



Optimization of Gradient Prediction
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Page: 1 of 20

GOCE High Level Processing Facility

Optimization of Gradient Prediction

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Abbreviations and Acronyms

AD	Applicable Document	EGG-C	European GOCE Gravity Consortium
ADD	Architectural Design Document	EGM	Earth Gravity Model
ADIR	Architectural Design and Interface Review	EGM96	EGM 1996
ADP	Auxiliary Data Provider	EM	Engineering Model
AIT	Acceptance, Integration, Test	EME2000	Equinox and Mean Equator of J2000.0
AO	Announcement of Opportunity	EO	Earth Observation
AR	Acceptance Review	EOEP	Earth Observation Envelope Programme
AS	Anti-Spoofing	EPAR	Extended mission Product Acceptance Review
ATP	Authorisation To Proceed	ESA	European Space Agency
ATR	Algorithm Test Review	FM	Flight Model
CAB	Change Appeal Board	FOCC	Flight Operations Control Centre
CBCP	Current Baseline Cost Plan	FOS	Flight Operations Segment
CCN	Contract Change Notice	GLONASS	GLOBAL NAVIGATION Satellite System
CDAF	Command and Data Acquisition Facility	GOCE	Gravity field and steady-state Ocean Circulation Explorer
CDP	Configuration and Documentation management Plan	GPS	Global Positioning System
CDR	Critical Design Review	GRACE	Gravity Recovery And Climate Experiment
CFI	Customer Furnished Item	GRF	Gradiometer Reference Frame
CHAMP	CHALLENGING Minisatellite Payload for geophysical research and application	GS	Ground Segment
CMF	Calibration and Monitoring Facility	GSOV	Ground Segment Overall Validation
CNL	Contract change Notices status List	GSRR	Ground Segment Readiness Review
COP	Commissioning Operations Phase	HK	House-Keeping
COS	Consortium Organisation Structure	HOP	Hibernation Operations Phase
CPF	Central Processing Facility	HPF	High level Processing Facility
CPR	Cycle Per Revolution	HW	Hardware
CPS	Company Project Structure	IAG	International Association of Geodesy
CR	Change Request	ICD	Interface Control Document
CRB	Change Review Board	IGS	International GPS Service
DCN	Document Change Notice	ILRS	International Laser Ranging Service
DDP	Design and Development Plan	IPF1	Instrument Processing Facility level 1
DFACS	Drag-Free and Attitude Control System	ISP	Instrument Source Packet
DPA	Data Processing Archive	ITT	Invitation To Tender
DPM	Detailed Processing Model	L	Level, L-band frequency
DSAT	Development Site Acceptance Test	LAN	Local Area Network
DTL	Documentation Tree and status List	LEOP	Launch and Early Orbit Phase
E2E	End-to-End Simulator	LORF	Local Orbital Reference Frame
ECMWF	European Centre for Medium-range Weather Forecast	LRR	Laser Retro-Reflector
ECP	External Calibration Products	LSC	Least Squares Collocation
ECSS	European Cooperation for Space Standardization	LTA	Long-Term Archive
EFRF	Earth Fixed Reference Frame	MBW	Measurement BandWidth
EGG	Electrostatic Gravity Gradiometer	MOP	Measurement Operational Phase
		MPS	Mission Planning System
		NA	Not Applicable



NRT	Near-Real Time	SMF	Software Maintenance Facility
OBCP	On-Board Control Procedures	SOW	Statement Of Work
OBT	On-Board Time	SPC	Satellite Prime Contractor
ORR	Operational Readiness Review	SPF	Sub-Processing Facility
OSAT	On-Site Acceptance Test	SPR	Software Problem Report
PAR	Product Acceptance Review	SPRL	Software PRoblems status List
PCD	Product Confidence Data	SRD	System Requirements Document
PDD	Product Definition Document	SRR	System Requirements Review
PDS	Payload Data Segment	SST	Satellite-to-Satellite Tracking
PF	Processing Facility	SSTI	Satellite-to-Satellite Tracking Instrument
PI	Principal Investigator	SSTR	Sub-System Test Review
POD	Precise Orbit Determination	STP	Software Test Plan
PSD	Packet Structure Definition; Power Spectral Density	SVT	System Validation Test
QL	Quick-Look	SW	SoftWare
QLP	Quick-Look Products	SWRD	SoftWare Requirements Document
RD	Reference Document	TBC	To Be Confirmed
RERF	Radial Earth-pointing Reference Frame	TBD	To Be Defined
RFQ	Request For Quotation	TC	TeleCommand
RMS	Root-Mean Square	TM	TeleMetry
RPF	Reference Planning Facility	TP	Test Plan
RSS	Root-Sum Square	TR	Test Report
S/C	Space-Craft	USF	User Services Facility
SCP	Secure Copy (remote file copy program)	UTC	Universal Time Coordinated
SDE	Software Development Environment	V0/1/2	Version 0/1/2
SFTP	Secure File Transfer Program	VC	Virtual Channel
SGG	Satellite Gravity Gradiometer	WAN	Wide Area Network
SLR	Satellite Laser Ranging	WBS	Work Breakdown Structure
		WP	Work Package
		XML	eXtensible Markup Language

Table of Contents

1. INTRODUCTION.....	7
1.1 PURPOSE AND SUMMARY	7
1.2 APPLICABILITY	7
2. APPLICABLE AND REFERENCE DOCUMENTS.....	8
2.1 APPLICABLE DOCUMENTS	8
2.2 REFERENCE DOCUMENTS.....	8
3. GOCE GROUND SEGMENT	10
3.1 OVERVIEW GOCE GROUND SEGMENT	10
3.2 HIGH-LEVEL PROCESSING FACILITY	10
4. OVERVIEW	11
5. SIMULTANEOUS PREDICTION OF ALL GRAVITY GRADIENTS.....	12
6. MULTIPROCESSING.	14
7. CONCLUSION.	18
8. APPENDIX. LSC AND CHOLESKY-FACTORISATION.....	19

		<p><i>Optimization of Gradient Prediction</i> Doc. Nr: GO-TN-HPF-GS-0214 Issue: 1.0 Date: 21.12.2007 Page: 7 of 20</p>
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1. INTRODUCTION

1.1 PURPOSE AND SUMMARY

The purpose of this document is to describe the tests that have been performed in the context of HPF BP1420. Improvements and the effects of the improvements which have made in the program GEOCOL for the prediction of gravity gradients are described.

1.2 APPLICABILITY

This document is part of V3 of the HPF and was requested during the AR-3 review process.

		<p style="text-align: right;"><i>Optimization of Gradient Prediction</i></p> <p>Doc. Nr: GO-TN-HPF-GS-0214 Issue: 1.0 Date: 21.12.2007 Page: 8 of 20</p>
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2.1 APPLICABLE DOCUMENTS

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 - [RD-3] GO-TN-ESA-GS-0017: GOCE Ground Segment Concept and Architecture
 - [RD-4] GO-SP-AI-0004: GPS Receiver Ground Processing Algorithms Specification
 - [RD-5] GO-SP-AI-0003: Gradiometer Ground Processing Algorithms Specification
 - [RD-6] GO-TN-AI-0067: Gradiometer Ground Processing Algorithms Documentation
 - [RD-7] GO-TN-AI-0068: Gradiometer Ground processing Analysis
 - [RD-8] GO-PL-AI-0039: Gradiometer Calibration Plan
 - [RD-9] GO-TN-AI-0069: Gradiometer On-Orbit Calibration Procedure Analysis
 - [RD-10] GO-RP-AI-0014: Mission Analysis Report
 - [RD-11] CS-MA-DMS-GS-0001: Earth Explorer Mission Conventions Document
 - [RD-12] GO-MA-AI-0002: GOCE User's Manual
 - [RD-13] GO-TN-AI-0027: Performance Requirements and Budgets for the Gradiometric Mission
 - [RD-14] GO-TN-IAPG-0001: Detailed Processing Model for EGG
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		<p style="text-align: right;"><i>Optimization of Gradient Prediction</i></p> <p>Doc. Nr: GO-TN-HPF-GS-0214 Issue: 1.0 Date: 21.12.2007 Page: 9 of 20</p>
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- [RD-16] GO-ID-ESC-FS-5070: FOS/PDS – PDS/SLR: Predicted Orbit File
 - [RD-17] GO-RS-ESA-GS-0052: Product Requirement Document
 - [RD-18] ECSS-M-00A: Policy Principles
 - [RD-19] ECSS-M-10A: Project Breakdown and Structures
 - [RD-20] ECSS-M-20A: Project Organization
 - [RD-21] ECSS-M-30A: Project Phasing and Planning
 - [RD-22] ECSS-M-40A: Configuration Management
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 - [RD-24] ECSS-M-60A : Cost Schedule Management
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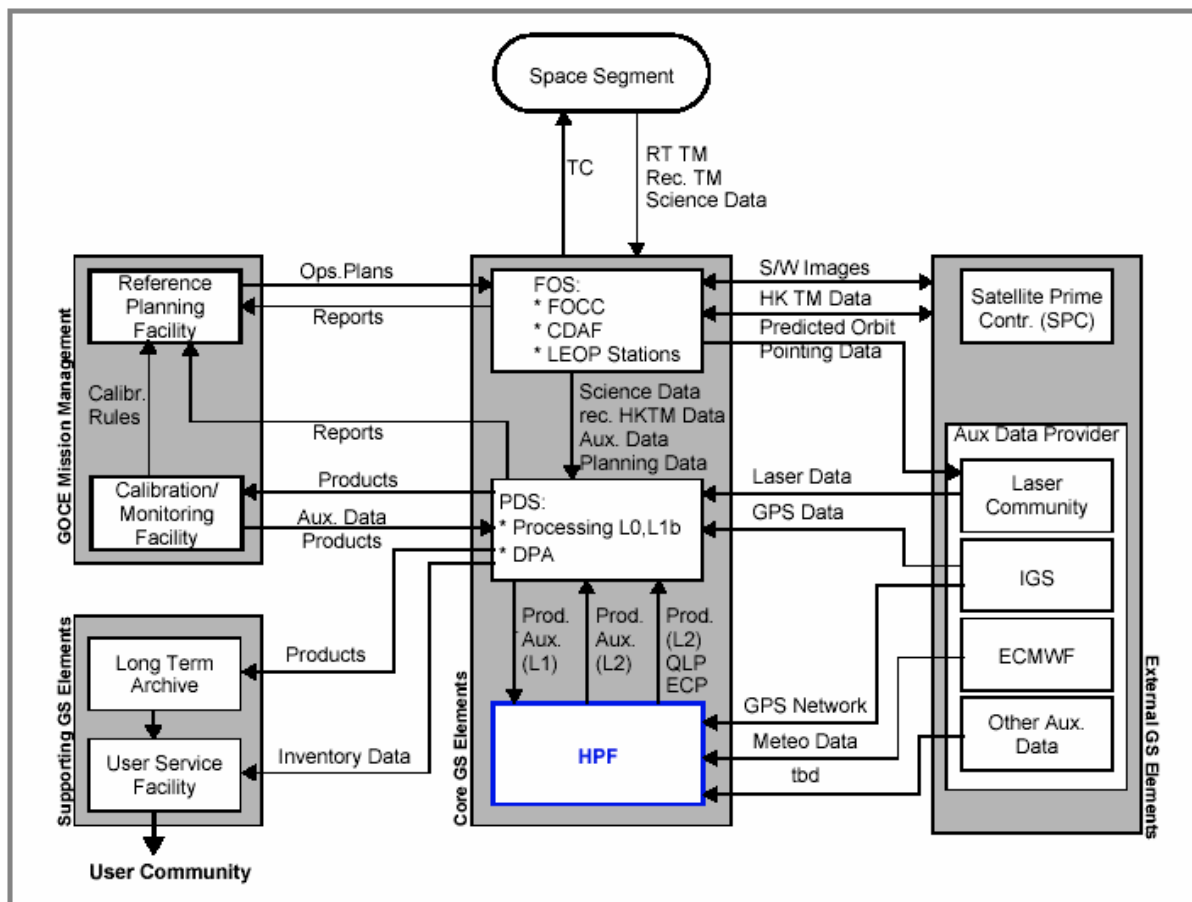


3. GOCE GROUND SEGMENT

3.1 OVERVIEW GOCE GROUND SEGMENT

The GOCE ground segment concept and architecture is described in [RD-3]. The following gives a brief summary of all ground segment elements, depicted in Figure 3-1.

Figure 3-1: GOCE Ground System



3.2 HIGH-LEVEL PROCESSING FACILITY

Within the GOCE GS the HPF is one of the Core GS Elements (ESA-controlled), and it is charged with the generation of L2 products and acquisition of the external (auxiliary) data needed to generate these products, the delivery of these products (auxiliary, intermediate and final) to the PDS/DPA and/or the LTA and the generation of QLP and ECP for the purpose of the activities of the CMF.

		<p style="text-align: right;"><i>Optimization of Gradient Prediction</i></p> <p>Doc. Nr: GO-TN-HPF-GS-0214 Issue: 1.0 Date: 21.12.2007 Page: 11 of 20</p>
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4. OVERVIEW

In the GOCE HPF WP3000 and WP7000 gravity gradients are predicted. In WP3000 gravity gradients in the gradiometer reference frame are predicted from ground gravity (Bouman et al. 2004) and in WP7000 the gradients are predicted in equi-angular grids from gradient values observed by GOCE, i.e. also in the gradiometer reference frame, (Tscherning, 2004).

The predictions are executed by the FORTRAN Program GEOCOL (Tscherning, 1974) using Least-Squares Collocation (Moritz, 1980) and the remove-restore principle (Tscherning, 1983). Basically all computations are carried out in a spherical reference frame, East, North, Radially-Up and then transformed to any other reference frame by rotations (Tscherning, 1979, 2004). In order to execute the transformations reference model values and covariances of all derivatives of the same degree must be available simultaneously.

One may take advantage of this both when establishing the normal-equations and when predicting all components of the gravity gradient and calculating their error-estimates. Here only improvements related to the prediction of gravity gradients have been made and will be described in Chapter 5.

Many computers are today equipped with several processors. For gravity gradient prediction we may take advantage of this when (a) establishing the normal equations, (b) executing Cholesky-reduction of the equations (c) predicting gravity gradients and (d) calculating error-estimates and error-correlations. Only improvements related to (b) and (d) have been implemented because the implementation in (a) and (c) requires very complicated changes of the GEOCOL program. The changes and the effect of the changes are reported in Chapter 6.

5.SIMULTANEOUS PREDICTION OF ALL GRAVITY GRADIENTS.

The prediction of all gradient components in a point requires the computation of several quantities which all sum up to the predicted quantity:

Function	Subroutine	Interface through
Reference system change	TRANS	COMMON Block
Contribution from EGM	GPOTDR	Same
LSC contribution from data	COPRED, PRED	Same
Error-estimation	NES, NES_MP	File on Disk
Output of result	COUT	COMMON Block

Table 1. Functionalities and subroutines changed to enable prediction of all gravity gradients

The usual prediction equation is based on that the anomalous potential is split into 3 parts

$$L_p T = L_p(T_{TRANS}) + L_p(T_{GPOTDR}) + L_p(T_{LSC})$$

where L_p is the linear functional associated with the predicted quantity,

e.g. a second order along track derivative.

The calculation will for gravity gradients involve a rotation to or from an East/North/Up local system, i.e. all components of the gradient will have to be computed in order to make the rotation.

The LSC contribution is computed as the product sum of the covariances between the observations and the individual gravity gradients:

$$L_p(T_{LSC}) = \sum_{i=1}^N b_i \text{cov}(L_p, L_i), \text{ where}$$

b_i are the solutions to the normal - equations

N the number of observations and

L_i the observation functionals.

The computational saving is due to that all covariances between the gradient components and the observation are available simultaneously. Obviously, the product sum between observations and predicted quantities has to be calculated in all cases.

The saving may be illustrated by the number from a typical WP3000 functionality (Arabelos and Tscherning, 2005, Tscherning and Veichert, 2006), where gravity gradients in the Scandinavian area are predicted from 14728 surface gravity anomalies. The computation time for one predicted component was 1.36 s while it for all 6 components was 1.41 s. I.e. as expected a six-fold gain in computational speed.

		<p><i>Optimization of Gradient Prediction</i> Doc. Nr: GO-TN-HPF-GS-0214 Issue: 1.0 Date: 21.12.2007 Page: 13 of 20</p>
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There are also computational savings when calculating error-estimates. These estimates are calculated as the Cholesky-reduced of the N-vector or the Nx6 matrix containing the covariances between the predicted quantities and the observations. The savings are again not in the formation of the product sums used when calculating the Cholesky-reduced, but in the reduction of disk-transports between the central memory of the computer and the disk where the reduced normal-equations are stored.

6.MULTIPROCESSING.

Multiprocessing has been implemented in order to speed-up the two most heavy tasks, namely the Cholesky factorisation of the normal equations and the computation of error-estimates and error-correlations.

The GEOCOL program has in its earlier versions exclusively been using Fortran77, but the use of multiprocessing requires the use of Fortran90. Three compilers have been available for the creation of executables: Portland Group pgf90, Intel and Power FORTRAN compilers.

The key to the parallelisation is that when a diagonal element in column j has been reduced may each element in the j 'th row be reduced separately

$$L_{ij} \cdot L_{jj} = (C_{ij} - \sum_{m=1}^{j-1} L_{im} L_{jm}),$$

since it involves only row - elements in same column j .

Hence each processor may take care of one column at a time.

Now the question is

- Should one row be reduced at a time ?
- Or should we make the factorisation of blocks of elements ?
- Out-of-core factorisation needed for large matrices, so let the processors work on blocked matrices.

The last option has been selected in order to handle very large systems of equations or matrices added to the reduced normal-equation system, and is illustrated in Fig. 6.1.

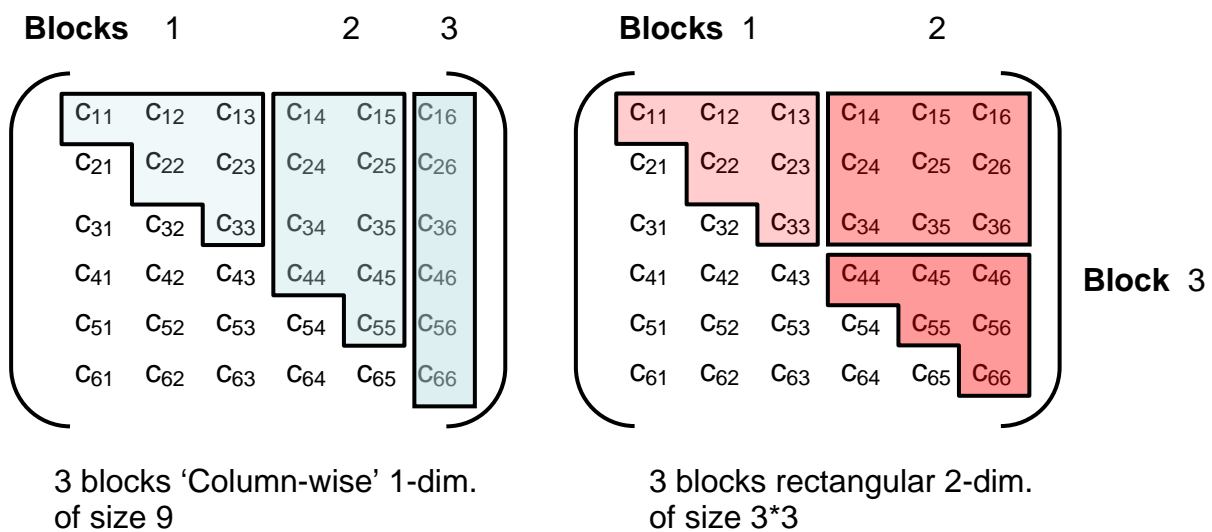


Fig. 6.1 Old and new blocking system used for the normal-equations.

The effect of the selected block-size was tested on systems of equations with 10000 and 20000 unknowns. A clear dependency on the block-size is seen in Fig. 6.2. It is probably related to the use of the cache when transferring data in and out of core.

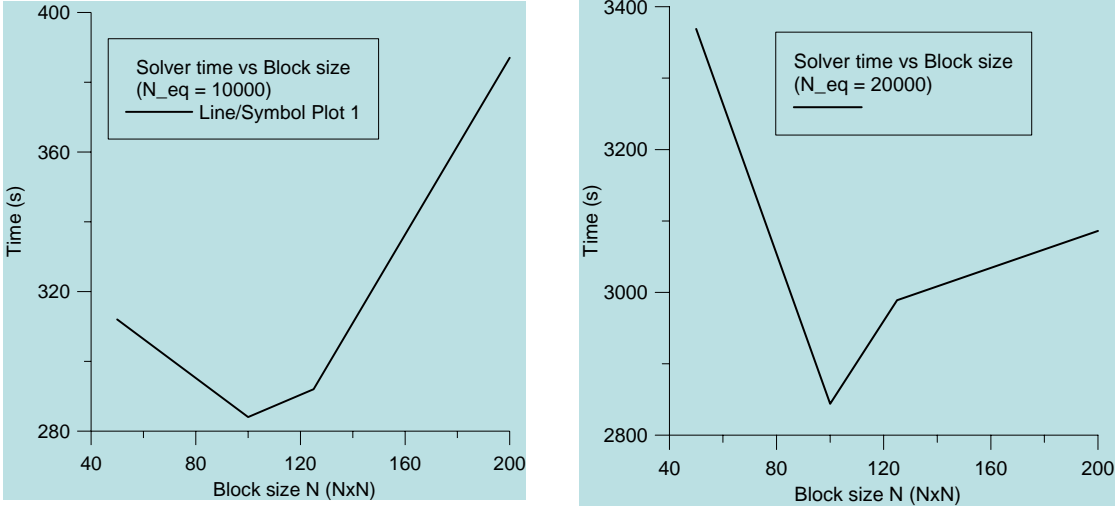


Fig. 6-2 Dependency of the execution time of the Cholesky factorisation of the normal-equations as a function of block-size.

The flow of the GEOCOL program segment which executes the Cholesky-factorisation is illustrated in Fig. 6-3. In the old versions of GEOCOL (17 and below) everything was executed by one subroutine “NES”. In the new version 18, a subroutine NES_MP is used to execute the factorisation, aided by a number of subroutines which executes the reduction or factorization.

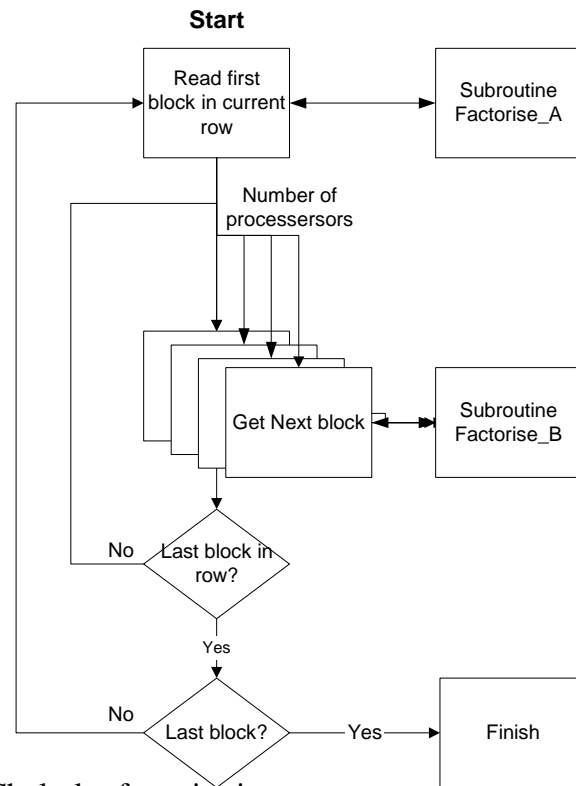


Fig. 6-3. Flow-chart of Cholesky-factorisation.

The performance of the subroutines have been tested on two computers of the Niels Bohr Institute, called “GOCE” and “IKOS”. In Table 6-1. results from using 1, 2 and 4 processors are shown.

		GOCE (4x3GHz, 2GB)		IKOS (4x2.66GHz, 4GB)	
PROC	NEQ.	NES	NES_MP	NES	NES_MP
1	6400	775	177		
2	6400		130	136	71
4	6400		87		
1	8100	1570	347		
2	8100		228		
4	8100		177		
1	10000	2966	650	586	290
2	10000		446		159
4	10000		369		

Table 6-1. Execution time for old and new subroutines NES and NES_MP in seconds as a function of number of equations (column headed NEQ) and number of processors shown for two different computers (Servers).

The saving in execution time is always somewhat below 50 % when 4 processors are used. It is however quite dependent on which optimization options have been used when compiling

the program. Other typical processing times for the Cholesky-factorisation are shown in Table 6-2.

Server	NEQ	Geocol17	Geocol18 Processors		
			1	2	4
GOCE	5000	370	80	46	24
	10000	2971	851	630	354
	20000	9464	4249	2844	2081
IKOS	5000		23		
	10000		330		

Table 6-2. Execution times using old (GEOCOL17) and new version (GEOCOL18) with respect to Cholesky factorisation.

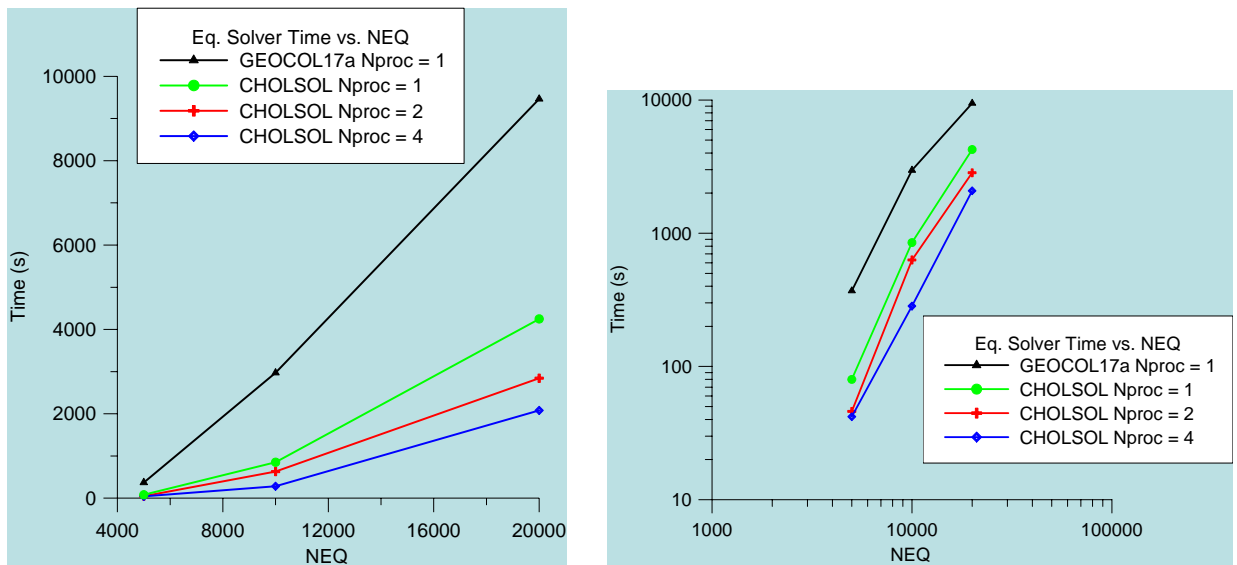


Fig. 6-4. Examples of execution times for the Cholesky-factorisation.

		<p style="text-align: right;"><i>Optimization of Gradient Prediction</i></p> <p>Doc. Nr: GO-TN-HPF-GS-0214 Issue: 1.0 Date: 21.12.2007 Page: 18 of 20</p>
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7.CONCLUSION.

The optimization has been successfully implemented with substantial savings in processing time for gravity gradient prediction. The application of the “analytic” property that all quantities related to the 6 independent gravity gradient components resulted in a factor 6 in saving when predicting all components.

The implementation and use of multiprocessing gives substantial savings in computational time for the Cholesky factorisation. Factorisation is computational time inversely proportional to the number of available processors are found in a number of tests. The reduction is slightly less than 50 % when 4 processors are used. It could in theory be 25 %, but it is not possible to use all processors all the time.

But let us mention a final result from a processing example where 39606 equations were solved. The execution time was 30360 seconds = 56 minutes. This opens up for a number of large computational experiments using GOCE data, see e.g. Arabelos and Tscherning, 2007.



8.APPENDIX. LSC AND CHOLESKY-FACTORISATION.

The basic observation equation for LSC is

$$y_i = L_i(T_{LSC}) + e_i + A_i^T X, \text{ where}$$

X are contingent parameters, A_i is a vector connecting parameters and the observations, e_i is the error contribution.

Here the contribution from a contingent datum-transformation and a EGM must have been subtracted.

The estimate of T_{LSC} is obtained by

$$\tilde{T}_{LSC}(P) = \{C_{Pi}\}^T \bar{C}^{-1} \{y - A^T X\}, \text{ where } \bar{C} = \{C_{ij} + \sigma_{ij}\}, \text{ and}$$

σ_{ij} is the variance - covariances of the errors.

The estimate of the (M) parameters are obtained by

$$\hat{X} = (A^T \bar{C}^{-1} A + W)^{-1} (A^T \bar{C}^{-1} y)$$

The error-estimates and error-covariances, ec_{kl} are found with:

$$H = \{COV(L_k, L_i)\}^T \bar{C}^{-1}, \text{ MxN matrix}$$

$$m_X^2 = (A^T \bar{C}^{-1} A + W)^{-1}$$

$$\{ec_{kl}\} = \{\sigma_{kl}\} - H \{cov(L_j, L_l)\} + HAM_X (HA)^T$$

W is the matrix of contributions from observations only related to parameters such as differences between geodetic coordinates in a geocentric and non-geocentric datum.

Despite the occurrence of the mathematical sign for inverse (-1) as a superscript, it is not necessary to compute the inverse. Only the solution is needed. This may be obtained by Cholesky factorisation. With L being a lower-triangular matrix we have

$$\bar{C} x = LL^T x = y$$

$$L^{-1}(LL^T)x = L^{-1}y$$

$$L^T x = L^{-1}y, \text{ now upper - triangular system.}$$

$$L_{ij} = (C_{ij} - \sum_{m=1}^{j-1} L_{im} L_{jm}) / L_{jj}$$

$$L_{jj} = (C_{jj} - \sum_{m=1}^{j-1} L_{jm}^2)^{1/2}$$

		<p style="text-align: right;"><i>Optimization of Gradient Prediction</i></p> <p>Doc. Nr: GO-TN-HPF-GS-0214 Issue: 1.0 Date: 21.12.2007 Page: 20 of 20</p>
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The procedure may be generalized to include also the parameter-estimates by regarding an extended matrix equation.

$$\begin{Bmatrix} \{\bar{C}_{ij}\}_{nn} & \{A_{jl}\}_{mn} \\ \{A_{ik}\}_{nm}^T & \{W_{kl}\}_{mm} \end{Bmatrix} \begin{Bmatrix} x \\ X \end{Bmatrix} = \begin{Bmatrix} y \\ p \end{Bmatrix},$$

This system may be Cholesky factorized using negative accumulation when subscript are $\geq n$.

It may be even further generalized, by extending the system with the matrices which contain the covariances between observations and quantities to be predicted.

$$\begin{Bmatrix} \{\bar{C}_{ij}\}_{pp} & \{A_{jl}\}_{pq} & \{C_{jm}\}_{pr} \\ \{A_{ik}\}_{qp}^T & \{W_{kl}\}_{qq} & \{A_{jk}\}_{qr} \\ \{C_{in}\}_{rp}^T & \{A_{ni}\}_{rq}^T & \{\sigma_{nm}\}_{rr} \end{Bmatrix}$$

when factorized with negative accumulation for subscript larger than p and no accumulation for subscript large than $p + q$ returns error - covariances.