336

Geoid Modeling Using Collocation in Scandinavia and Greenland

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Abstract Collocation has been used for the construction of approximations to the anomalous gravitational potential in Scandinavia and Greenland, combining geopotential coefficients to degree and order 180, gravity anomalies, deflections of the vertical, and Doppler-satellite and satellite-altimeter-derived geoid heights. The method was also simultaneously used to determine a longitude bias of the deflections corresponding to a rotation around the Earth's Z-axis of -0.50 ± 0.26 arcsec.

Approximations were also determined for test purposes without using the (approximate) geoid heights determined by satellite altimetry. Using a semi–major axis of 6,378,136 m, the rms difference between observed and computed values was $\pm$ 0.4 m for the sea areas bordering Scandinavia up to a distance of 150 km from the coast, with an even better agreement in areas with available sea gravity data.

Introduction

The gravitational potential (W) is equal to the sum of the gravity potential (V) and the centrifugal potential (Φ), $W = V + \Phi$.

In principle, W is varying with time, but we will here regard W at a specific epoch. The geoid is then an equipotential surface of W, which agrees most closely with mean sea level at the same epoch. Outside the oceans, this surface will generally pass

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through the solid earth surface, and this fact makes it necessary to introduce the concept of the quasi-geoid.

Let U be an approximation to W, the normal potential, for which the reference ellipsoid is an equipotential surface with $U = W_0$, where W_0 is the value of W at the geoid. It is furthermore constructed so that the difference $T = W - U$, the anomalous potential, is a harmonic function regular at infinity.

To a point in space, P, corresponds a point Q located on the same normal to the ellipsoid as P, for which $U(Q) = W(P)$ (see Figure 1). The ellipsoidal height distance between P and Q is called the height anomaly $\zeta(P)$. This quantity can be used to construct another point, P', as the point on the ellipsoidal normal having the height $\zeta(P)$. The surface formed by taking all points P' corresponding to P at the Earth's surface is called the quasi-geoid. (That this construction is not always unique causes a small problem in physical geodesy.)

Therefore, if we are able to determine T, we can determine the quasi-geoid. Obviously, the geoid and the quasi-geoid are nearly identical at the oceans.

The advantage of dealing with T and the corresponding quasi-geoid stems from the fact that T is a harmonic function. It may be determined by solving a boundary value problem. Or ap-

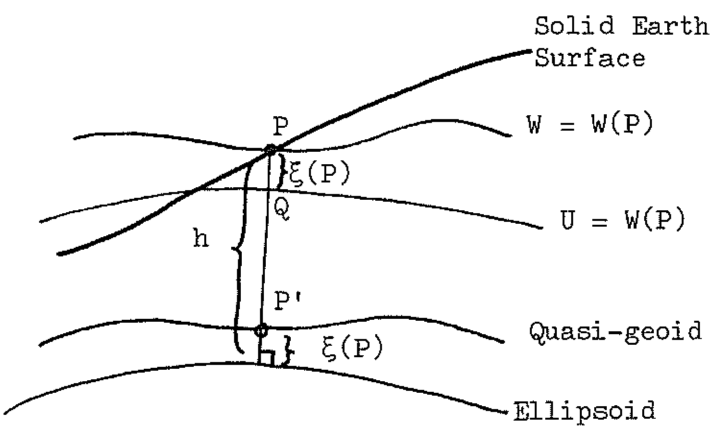

FIGURE 1. The relationship between the quasi-geoid and the height anomaly ξ (P).

proximations T' may be determined by numerical techniques for solving boundary-value problems.

The purpose of this paper is to describe some recent results obtained using collocation for the determination of an approximation T' and thereby of the quasi-geoid. I also want to point out some of the problems which arise when using this method. In the next section, a short review of collocation is given and the main problems arising when the method is used are described. The solutions to these problems and the results obtained in two areas, Greenland and Scandinavia, are described in the section after that. The results include comparisons with SEASAT-A sea surface heights.

Method of Collocation and Its Problems

We suppose that we have observed values $x(i)$ related to the gravity field, which may be expressed as the sum of a linear function L_i of T, the scalar product of a vector $\mathbf{A}_i$ and a parameter vector $\mathbf{X}$, and the noise $n(i)$,

$$x(i) = L_i(T) + \mathbf{A}_i^T \cdot \mathbf{X} + n(i) \quad (1)$$

The parameters may represent the effect of a datum-shift correction or some other systematic effect.

Collocation is then a technique that determines a (harmonic) function T' and estimates of parameters $\mathbf{X}'$ so that, for $n(i) = 0$, we have an exact agreement,

$$x(i) = L_i(T') + \mathbf{A}_i^T \cdot \mathbf{X}'$$

A unique solution is found, for example, by requiring T' to be an element of a reproducing-kernel Hilbert space, consisting of harmonic functions with norm $\| \cdot \|_H$, so that $\| T' \|_H^2 + \lambda\,(\mathbf{n}^T \cdot \mathbf{R}^{-1} \cdot \mathbf{n})$ is minimized. Here $\mathbf{R} = \{R_{ij}\}$ is the variance-covariance matrix of the noise vector $\mathbf{n}$ and λ is a positive constant.

If the so-called empirical covariance function is used as the reproducing kernel, then T' will be optimal in a least-squares sense (Moritz, 1980). In this case the method is called least-squares collocation. (Also, $\lambda = 1$ here.) We will discuss only this form for collocation in the following.

The covariance function of two values $L_P(T) = T(P)$ and $L_Q(T) = T(Q)$ is here equal to

$$\operatorname{cov}(L_P, L_Q) = K(P, Q) = \sum_{n=2}^{\infty} \sigma_n \left(\frac{R_E^2}{r_P r_Q}\right)^{n+1} P_n(\cos \psi_{PQ}) \quad (2)$$

where

- r_P, r_Q = the radial distances of P and Q, respectively, from the origin,
- ψ_{PQ} = the spherical distance between P and Q,
- σ_n = so-called degree-variances,
- R_E = the mean radius of the Earth,
- σ_n = the square sum of the nth degree fully normalized spherical-harmonic coefficients $\overline{C}_{nm}$ and $\overline{S}_{nm}$ occurring in the spherical harmonic expansion of T, multiplied by the gravitational constant and divided by R_E.

We will denote one or two linear functionals applied on K by

$$L_i[K(P, Q)] = K(L_i, Q) = K_i(Q)$$

$$L_i L_j[K(P, Q)] = K(L_i, L_j) = K_{ij}$$

Let us now regard a set of observations $x(i)$, $i = 1, \ldots, N$, and put

$$\overline{\mathbf{C}} = \{K_{ij}\} + \{R_{ij}\}, \qquad \mathbf{A} = \{A_{ik}\}, \qquad \mathbf{x} = \{x(i)\}$$

Then

$$T'(P) = \sum_{i=1}^{N} b_i \cdot K_i(P) \quad (3)$$

$$\mathbf{X}' = (\mathbf{A}^T \cdot \overline{\mathbf{C}}^{-1} \cdot \mathbf{A})^{-1} \cdot \mathbf{A}^T \cdot \overline{\mathbf{C}}^{-1} \cdot \mathbf{x} \quad (4)$$

$$\{b_i\} = \overline{\mathbf{C}}^{-1} (\mathbf{x} - \mathbf{A}^T \cdot \mathbf{X}') \quad (5)$$

Error estimates may also be computed for predictions $L(T')$,

$$\sigma^2[L(T')] = [K(L, L) - \{K(L, L_i)\}^T \cdot \overline{C}^{-1} \cdot \{K(L_j, L)\}] \quad (6)$$

in the simplest case with no parameters. Error expressions for the parameters can be found in Moritz, 1980. (Note that, as a condition for obtaining a solution, we must require at least $\overline{C}$ to be positive definite.)

The most severe problem should, in principle, be the determination of the empirical covariance function, Equation (2), as it involves the knowledge of the coefficients of the spherical-harmonic expansion of T, the function we are trying to approximate. However, in the use of collocation as a kind of interpolation method (e.g., predicting gravity anomaly values from other gravity values), the results have not been very sensitive to the kind of model used to describe the variation of the degree-variances. But problems have been found, for example, when computing geoid heights from gravity anomalies (Arabelos, 1980).

Another problem arises because of the global character of Equation (2). The solution becomes optimal only in the mean for all data point configurations, which, by a rotation of the coordinate system, will coincide The use of empirical covariance functions, which better describes the local behavior of the gravity field, may help in solving this problem (Schwarz and Lachapelle, 1980; Goad et al., 1984). Furthermore, the main causes of the anisotropies of the gravity field, namely the attraction of the topographic masses, should be removed, and later on restored (Forsberg and Tscherning, 1981a, 1981b).

In practice, a severe problem occurs due to the need for solving the system of linear equations, Equation (5), which in principle has as many equations as the number of observations, N. However, using Equation (6) with varying data configurations shows that the addition of new data will not always give any essential improvement of the needed result, such as the geoid in a specific local area.

The result will, to a great extent, depend on the local data distribution. Hence, it is reasonable to try to construct local, but overlapping, solutions as proposed in Tscherning (1975). Alternatively, one may use a covariance function that is put equal to zero for $\psi > \psi_0$, and then solve the equations using sparce-matrix techniques. (This alternative is presently being investigated.) But what should be the size of the overlap area? And how should the solutions be "glued" together? (Figure 4 illustrates this problem.)

Finally, the real practical problems are connected to the data:

1. lack of sufficient data;
2. data is disturbed by systematic effects or by random effects with (sometimes unknown) statistical characteristics;
3. data are disturbed by gross errors;
4. data are not properly related to a well-defined datum.

However, using collocation, we have the possibility to combine various data types and to model systematic and random effects in a consistent manner.

Recent Quasi-geoid Determinations in Greenland and Scandinavia

In the last years we have seen a significant improvement in our knowledge of the global gravity field variations, primarily due to the collection of sea surface heights by the satellites GEOS-3 and SEASAT (e.g., Rapp, 1981, 1982). The 180 × 180 degree and order potential coefficient sets (Lerch et al., 1981; Rapp, 1979, 1981) have been especially valuable for the construction of local approximations to the gravity field. By subtracting out the effect represented by one of these 180 × 180 sets, the covariance function of the residuals will get a correlation distance much smaller than before (Tscherning and Rapp, 1974; Tscherning, 1982, Figure 1). In the results discussed below, we have throughout used the Rapp (1979) solution, which seems to be the best in the investigated areas (Tscherning and Forsberg, 1982).

Greenland

In Greenland geoid undulations are determined at the coast as the difference between ellipsoidal heights determined by satellite

Doppler technique and heights above mean sea level determined by traditional geodetic techniques. The problem is to "extrapolate" these values to points far away from the coast (see Figure 2) so that the ellipsoidal heights can be "converted" into heights above mean sea level (Forsberg and Madsen, 1981).

Besides the potential coefficients and the height anomalies at the coast, we have point gravity values (20–30 km spacing) and preliminary 10 × 10 km mean topographic heights. Also, SEASAT sea surface heights have recently become available, but we have not yet been able to take advantage of this information.

Now, the point is that, without any special problems, it was possible to combine various data types in a rigorous way. The resulting geoid is shown in Figure 2. The geoid is estimated to have a standard deviation of 1.1 m. The magnitude of the value is caused by the quality of the Doppler-derived geoid heights and the (probably) rather large error in the contribution from the potential coefficients in North Greenland (see Table 1). The Doppler coordinates originally given in NWL9D were transformed using a scale change $\Delta L = -0.4 \times 10^{-6}$ and a rotation around the Z-axis, $\omega_Z = -.08''$. The main advantage of using collocation here has been that it was possible to combine all data types in a rigorous way.

Table 1
Survey of Comparison between Doppler-Derived Geoid Heights and Values Computed from Various Data Combinations—58 Stations Used

	Difference (Observed − Computed), m	
Data Used	Mean	Standard Deviation
None	28.85	9.31
Rapp, 1979 coefficients	−0.78	4.23
Rapp, 1979 and gravity	−0.80	3.01
Rapp, 1979, gravity, topography	−0.42	2.43

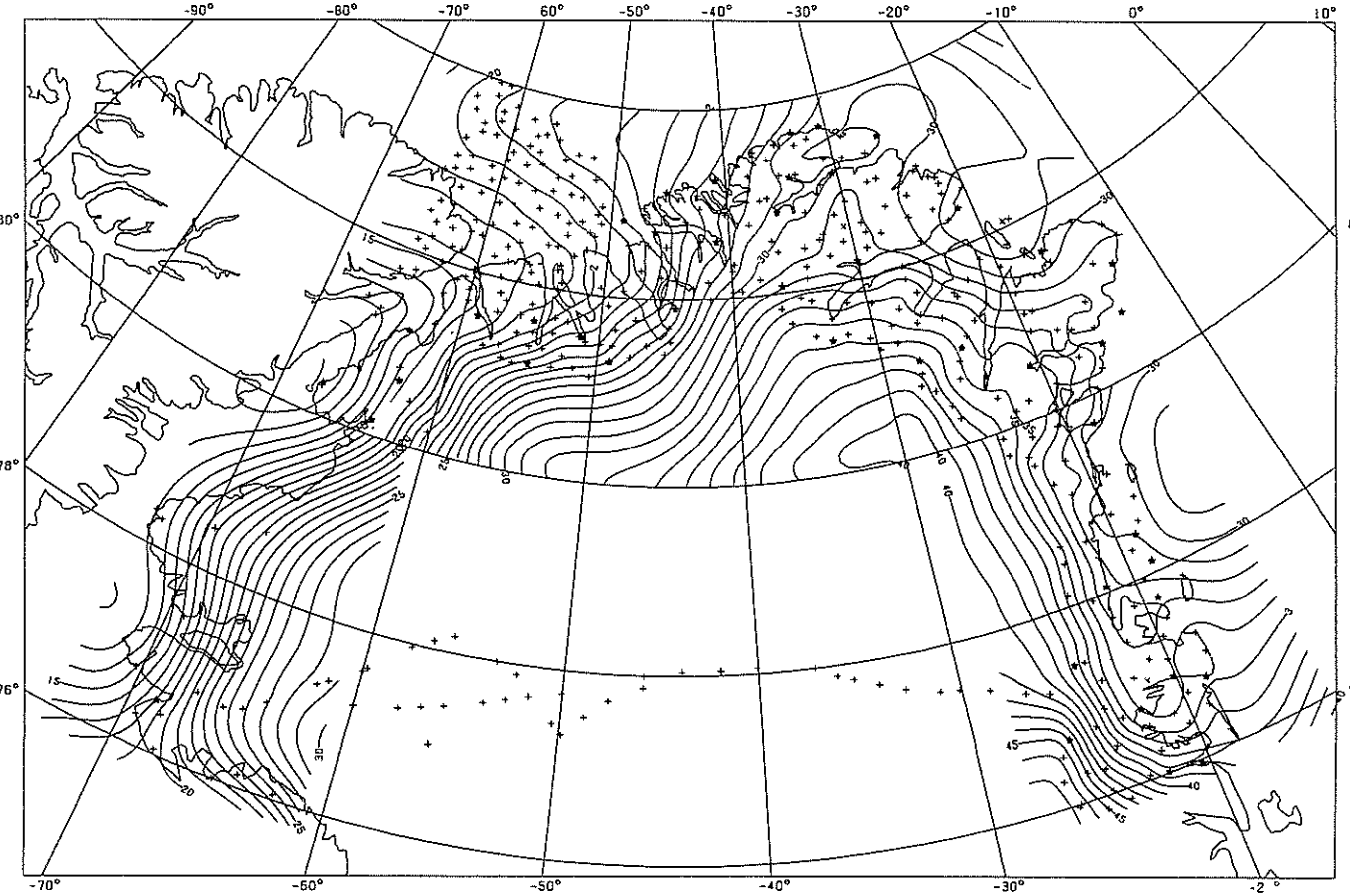

FIGURE 2. Geoid heights in GRS1980 based on gravity anomalies (+), Doppler-derived geoid undulations (*), and digitized mean topographic heights computed December 1981. (Contour interval = 1 m.)

Scandinavia

The present author is responsible for the computation of a standard geoid for the area of the Nordic countries. The object is to compute quasi-geoidal heights, for which the standard deviation of the differences is below 0.5 m. The project is described in detail in Tscherning (1982, 1983).

Many data are available, but they are rather irregularly distributed. In most of the Baltic Sea, no gravity values are available due to the political situation in this area. However, deflections of the vertical make it possible to "extrapolate" the geoid from the coast to the sea. And, fortunately, sea surface heights determined from satellite radar altimetry have now also become available. They have, however, not been used to their full extent because we must first investigate the magnitude of possible oceanographic effects. This may be done by computing geoid heights from independent data, such as topography, gravity, deflections of the vertical, and doppler-derived geoid undulations.

First approximations T', valid in blocks of approximately 220 $\times$ 220 km or larger, were computed using gravity data and gravity data combined with deflections of the vertical. The gravity data used were located as close as possible to the knots of a $3' \times 3'$ grid, with a denser spacing in areas with a strongly varying field. Data located within 0.5° distance from the area were used when available. The deflections of the vertical, which originally were given in ED1950, were transformed to an approximate geocentric reference system using parameters given in Ordnance Survey (1981) and by T. Vincenty (personal communication, 1980). This preliminary transformation was defined by the following values:

$$\Delta X = -89.5 \text{ m}, \qquad \Delta Y = -93.8 \text{ m}$$

$$\Delta Z = -127.6 \text{ m}, \qquad \Delta L = 1.4 \times 10^{-6}$$

$$\omega_X = 0.0'', \qquad \omega_Y = 0.0'', \qquad \omega_Z = 0.17''$$

A comparison with a set of Doppler-derived geoid heights and a set of SEASAT sea surface heights spaced 1° apart (Rapp, 1981, personal communication) showed that the best-fitting ellipsoid had to have a semi-major axis a = 6,378,136 m, and that the Doppler coordinate system needed a shift of the equatorial plane of 3.5 m. Using these parameters, the difference between 31 Doppler-derived geoid heights and geoid heights computed from Rapp's 1979 solution had a mean of 0.49 m and a standard deviation of ±0.78 m.

The approximations were used to compute the geoid heights in the points where SEASAT heights were available within the area of validity of the solutions (Tscherning, 1983, Table 5). In the southernmost block, which had an east–west length of 600

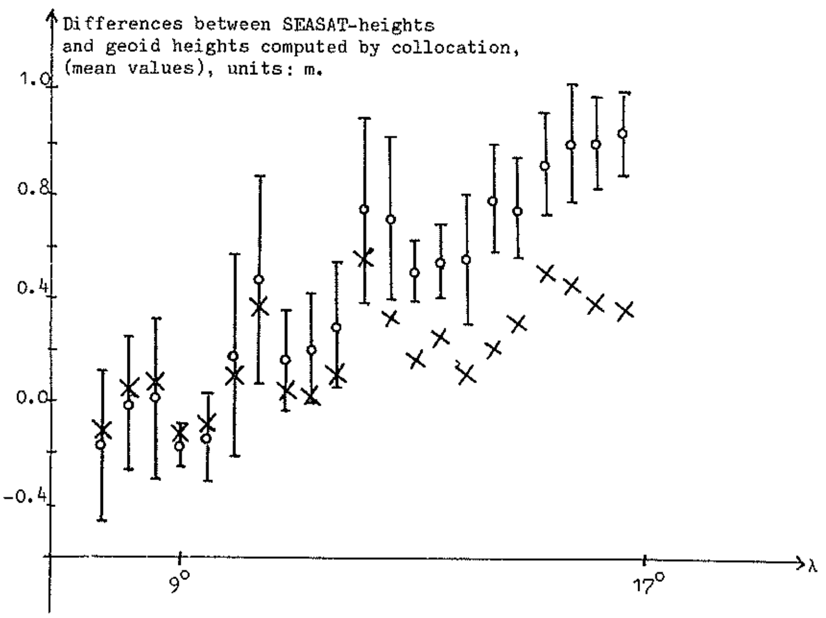

FIGURE 3. Mean values of differences between SEASAT sea surface heights and geoid heights determined using (O) gravity anomalies *and* deflections of the vertical and (×) only gravity anomalies. The differences have been samples in equidistant belts having a longitude extent of 0.25° (at φ = 56°) bounded by 54.5° ≤ φ ≤ 57.5°. The standard deviation for the differences in each class is shown as a bar for the first kind of differences (O). The 0.5″ tilt of the computed astrogravimetric geoid is seen clearly.

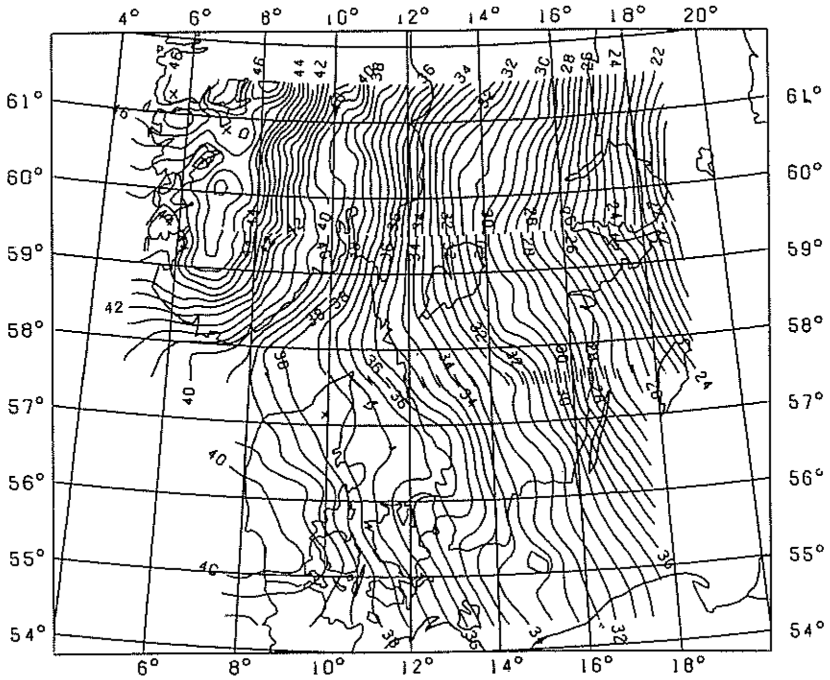

FIGURE 4. Geoid heights in GRS1980.

km, it was recognized that the computed geoid had a large tilt. This tilt was not present in the solution where only gravity data had been used. The tilt is illustrated in Figure 3.

The reason for this tilt is that the zero-meridian plane for the NWL9D system is not well defined (Hothem et al., 1982). Collocation was then used to determine a correction of the meridian plane, described as an unknown rotation ω_Z around the Z-axis. The value was first determined using only gravity and deflection data, and a value of $\omega_Z = -0.3''$ was found. However, this removed only part of the tilt. Second, SEASAT heights in the area bounded by $56° < \varphi < 58°$ and $7.8° < \lambda < 12°$ were used and treated as geoid heights, resulting in a value of $\omega_Z = -0.50'' \pm 0.26''$. A comparison between the heights used and geoid heights determined with only gravity data showed that the oceanographic effects in this area were probably small, while large ef-

Table 2
Results of Comparisons between Computed Geoid Heights and SEASAT Sea Surface Heights

Area				SEASAT Heights		Differences		
Latitude		Longitude						
min, °	max, °	min, °	max, °	Mean, m	Standard Deviation, m	Mean, m	Standard Deviation, m	Number of Values
61.5	63.5	14	19	20.81	1.59	−0.12	0.39	277
59.5	61.5	4	10	43.93	0.53	−1.34	0.42	65
59.5	61.5	14	19	22.31	1.13	−0.03	0.33	104
57.5	59.5	5	10	39.36	1.50	0.01	0.22	436
57.5	59.5	10	14	37.28	0.55	−0.11	0.19	57
57.5	59.5	14	19	24.31	2.06	−0.49	0.37	184
54.5	57.5	7	17	37.20	2.29	0.12	0.27	426

fects, up to 30 cm, are known to exist in the Danish Straits (Borre, 1970).

Using the updated parameters, new comparisons with the SEASAT heights were made. (Note that it was not necessary to compute the whole set of solutions to Equations (5) again, as the reduced normal equations had been saved on magnetic tape.) The comparison has not yet been finished, but the results obtained as of January 10, 1983, are given in Table 2.

The computed geoid is shown in Figure 4. Note the small disagreement between the solutions from the individual blocks. The discrepancy would have been slightly larger if we had not used, as "artificial" observations, geoid heights from the neighboring blocks on the south and east (when available).

The estimated standard deviations are shown in Figure 5. The values have been computed using local empirical covariance

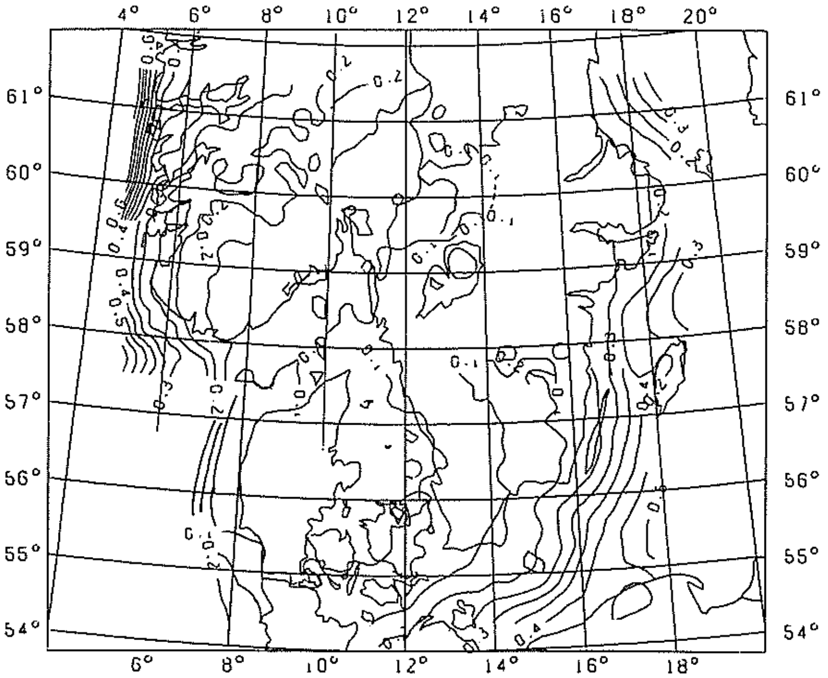

FIGURE 5. Estimated error, contour interval = 0.05 m.

functions, so that the local gravity field variation is reflected in a reasonable way.

The results are very satisfactory in all areas where sea gravity is available (Tscherning, 1983, Figure 1). The large error in one of the blocks is primarily due to the fact that no gravity values east of longitude 5° were used and to the local, very strong gravity variation. Fortunately, this is reflected very well in Figure 5, which shows the estimated prediction error.

Conclusion

The method of collocation has been used successfully for the determination of the geoid in large areas and for the determination of a significant longitude bias. The problems arising when using the method have not yet been completely solved. Whether the method will give better or worse results than other techniques will still have to be investigated.

An agreement with SEASAT sea surface heights is found at the 0.4-m level, with an even better agreement in areas with available sea gravity. Improved results can be expected if more data are used and the topographic and geological variations are taken into account, as we were forced to do in Greenland. However, the long-wavelength information represented by the potential coefficients should be improved in order to get satisfactory results in areas of high latitude, like Greenland.

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My colleagues R. Forsberg and F. Madsen have supplied the information concerning Greenland and the results presented in Figure 2. This paper is a contribution to a project supported by NATO Grant no. 320/82.

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