction tests using collocation compared to results obtained by (Katsambalos, 1981, Table 6.7 and 6.8). ϕ is the distance of the data point from the axis of the cone.

Value of $\partial T/\partial r$ computed from MMM			Difference $\partial T/\partial r - \partial\tilde{T}/\partial r$ using			
			"Dirac"	collocation, $R-R_B = 10$ m		
	height			k=20	k=45	k=50
ϕ	(km)	mgal	mgal	mgal	mgal	mgal
0'	20	12.07	-0.34	-0.59	-0.34	-0.30
2'	-	11.85	-0.35	-0.57	-0.32	-0.28
4'	-	11.22	-0.35	-0.53	-0.28	-0.24
6'	-	10.30	-0.37	-0.47	-0.21	-0.17
8'	-	9.20	-0.40	-0.41	-0.15	-0.11
10'	-	8.06	-0.41	-0.35	-0.08	-0.04
12'	-	6.95	-0.44	-0.31	-0.03	-0.01
0'	100	0.63	-0.08	-0.01	0.08	0.09
2'	-	0.63	-0.08	-0.01	0.08	0.09
4'	-	0.63	-0.08	-0.01	0.08	0.09
6'	-	0.62	-0.08	-0.02	0.07	0.08
8'	-	0.62	-0.08	-0.01	0.08	0.09
10'	-	0.61	-0.07	-0.02	0.07	0.08
12'	-	0.61	-0.08	-0.01	0.08	0.09

(Note, that the errors mainly have the character of a bias. This is a typical phenomena related to a too large or too small smothing. In fact experiments with a varying value of R_B showed that a decrease in its value gave larger errors.)

5. REMARKS ON THE USE OF NUMERICAL SIMULATION STUDIES

It is well-known, that the operational approach to gravity field determination (Moritz, 1978) (to which both methods (A) and (B) belong) inevitably make us introduce modifications to the ideal mathematical model. We use spherical approximation, regularized topography and too large sets of harmonicity for $\tilde{T}$. Numerical studies could be most useful in illustrating the effects of such model defects. However, in cases where these types of errors can be expressed mathematically, the errors will have bounds which depend linearily on some type of norm of T. (This is well-known from the error expression related to a Taylor expansion). Therefore, the simulation model should as far as possible resemble the Earth's gravity field and topography.

An alternative to the MMM is described in (Schwarz, 1977), but this model is also based on the use of point mass potentials. A more appropriate model could maybe be made using " mass dipoles", which then would enable us to simulate various types of isostatic models, simultaneously.

I have not with this paper tried to issue anything else than a warning against the overinterpretation of results of numerical studies. There are many problems, of actual importance which could be studies including

- the effect of using isotropic covariance functions, when the gravity field is anisotropic
- the effects of various types of terrain reductions
- the effects of strong correlations between various data types or between subsets of data
- the effect of using data with correlated errors
- the validity of certain types of error estimates
- the effect of spherical approximation
- the effect of using a set of harmonicity for $\tilde{T}$ larger than this of T
- the effect of the choice of various types of base functions or of various norms on upward or downward continued quantities,

just to mention a few. Some may be studied using the MMM (for example the first one) and some can only be studied using real data. But some, like the last one mentioned, can <u>only</u> be studied using simulated data, because real data are not yet available.

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