Note on Clenshaw summation on matrix form.

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$$V(\overline{\varphi}.\lambda, r) = \frac{GM}{r}\sum_{i=0}^{N}\left(\frac{a}{r}\right)^{i}\sum_{j=-i}^{i}\overline{P}_{ij}(\sin\overline{\varphi})\overline{C}_{ij}e_j(\lambda)$$

$$= \frac{GM}{r}\sum_{j=-N}^{N}e_j(\lambda)\sum_{i=|j|}^{N}\left(\frac{a}{r}\right)^{i}\overline{P}_{ij}(\sin\overline{\varphi})\overline{C}_{ij} = \frac{GM}{r}\sum_{j=-N}^{N}e_j\sum_{i=|j|}^{N}q^i\overline{P}_{ij}(t)\overline{C}_{ij},$$

with $t = \sin(\overline{\varphi}), \quad e_j = \mathrm{cov}(j\lambda)$ or $\sin(|j|\lambda)$, $q = a/r$

$$V(\overline{\varphi}.\lambda, a) = \frac{GM}{r}\sum_{j=-N}^{N}e_j(\lambda)S_i(t), \text{ with } S_0(t) = \sum_{j=0}^{N}\overline{P}_{j0}(t)\overline{C}_{j0}$$

We see we have to sum 2N+1 series with (fully normalized) Legendre functions.
Let us look at the most simple with q=1, j=0 and $\overline{C}_{0j} = c_j$ and un-normalized functions.

Then

$$S_0(t) = \{P_j(t)\}^T\{c_j\},$$

and using the recursion formula

$$P_j(t) = a_j(t)P_{j-1}(t) + b_jP_{j-2}, \text{ with } a_j(t) = -t(2j-1)/j, \quad b_j = (j-1)/j \text{ and } P_0(t) = 1$$

we have for N = 3

$$\begin{Bmatrix} 1 & -5t/3 & 2/3 & 0 \\ 0 & 1 & -3t/2 & 1/2 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{Bmatrix}\begin{Bmatrix} P_3(t) \\ P_2(t) \\ P_1(t) \\ P_0(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}, \text{ or } AP = I, \text{ with } I = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

This means that

$$S_0(t) = P^Tc = c(A^T)^{-1}I^T = sI^T = \{s\}\{I\}^T = s_0 \qquad \text{or}$$

since the sum is a (real) number, i.e. 1-dimensional matrix,

$$\begin{Bmatrix} s_3 \\ s_2 \\ s_1 \\ s_0 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ -5t/3 & 1 & 0 & 0 \\ 2/3 & -3t/2 & 1 & 0 \\ 0 & 1/2 & -t & 1 \end{Bmatrix}^{-1}\begin{Bmatrix} c_3 \\ c_2 \\ c_1 \\ c_0 \end{Bmatrix}$$

The system is easily solved, since we have an lower-triangular matrix, and the determinant is equal to 1 (so the system is never singular).

$$\begin{Bmatrix} s_3 \\ s_2 \\ s_1 \\ s_0 \end{Bmatrix} = \begin{Bmatrix} c_3 \\ c_2 + 5t/3s_3 \\ c_1 - 2t/3s_2 \\ c_0 - 1/2s_2 + ts_1 \end{Bmatrix} = c_0 + ts_1 - 1/2s_2$$

We see that the general algorithm is

$$s_j = c_j - a_{j-1}s_{j-1} - b_{j-2}s_{j-2}$$

Now, both s_j and a_j depends on t. Hence the derivative with respect to t is computed using

$$\frac{ds_j}{dt} = s'_j = -\frac{da_{j-1}}{dt}s_{j-1} - a_{j-1}s'_{j-1} - b_{j-2}s'_{j-2}$$

with $\frac{da_{j-1}}{dt} = (2\mathrm{j}-1)/(\mathrm{j}-1)$

Similar equations for the higher derivatives are easily derived. These simple equations are also in use when computing covariance functions for the (anomalous) gravity field.

Since the associated Legendre functions fulfill recursion formula similar to these of the Legendre functions, the algorithm is also valid for the partial sums $S_i(t)$. In the subroutine “gpotdr” they are computed one by one using that the same recursion formula are also valid for the quantities multiplied with q_i. For the Legendre polynomials we have

$$q^j P_j(t) - qa_j(t)(q^{j-1}P_{j-1}(t)) - q^2 b_j(q^{j-2}P_{j-2}(t)) = 0,$$

i.e. the reqursion algorithm coefficients are multiplied with powers of q.

Recursion formula are also easily written down for definite integrals (with respect to t) of the functions, and simple reqursion formulae for the sums are then derived.

Exercise: write down the algoritm for usual polynomials.

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