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NOTE ON COVARIANCE EXPRESSIONS FOR SATELLITE TO SATELLITE TRACKING

1. Introduction

In the following we will discuss the computation of the covariances needed when the method of collocation is used to extract gravity field information from a satellite to satellite tracking (SST)-experiment. We will use the term "covariance" synonymeous with "two linear functionals applied on a reproducing kernel", i.e.

$$\text{cov}(L_i(T),L_j(T)) = \text{cov}(L_i,L_j) = L_iL_jK(P,Q), \qquad (1)$$

where K(P,Q) is the reproducing kernel, L_i, L_j are the two linear functionals and P, Q are two points in R^3. The spherical coordinates of these two points will be denoted (r_P, φ_P, λ_P), (r_Q, φ_Q, λ_Q), respectively (r = distance from the origin, φ = (geocentric) latitude and λ = longitude). With each point P and Q we will also introduce a local orthonormal frame with coordinates (x, y, z), with zero point P, (x', y', z') with zero point Q. The z-axis will point in the direction of the radius vector, x will be North and y East.

We will here only regard rotationally invariant (isotropic) reproducing kernels. These kernels can then be expressed as a Legendre series.

$$K(P,Q) = \sum_{i=0}^{\infty} \sigma_i \left(\frac{R^2}{r_P r_Q}\right)^{i+1} P_i(\cos\psi), \qquad (2)$$

where R is the radius of the Bjerhammar-sphere, ψ is the spherical distance between P and Q, P_i are the Legendre polynomials and σ_i are positive constants, the so-called degree-variances. We also have

$$K(P,Q) = \sum_{i=o}^{\infty} \frac{\sigma_i}{2i+1} \left(\frac{R^2}{r_P r_Q}\right)^{i+1} \sum_{j=o}^{i} \bar{P}_{ij}(\sin\varphi_P)\, \bar{P}_{ij}(\sin\varphi_Q)$$

$$\cdot\ (\cos(j\lambda_P)\cos(j\lambda_Q) + \sin(j\lambda_P)\sin(j\lambda_Q)), \qquad (3)$$

where $\bar{P}_{ij}$ are the fully normalized associated Legendre polynomials, cf. Heiskanen and Moritz (1967, eq. (1-77 a, b)).

The linear functionals which are of interest for SST are all *linear combinations* of the following:
the evaluation functional

$$L_P(T) = T(P), \qquad (4)$$

the radial derivative in P

$$L_z(T) = \frac{\partial}{\partial r_P}(T) \qquad (5)$$

the derivative in northern direction

$$L_x(T) = \frac{1}{r_P} \frac{\partial}{\partial \varphi_P}(T) \qquad (6)$$

the derivative in eastern direction

$$L_y(T) = \frac{1}{r_P \cos\varphi_P} \frac{\partial}{\partial \lambda_P}(T). \qquad (7)$$

The most used linear combinations are the derivatives with respect to cartesian coordinates, X, Y, Z. With

$$X = r\cos\varphi\cos\lambda$$
$$Y = r\sin\varphi\sin\lambda$$
$$Z = r\sin\varphi$$

we have

$$\frac{\partial}{\partial X} = \cos\varphi\cos\lambda\frac{\partial}{\partial r} - \sin\varphi\cos\lambda\frac{1}{r}\frac{\partial}{\partial\varphi} - \sin\lambda\frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda}$$

$$= \cos\varphi\cos\lambda\, L_z - \sin\varphi\cos\lambda\, L_x - \sin\lambda\, L_y$$

$$\frac{\partial}{\partial Y} = \cos\varphi \sin\lambda \frac{\partial}{\partial r} - \frac{1}{r} \sin\varphi \sin\lambda \frac{\partial}{\partial \varphi} + \cos\lambda \cdot \frac{1}{r \cos\varphi} \frac{\partial}{\partial \lambda}$$

$$= \cos\varphi \sin\lambda \, L_z - \sin\varphi \sin\lambda \, L_x + \cos\lambda \cdot L_y$$

$$\frac{\partial}{\partial Z} = \sin\varphi \frac{\partial}{\partial r} + \cos\varphi \frac{1}{r} \frac{\partial}{\partial \varphi} = \sin\varphi \, L_z + \cos\varphi \, L_x .$$

Integration of $\frac{\partial T}{\partial X}, \frac{\partial T}{\partial Y}, \frac{\partial T}{\partial Z}$ along a satellite orbit give as result the velocity change of the satellite due to the potential T. We will denote the curve integral from a point a to a point b along a specific orbit by I_a^b. Then

$$I_a^b \left(\frac{\partial T}{\partial X}\right) = \int_a^b \frac{\partial T}{\partial X} \Big|_P \, ds_P =$$

$$I_a^b(\cos\varphi \cos\lambda \, L_z T) - I_a^b(\sin\varphi \cos\lambda \, L_x T) - I_a^b(\sin\lambda \, L_y T), \quad (8)$$

$$I_a^b \left(\frac{\partial T}{\partial Y}\right) = \int_a^b \frac{\partial T}{\partial Y} \Big|_P \, ds_P =$$

$$I_a^b(\cos\varphi \sin\lambda \, L_z T) - I_a^b(\sin\varphi \sin\lambda \, L_x T) + I_a^b(\cos\lambda \, L_y T), \quad (9)$$

$$I_a^b \left(\frac{\partial T}{\partial Z}\right) = \int_a^b \frac{\partial T}{\partial Z} \Big|_P \, ds_P$$

$$I_a^b(\sin\varphi \, L_z T) + I_a^b(\cos\varphi \, L_x T) \quad (10)$$

We will in the following suppose that the "true" orbit (or parts of the orbit) in an Earth-fixed coordinate system has been approximated by (a part of) a circle with centre in the mass center of the Earth. This will enable us to perform a change of variables, so that the orbit is (a part of) a circle in the equatorial plane, i.e. the arc-length s becomes equal to the longitude times r_P. Hence e.g.

$$I_a^b \left(\frac{\partial T}{\partial X}\right) = I_a^b(\cos\lambda \, L_z T) - I_a^b(\sin\lambda \, L_y T) \quad (11a)$$

$$I_a^b \left(\frac{\partial T}{\partial Y}\right) = I_a^b(\sin\lambda \, L_z T) + I_a^b(\cos\lambda \, L_y T) \quad (11b)$$

$$I_a^b \left(\frac{\partial T}{\partial Z}\right) = I_a^b(L_x T). \quad (11c)$$

2. *The linear functionals applied on the reproducing kernel of a finite dimensional subspace*

We will suppose that the point P has spherical coordinates ($r_p = r$, $\varphi_p = 0$, $\lambda_p = \lambda$). The (new) coordinates of Q will again be denoted (r_Q, φ_Q, λ_Q). Expressions for the functionals eq. (4) - (7) applied on different K(P,Q) can be found in Tscherning (1976). The functionals were applied on closed expressions representing infinite series like eq. (2). The derived expressions contained logarithmic terms for which the curve integrals eq. (8) - (10) can not be expressed analytically. As an alternative, the corresponding infinite series may be cut off at a certain degree I and this finite series can then be integrated term by term. Mathematically this corresponds to that we operate in a finite dimensional Hilbert space. (Dimension equal to $(I + 1)^2$).

We must therefore be able to apply the functionals on the single terms occurring in eq. (3). We put

$$V_{nm}(P) = (\frac{R}{r})^{n+1} \bar{P}_{nm}(\sin\varphi) \left\{ \begin{matrix} \cos m\lambda \\ \sin m\lambda \end{matrix} \right\},$$

$$SC(n) \int_a^b \sin n\lambda \cdot \cos\lambda \, d\lambda = \left[- \frac{\cos(n-1)\lambda}{2(n+1)} - \frac{\cos(n+1)\lambda}{2(n+1)} \right]_b^a \qquad n > 1$$

$$= \tfrac{1}{2}\left[\sin^2\lambda \right]_a^b \qquad n = 1$$

$$= 0 \qquad n = 0$$

$$SS(n) = \int_a^b \sin n\lambda \cdot \sin\lambda \, d\lambda = \left[\frac{\sin(n-1)\lambda}{2(n-1)} - \frac{\sin(n+1)\lambda}{2(n+1)} \right]_a^b \qquad n > 1$$

$$= \tfrac{1}{2}\left[\lambda - \sin\lambda \cos\lambda \right]_a^b \qquad n = 1$$

$$= 0$$

$$CC(n) = \int_a^b \cos n\lambda \cos\lambda \, d\lambda = \left[\frac{\sin(n+1)\lambda}{2(n+1)} + \frac{\sin(n-1)\lambda}{2(n-1)} \right]_a^b \qquad n > 1, \; n = 0$$

$$= \tfrac{1}{2}\left[\cos\lambda \sin\lambda + \lambda \right]_a^b \qquad n = 1$$

$$CS(n) = \int_a^b \cos n\lambda \sin\lambda \, d\lambda = \left[-\frac{\cos(1-n)\lambda}{2(1-n)} - \frac{\cos(n-1)\lambda}{2(n+1)} \right]_a^b \qquad \begin{matrix} n > 1 \\ n = 0 \end{matrix}$$

$$= \tfrac{1}{2} \left[\cos^2\lambda \right]_a^b \qquad n = 1 .$$

Then

$$\cos\lambda \cdot L_z V_{nm} = -\frac{n+1}{r} \left(\frac{R}{r}\right)^{n+1} \bar{P}_{nm}(0) \begin{Bmatrix} \cos\lambda \cdot \cos m\lambda \\ \cos\lambda \cdot \sin m\lambda \end{Bmatrix} \tag{12a}$$

$$\sin\lambda \; L_z V_{nm} = -\frac{n+1}{r} \left(\frac{R}{r}\right)^{n+1} \bar{P}_{nm}(0) \begin{Bmatrix} \sin\lambda \; \cos m\lambda \\ \sin\lambda \; \sin m\lambda \end{Bmatrix} \tag{12b}$$

$$L_x V_{nm} = \frac{1}{r} \left(\frac{R}{r}\right)^{n+1} \frac{\partial}{\partial\varphi} \bar{P}_{nm}(0) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \tag{13}$$

$$\cos\lambda \cdot L_y V_{nm} = \frac{1}{r} \left(\frac{R}{r}\right)^{n+1} \bar{P}_{nm}(0) \begin{Bmatrix} -\cos\lambda \; \sin(m\lambda) \cdot m \\ \cos\lambda \; \cos(m\lambda) \cdot m \end{Bmatrix} \tag{14a}$$

$$\sin\lambda \cdot L_y V_{nm} = \frac{1}{r} \left(\frac{R}{r}\right)^{n+1} \bar{P}_{nm}(0) \begin{Bmatrix} -\sin\lambda \; \sin(m\lambda) \cdot m \\ \sin\lambda \; \cos(m\lambda) \cdot m \end{Bmatrix} \tag{14b}$$

$$I_a^b(\cos\lambda L_z V_{nm}) = -(n+1) \left(\frac{R}{r}\right)^{n+1} \bar{P}_{nm}(0) \begin{Bmatrix} CC(m) \\ SC(m) \end{Bmatrix} \tag{15a}$$

$$I_a^b(\sin\lambda L_z V_{nm}) = -(n+1) \left(\frac{R}{r}\right)^{n+1} \bar{P}_{nm}(0) \begin{Bmatrix} CS(m) \\ SS(m) \end{Bmatrix} \tag{15b}$$

$$I_a^b(L_x V_{nm}) = \left(\frac{R}{r}\right)^{n+1} \cdot m \cdot \frac{\partial}{\partial\varphi} \bar{P}_{nm}(0) \begin{Bmatrix} -\sin m\lambda_b + \sin m\lambda_a \\ \cos m\lambda_b - \cos m\lambda_a \end{Bmatrix} \tag{16}$$

$$I_a^b(\cos\lambda \; L_y V_{nm}) = \left(\frac{R}{r}\right)^{n+1} \bar{P}_{nm}(0) \begin{Bmatrix} -m \; SC(m) \\ m \; CC(m) \end{Bmatrix} \tag{17a}$$

$$I_a^b(\sin\lambda \; L_y V_{nm}) = \left(\frac{R}{r}\right)^{n+1} \bar{P}_{nm}(0) \begin{Bmatrix} -m \; SS(m) \\ m \; CS(m) \end{Bmatrix} \tag{17b}$$

Expressions like $L_x'(V_{nm}(Q)) \cdot V_{ab}\left(\begin{Bmatrix} \cos\lambda \\ \sin\lambda \end{Bmatrix} \begin{Bmatrix} L_z \\ L_y \end{Bmatrix} V_{nm}(P)\right)$ can then easily be written down combining eq. (12) - (17). Difficulties will first arrive when we want to compute quantities like $I_c^d(V_{nm}(Q)) \cdot I_a^b(V_{nm}(P))$, where c and d are points on a second orbit, cf. Figure 1. This is because the arc-length of the orbit from c to d will not be related in a simple way to the coordinates φ_Q and λ_Q of the points Q of the orbit. We may, however, perform a change of the system of spherical coordinates so that the "Q-orbit" lies in the new equatorial plane. Fortunately the solid spherical harmonics $V_{nm}(Q)$ will be transformed into a linear combination of spherical harmonics of the same degree, n. We have

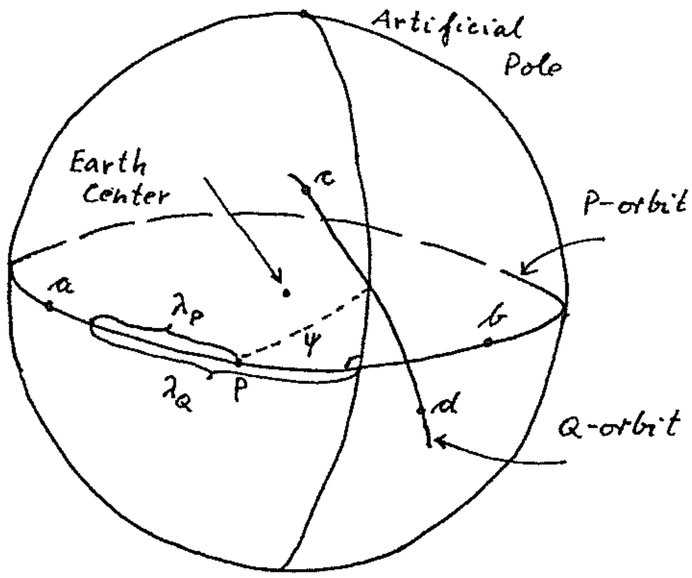

Figure 1. Two circular orbits projected on the unit-sphere, with the P-orbit being the Equator-circle.

$$V_{nm}(r_Q,\varphi_Q,\lambda_Q) = \sum_{m=-n}^{n} V_{nm}(r_Q,0,\lambda_0')k_{nm} \qquad (18)$$

where the constants k_{nm} are given in e.g. Aardoom (1969).

Using eq. (18) and eq. (15) -(17) we can evaluate

$$I_c^d(\left\{ \begin{matrix} \cos\lambda \\ \sin\lambda \end{matrix} \right\} \left\{ \begin{matrix} L_z \\ L_y \end{matrix} \right\} V_{nm}(Q)) \text{ and } I_c^d(L_x V_{nm}(Q)).$$

Hence we are able to compute the value of the linear functionals applied on reproducing kernels of finite dimensional subspaces, i.e., e.g.

$$I_a^b(\cos\varphi_P L_x(I_c^d(\sin\lambda_Q L_y' (K_I(P,Q))))) =$$

$$\sum_{n=o}^{I} \frac{\sigma_n}{2n+1} \sum_{m=-n}^{n} I_a^b(\cos\varphi_P L_x V_{nm}(P))\cdot I_c^d (\sin\lambda_Q\cdot L_y' V_{nm}(Q)). \qquad (19)$$

3. *The linear functionals applied on a simple reproducing kernel*

In a few cases it is possible to perform the operations directly on a reproducing kernel, which can be expressed by a closed formula. We will here go through a simple example, where all the degree-variances σ_n are equal to a constant, (here put equal to one). The kernel we will regard is the one suggested by Krarup (1969, eq. (101)),

$$K(P,Q) = \sum_{i=o}^{\infty} \left(\frac{R^2}{r_P r_Q}\right)^{i+1} P_i(\cos\psi) = s/\sqrt{s^2-2st+1}, \tag{20}$$

where $s = R^2/(r_P \cdot r_Q)$ and $t = \cos\psi$. This kernel may seem rather special. But linear combinations of such kernels with different values of R may be used for the approximation of empirical covariance functions, cf. Jordan (1978). Let us first apply the linear functionals eq. (4) - (7) on K(P,Q), with respect to both P and Q. First we need some auxiliary quantities.

$$\frac{\partial}{\partial r_P}(s) = -\frac{1}{r_P} \cdot s, \quad \frac{\partial}{\partial r_Q}(s) = -\frac{1}{r_Q}s \tag{21}$$

$$\begin{aligned}\frac{\partial}{\partial \varphi_P}(t) &= \frac{\partial}{\partial \varphi_P}(\sin\varphi_P \sin\varphi_Q + \cos\varphi_P \cos\varphi_Q \cos(\lambda_P - \lambda_Q)) \\ &= \cos\varphi_P \sin\varphi_Q - \sin\varphi_P \cos\varphi_Q \cos(\lambda_P - \lambda_Q)\end{aligned} \tag{22}$$

$$\frac{\partial}{\partial \varphi_Q}(t) = \sin\varphi_P \cos\varphi_Q - \cos\varphi_P \sin\varphi_Q \cos(\lambda_P - \lambda_Q) \tag{23}$$

$$\frac{\partial}{\partial \lambda_P}(t) = -\cos\varphi_P \cos\varphi_Q \sin(\lambda_P - \lambda_Q) \tag{24}$$

$$\frac{\partial}{\partial \lambda_Q}(t) = \cos\varphi_P \cos\varphi_Q \sin(\lambda_P - \lambda_Q) \tag{25}$$

We will now put

$$L = (1-2st + s^2)^{-1/2}, \tag{26}$$

hence

$$L_z K = \frac{\partial}{\partial r_P} K = \frac{\partial s}{\partial r_P} \frac{\partial}{\partial s} K = - \frac{s}{r_P}(L - s(s-t)L^3)$$

$$= - \frac{s}{r_P}(1-st)L^3 \tag{27}$$

$$L_z' K = \frac{\partial}{\partial r_Q} K = - \frac{s}{r_Q}(1-st)L^3 \tag{28}$$

$$L_y K = \frac{1}{\cos\varphi_P r_P} \frac{\partial K}{\partial \lambda_P} = \frac{1}{\cos\varphi_P r_P} \frac{\partial t}{\partial \lambda_P} \frac{\partial K}{\partial t} = - \frac{s^2}{r_P} \cos\varphi_Q \sin(\lambda_P - \lambda_Q) L^3, \tag{29}$$

$$L_y' K = \frac{1}{\cos\varphi_Q r_Q} \frac{\partial K}{\partial \lambda_Q} = \frac{s^2}{r_Q} \cos\varphi_P \sin(\lambda_P - \lambda_Q) L^3, \tag{30}$$

$$L_x K = \frac{1}{r_P} \frac{\partial}{\partial \varphi_P} K = \frac{s^2}{r_P} \frac{\partial t}{\partial \varphi_P} L^3, \tag{31}$$

$$L_x' K = \frac{1}{r_Q} \frac{\partial}{\partial \varphi_Q} K = \frac{s^2}{r_Q} \frac{\partial t}{\partial \varphi_Q} L^3, \tag{32}$$

$$L_z L_z' K = \frac{\partial^2}{\partial r_P \partial r_Q} K = \frac{s}{r_P r_Q}[(1-2st)L^3 - (s-s^2t)3(s-t)L^5]$$

$$= \frac{s}{r_P r_Q}[L - s^2 L^3 - 3(s-s^2t)(s-t)L^5], \tag{33}$$

$$L_x' L_z K = \frac{1}{r_Q} \frac{\partial^2}{\partial \varphi_Q \partial r_P} K = - \frac{s}{r_P r_Q} \frac{\partial t}{\partial \varphi_Q}[- sL^3 + 3(1-st)sL^5]$$

$$= \frac{s^2}{r_P r_Q} \frac{\partial t}{\partial \varphi_Q}[L^3 - 3(1-st)L^5], \tag{34}$$

Control:

$$\frac{\partial}{\partial r_P}\left(\frac{1}{r_Q} \frac{\partial}{\partial \varphi_Q} K\right) = - \frac{s}{r_P r_Q} \frac{\partial t}{\partial \varphi_Q}[2sL^3 - 3s^2(s-t)L^5]$$

$$= - \frac{s^2}{r_P r_Q} \frac{\partial t}{\partial \varphi_Q}[2L^3 - 3L^3 - 3(st-1)L^5]),$$

$$L_x L_z' K = \frac{1}{r_P} \frac{\partial^2}{\partial\varphi_P \partial r_Q} K = \frac{s^2}{r_P r_Q} \frac{\partial t}{\partial\varphi_P} [L^3 - 3(1-st)L^5], \tag{35}$$

$$L_y L_z' K = \frac{1}{r_P \cos\varphi_P} \frac{\partial}{\partial\lambda_P} \frac{\partial}{\partial r_Q} K = -\frac{s}{r_P r_Q} \frac{\partial t}{\partial\lambda_P} \frac{1}{\cos\varphi_P} [-sL^3 + (1-st)3sL^5]$$

$$= -\frac{s^2}{r_P r_Q} \cos\varphi_Q \sin(\lambda_P - \lambda_Q)[L^3 - 3(1-st)L^5], \tag{36}$$

$$L_y' L_z K = \frac{1}{r_Q \cos\varphi_Q} \frac{\partial}{\partial\lambda_Q} \frac{\partial}{\partial r_P} K = \frac{s^2}{r_P r_Q} \cos\varphi_P \sin(\lambda_P - \lambda_Q)[L^3 - 3(1-st)L^5], \tag{37}$$

$$L_x' L_x K = \frac{1}{r_P r_Q} \frac{\partial^2}{\partial\varphi_P \partial\varphi_Q} K = \frac{s^2}{r_P r_Q} [\frac{\partial t}{\partial\varphi_P} \frac{\partial t}{\partial\varphi_Q} (3sL^5) + \frac{\partial^2 t}{\partial\varphi_P \partial\varphi_Q} L^3] \tag{38}$$

$$L_y L_x' K = \frac{1}{\cos\varphi_P r_P} \frac{\partial}{\partial\lambda_P} \frac{1}{r_Q} \frac{\partial K}{\partial\varphi_Q} = \frac{s^2}{r_P r_Q} \frac{1}{\cos\varphi_P} [\frac{\partial t}{\partial\varphi_Q} \frac{\partial t}{\partial\lambda_P} 3sL^5 + \frac{\partial^2 t}{\partial\lambda_P \partial\varphi_Q} L^3] \tag{39}$$

$$L_y' L_x K = \frac{1}{\cos\varphi_Q r_Q} \frac{\partial}{\partial\lambda_Q} \frac{1}{r_P} \frac{\partial K}{\partial\varphi_P} = \frac{s^2}{r_P r_Q} \frac{1}{\cos\varphi_Q} [\frac{\partial t}{\partial\varphi_P} \frac{\partial t}{\partial\lambda_Q} 3sL^5 + \frac{\partial^2 t}{\partial\lambda_Q \partial\varphi_P} L^3] \tag{40}$$

$$L_y L_y' K = \frac{1}{\cos\varphi_P r_P} \frac{\partial}{\partial\lambda_P} (\frac{1}{\cos\varphi_Q \; r_Q} \frac{\partial}{\partial\lambda_Q} K) = \frac{s^2}{r_P r_Q} [\cos(\lambda_P - \lambda_Q) L^3 - \sin^2(\lambda_P - \lambda_Q)$$

$$\cos\varphi_P \cos\varphi_Q \; 3sL^5] \tag{41}$$

We are now able to evaluate $L'(I_a^b(\{{\cos\lambda \atop \sin\lambda}\} \{{L_z \atop L_y}\} K(P,Q)))$ or $L'(I_a^b(L_x K(P,Q)))$ where L' is one of the functionals given by eq.
(4) - (7), evaluated in Q. Remember, that $\varphi_P = 0$, $r = r_P$. We will also put $s = \lambda = \lambda_P - \lambda_Q$ and shift a and b correspondingly (i.e. $a := a + \lambda_Q$ and $b := b + \lambda_Q$). Hence

$$t = \cos\varphi_Q \cos(\lambda_P - \lambda_Q) = \cos\varphi_Q \cos\lambda \tag{42}$$

$$L = 1/(1-2s \cos\varphi_Q \cos\lambda + s^2)^{1/2} \tag{43}$$

For $\lambda := 2\alpha$ we have $\cos 2\alpha = 1 - 2\sin^2\alpha = \cos^2\alpha - \sin^2\alpha$. So

$$L = (1-2s \cos\varphi_Q + s^2 + 4s \cos\varphi_Q \sin^2\alpha)^{-1/2}$$

$$= (1-2s \cos\varphi_Q + s^2)^{-1/2}(1-k^2 \sin^2\alpha)^{-1/2}, \tag{44}$$

where

$$k^2 = -4s\cos\varphi_Q/(1-2s\cos\varphi_Q+s^2). \qquad (45)$$

In the following we will frequently encounter integrals of $\sin^m\lambda L^n$, $\cos^m\lambda L^n$, and we will therefore write some of these down, cf. Ryshik and Gradstein (1957, section 2.47).

$$E(x,k) = \int_0^x (1-k^2\sin^2 y)^{\frac{1}{2}}dy \qquad (46)$$

$$F(x,k) = \int_0^x (1-k^2\sin^2 y)^{-\frac{1}{2}}dy \qquad (47)$$

(A fast algorithm for the evaluation of these integrals can be found in Burlirsch (1965)). The values of these two integrals are used when evaluating integrals of more complex quantities. These integrals will in the following be denoted E_o, E_1, E_2 etc.

We now put

$$k_c^2 = 1-k^2,\ k_1 = 1-\frac{k^2}{2},\ k_2 = \frac{k^2}{2},$$

$$k_o = (1-2s\cos\varphi_Q+s^2)^{-\frac{1}{2}} \text{ and } p = \cos\lambda,$$

then

$$1 - k^2\sin^2\alpha = k_1 + k_2\cos\lambda$$

and

$$L = k_o(k_1+k_2\cos\lambda)^{-\frac{1}{2}}.$$

Hence

$$E_o = \int_a^b L\,d\lambda = 2\int_\alpha^\beta k_o(1-k^2\sin^2\alpha)^{-\frac{1}{2}}d\alpha$$

$$= 2k_o(F(\beta,k) - F(\alpha,k)). \qquad (48)$$

From

$$\cos\lambda \cdot L = k_o \frac{\cos\lambda}{(k_1+k_2\cos\lambda)^{\frac{1}{2}}} = k_o \frac{k_2\cos\lambda + k_1-k_1}{(k_1+k_2\cos\lambda)^{\frac{1}{2}}k_2} =$$

$$= k_o \left[\frac{1}{k_2} (k_1+k_2\cos\lambda)^{\frac{1}{2}} - \frac{k_1}{k_2} (k_1+k_2\cos\lambda)^{-\frac{1}{2}} \right] \qquad (49)$$

we get

$$E_1 = \int_a^b \cos\lambda \; Ld\lambda = 2k_o \left[\frac{1}{k_2}\int_\alpha^\beta (1-k^2\sin^2\alpha)^{\frac{1}{2}}d\alpha - \frac{k_1}{k_2}\int_\alpha^\beta (1-k^2\sin^2\alpha)^{-\frac{1}{2}}d\alpha\right]$$

$$= 2\frac{k_o}{k_2} (E(\beta,k)-E(\alpha\;,k)) - \frac{k_1}{k_2} E_o . \qquad (50)$$

Integrals which contain $\sin\lambda$ are somewhat easier:

$$E_2 = \int_a^b \sin\lambda \; Ld\lambda = \int_a^b k_o (k_1+k_2p)^{-\frac{1}{2}}dp = k_o\left[\frac{2}{k_2}(k_1+k_2\cos\lambda)^{\frac{1}{2}}\right]_a^b$$

$$= \frac{4k_o}{k^2} ((k_1+k_2\cos b)^{\frac{1}{2}} - (k_1+k_2\cos a)^{\frac{1}{2}}) \qquad (51)$$

Furthermore

$$E_3 = \int_a^b L^3 d\lambda = 2k_o{}^3 \int_\alpha^\beta (1-k^2\sin^2\alpha)^{-3/2}d\alpha$$

$$= 2k_o{}^3\left[\frac{1}{k_c^2}(E(\beta,k)-E(\alpha\;,k)) - (\frac{k}{2k_c})^2(\sin b(k_1+k_2\cos b)^{\frac{1}{2}}-\sin a\cdot (k_1+k_2\cos a)^{\frac{1}{2}})\right] \qquad (52)$$

Using eq. (49) we have

$$E_4 = \int_a^b \cos\lambda \; L^3 d\lambda = 2k_o^3(\int_\alpha^\beta \frac{1}{k_2}(1-k^2\sin^2\alpha)^{-\frac{1}{2}}d\alpha - \frac{k_1}{k_2}\int_\alpha^\beta (1-k^2\sin^2\alpha)^{-3/2}d\alpha$$

$$= \frac{k_o{}^2}{k_2} E_o - \frac{k_1}{k_2} E_3 . \qquad (53)$$

$$E_5 = \int_a^b \sin\lambda \; L^3 d\lambda = \int_a^b k_o{}^3 (k_1+k_2p)^{-3/2} dp = [k_o^3(-\frac{2}{k_2}(k_1+k_2p)^{-\frac{1}{2}}]_a^b$$

$$= \frac{2k_o{}^3}{k_2} ((k_1+k_2\cos a)^{-\frac{1}{2}} - (k_1+k_2\cos b)^{-\frac{1}{2}}) \quad . \tag{54}$$

From eq. (49) we have

$$\cos^2\lambda \cdot L^3 = \cos\lambda \cdot k_o{}^3 [\frac{1}{k_2} (k_1+k_2\cos\lambda)^{-\frac{1}{2}} - \frac{k_1}{k_2} (k_1+k_2\cos\lambda)^{-3/2}]$$

$$= k_o^3 [\frac{1}{k_2} (\frac{1}{k_2} (k_1+k_2\cos\lambda)^{\frac{1}{2}} - \frac{k_1}{k_2} (k_1+k_2\cos\lambda)^{-\frac{1}{2}})$$

$$- \frac{k_1}{k_2} (\frac{1}{k_2} (k_1+k_2\cos\lambda)^{-\frac{1}{2}} - \frac{k_1}{k_2} (k_1+k_2\cos\lambda)^{-3/2})]$$

$$= k_o{}^3 [\frac{1}{k_2^2} (k_1+k_2\cos\lambda)^{\frac{1}{2}} - 2\frac{k_1}{k_2^2} (k_1+k_2\cos\lambda)^{\frac{1}{2}} + \frac{k_1^2}{k_2^2} (k_1+k_2\cos\lambda)^{-3/2}] \tag{55}$$

then

$$E_6 = \int_a^b \cos^2\lambda \; L^3 d\lambda = \frac{k_o{}^3}{k_2{}^2} (E(\beta,k) - E(\alpha,k))$$

$$-2k_o{}^2 \frac{k_1}{k_2{}^2} E_o + \frac{k_1{}^2}{k_2{}^2} E_3 \tag{56}$$

$$E_7 = \int_a^b \sin\lambda\cos\lambda \; L^3 d\lambda = k_o{}^3 \int_a^b p(k_1+k_2p)^{-3/2} dp$$

$$= k_o{}^3 (\frac{1}{k_2} \int_a^b (k_1+k_2p)^{-\frac{1}{2}} dp - \frac{k_1}{k_2} \int_a^b (k_1+k_2p)^{-3/2} dp$$

$$= k_o{}^2/k_2 \cdot E_2 - k_1/k_2 \cdot E_5 \tag{57}$$

$$E_8 = \int_a^b L^5 d\lambda = 2k_o{}^5 \int_\alpha^\beta (1-k^2\sin^2\alpha)^{-5/2} d\alpha = - \frac{k_o{}^4}{3k_c{}^2} E_1 + \frac{2}{3} \frac{(2-k^2)}{k_c{}^2} k_o^2 \; E_3$$

$$- (\frac{k}{k_c})^2 \cdot \frac{1}{6} (\sin b(k_1+k_2\cos b)^{-3/2} - \sin a(k_1-k_2\cos a)^{-3/2}) k_o{}^5 . \tag{58}$$

$$E_9 = \int_a^b \cos\lambda\ L^5 d\lambda = k_o^{\,5}\left(\frac{1}{k_2}\int_a^b (k_1+k_2\cos\lambda)^{-3/2} d\lambda - \frac{k_1}{k_2}\int_a^b (k_1+k_2\cos\lambda)^{-5/2} d\lambda\right)$$

$$= k_o^{\,2}/k_2 \cdot E_3 - k_1/k_2\ E_8 \qquad (59)$$

$$E_{10} = \int_a^b \sin\lambda\ L^5 d\lambda = k_o^{\,5}\int_a^b (k_1+k_2 p)^{-5/2} dp$$

$$= k_o^{\,5}\frac{1}{3k_2}\Big[(k_1+k_2\cos\lambda)^{-3/2}\Big]_a^b = k_o^{\,5}/(3k_2)((k_1+k_2\cos b)^{-3/2} - (k_1+k_2\cos a)^{-3/2}) \qquad (60)$$

$$E_{11} = \int_a^b \cos^2\lambda\ L^5 d\lambda = k_o^{\,5}\int_a^b \Big(\frac{1}{k_2^{\,2}}(k_1+k_2\cos\lambda)^{-\frac{1}{2}} - 2\frac{k_1}{k_2^{\,2}}(k_1+k_2\cos\lambda)^{-3/2}$$

$$+ k_1^{\,2}/k_2^{\,2}(k_1+k_2\cos\lambda)^{-5/2}\Big) d\lambda$$

$$= k_o^{\,4}/k_2^{\,2} E_o - k_o^{\,2}\cdot 2\cdot k_1/k_2^{\,2}\cdot E_3 + k_1^{\,2}/k_2^{\,2} E_8 \qquad (61)$$

$$E_{12} = \int_a^b \cos\lambda\ \sin\lambda\ L^5 d\lambda = k_o^{\,5}\Big(\int_a^b \Big(\frac{1}{k_2}(k_1+k_2 p)^{-3/2} - \frac{k_1}{k_2}(k_1+k_2 p)^{-5/2}\Big) dp$$

$$= k_o^{\,2}/k_2\ E_5 - k_1/k_2\cdot E_{10} \qquad (62)$$

$$E_{13} = \int_a^b \cos^3\lambda\ L^5 d\lambda = k_o^{\,5}\int_a^b \Big(\frac{1}{k_2}\cos^2\lambda(k_1+k_2\cos\lambda)^{-3/2} - \frac{k_1}{k_2}\cos^2\lambda(k_1+k_2\cos\lambda)^{-5/2} d\lambda$$

$$= k_o^{\,2}/k_2\cdot E_6 - k_1/k_2\cdot E_{11} \qquad (63)$$

$$E_{14} = \int_a^b \cos^2\lambda\sin\lambda\ L^5 d\lambda = k_o^{\,5}\int_a^b \Big(\frac{1}{k_2^{\,2}}(k_1+k_2 p)^{-\frac{1}{2}} - 2\frac{k_1}{k_2^{\,2}}(k_1+k_2 p)^{-3/2} + \frac{k_1^{\,2}}{k_2^{\,2}}(k_1+k_2 p)^{-5/2}\Big) dp$$

$$= k_o^{\,4}/k_2^{\,2}\cdot E_2 - k_o^{\,2}\cdot 2k_1/k_2^{\,2}\cdot E_5 + k_1^{\,2}/k_2^{\,2}\cdot E_{10} \qquad (64)$$

Hence from eq. (27), (53) and (56)

$$L_Q I_a^b(\cos\lambda\ L_z K(P,Q)) = -r\int_a^b [\frac{s}{r}(1-s\ \cos\varphi_Q \cos\lambda) L^3 \cos\lambda] d\lambda$$

$$= -s[\int_a^b \cos\lambda\ L^3 d\lambda - s\cos\varphi_Q \int_a^b \cos^2\lambda\ L^3 d\lambda\ ] =$$

$$= -sE_4 + s^2 \cos\varphi_Q\ E_6\ , \qquad (65)$$

from eq. (27), (54), (52) and (56),

$$L_Q I_a^b(\sin\lambda\ L_z K(P,Q)) = -\int_a^b s(1-s\cos\varphi_Q \sin\lambda) L^3 \sin\lambda\ d\lambda$$

$$= -s\ E_5 + s\cos\varphi_Q (E_3 - E_6) \qquad (66)$$

from eq. (29), (51)

$$L_Q I_a^b(\cos\lambda\ L_y K(P,Q)) = r\int_a^b \frac{1}{r}\cos\lambda\ \frac{\partial}{\partial\lambda} K(P,Q) d\lambda$$

$$= [\cos\lambda\cdot s\cdot L\ ]_a^b + \int_a^b \sin\lambda\cdot s\cdot L\cdot d\lambda$$

$$= s(\cos b(k_1+k_2\cos b)^{-\frac{1}{2}} - \cos a(k_1+k_2\cos a)^{-\frac{1}{2}}) + s\cdot E_2 \qquad (67)$$

from eq. (29), (50)

$$L_Q I_a^b(\sin\lambda\ I_y\ K(P,Q)) = r\int_a^b \frac{1}{r}\sin\lambda\ \frac{\partial}{\partial\lambda} K(P,Q) d\lambda$$

$$= [\sin\lambda\cdot s\cdot L\ ]_a^b - s\int_a^b (\cos\lambda\cdot L) d\lambda$$

$$= s(\sin b(k_1+k_2\cos b)^{-\frac{1}{2}} - \sin a(k_1+k_2\cos a)^{-\frac{1}{2}}) - s\ E_1 \qquad (68)$$

from eq. (31), (52)

$$L_Q I_a^b(L_x K(P,Q)) = r \int_a^b \frac{s^2}{r} \sin\varphi_Q L^3 d\lambda = s^2 \sin\varphi_Q \; E_3 \qquad (69)$$

from eq. (33), (50), (53), (59), (61), (63)

$$L_z' I_a^b(\cos\lambda \; L_z K(P,Q)) = r \int_a^b \cos\lambda \left[\frac{s}{r\, r_Q} (L - s^2 L^3 - 3(s - t(1-s^2) + st^2) L^5 s \right] d\lambda$$

$$= \frac{s}{r_Q} (E_1 - s^2 E_4 - 3s(E_9 \cdot s - \cos\varphi_Q (1-s^2) E_{11} + s \cos^2\varphi_Q \; E_{13})) \qquad (70)$$

from eq. (33), (51), (54), (60), (62), (64)

$$L_z' I_a^b(\sin\lambda \; L_z K(P,Q)) = r \int_a^b \sin\lambda \left[\frac{s}{r\, r_Q} (L - s^2 L^3 - 3(s - t(1-s^2) + st^2) L^5 s) \right] d\lambda$$

$$= \frac{s}{r_Q} (E_2 - s^2 E_5 - 3s(E_{10} - (1-s^2)\cos\varphi_Q \cdot E_{12} + s\cos^2\varphi_Q E_{14})) \qquad (71)$$

from eq. (34), (53), (59), (61)

$$L_x' I_a^b(\cos\lambda \cdot L_z \; K(P,Q)) = -\frac{s^2}{r_Q} \sin\varphi_Q \int_a^b \cos\lambda (L^3 - 3(1-st) L^5) d\lambda$$

$$= -\frac{s^2}{r_Q} \left[E_4 - 3(E_9 - s\cos\varphi_Q E_{11}) \right] \cdot \sin\varphi_Q \qquad (72)$$

from eq. (34), (54), (60), (62)

$$L_x' I_a^b(\sin\lambda \; L_z K(P,Q)) = -\frac{s^2}{r_Q} \sin\varphi_Q \int_a^b \sin\lambda \left[L^3 - 3(1-st) L^5 \right] d\lambda$$

$$= -\frac{s^2}{r_Q} \sin\varphi_Q \left[E_5 - 3(E_{10} - s\cos\varphi_Q \; E_{12}) \right] \qquad (73)$$

from eq. (37), (57), (62), (64)

$$L_y' I_a^b(\cos\lambda \; L_z K(P,Q)) = \frac{s^2}{r_Q} \int_a^b \cos\lambda \; \sin\lambda \left[L^3 - 3(1 - s\cos\varphi_Q \; \cos\lambda) L^5 \right] d\lambda$$

$$= \frac{s^2}{r_Q} (E_7 - 3(E_{12} - s\cos\varphi_Q \; E_{14})) \qquad 74)$$

from eq. (37), (52), (58), (59), (56), (61), (63)

$$L_y' I_a^b(\sin\lambda \ L_z K(P,Q)) = \frac{s^2}{r_Q} \int_a^b \sin^2\lambda \ [L^3 - 3(1 - s\cos\varphi_Q \cos\lambda)L^5 \ ] d\lambda$$

$$= \frac{s^2}{r_Q} \ (E_3 - 3(E_8 - s\cos\varphi_Q \ E_9) - E_6 + 3(E_{11} - s\cos\varphi_Q \ E_{13})) \qquad (75)$$

from eq. (36), (74), (75)

$$L_z' I_a^b(\cos\lambda \ L_y \ K(P,Q)) = - \frac{s}{r_Q} \int_a^b \cos\lambda \ \sin\lambda \ \cos\varphi_Q \ [L^3 - 3(1 - st)L^5 \ ] d\lambda$$

$$= - \cos\varphi_Q \cdot \frac{s^2}{r_Q} \ (E_7 - 3(E_{12} - s\cos\varphi_Q \ E_{14})) \qquad (76)$$

$$L_z' I_a^b(\sin\lambda \ L_y K(P,Q)) = \frac{s}{r_Q} \int_a^b \sin^2\lambda \ \cos\varphi_Q \ [L^3 - 3(1 - st)L^5 \ ] d\lambda$$

$$= - \frac{s^2}{r_Q} \cos\varphi_Q (E_3 - 3(E_8 - s \ \cos\varphi_Q \ E_9) - \ E_6 + 3(E_{11} - s\cos\varphi_Q \ E_{13})) \qquad (77)$$

from eq. (39), (64)

$$L_x' I_a^b(\cos\lambda \cdot L_y K(P,Q)) = \frac{s^2}{r_Q} \int_a^b (-\sin\varphi_Q \ \cos^2\lambda \ \cos\varphi_Q \ \sin\lambda \cdot 3sL^5) d\lambda$$

$$= - 3 \ \frac{s^3}{r_Q} \sin\varphi_Q \ \cos\varphi_Q \cdot E_{14} \qquad (78)$$

from eq. (39, (59), (63)

$$L_x' I_a^b(\sin\lambda \ L_y K(P,Q)) = \frac{s^2}{r_Q} \int_a^b (-\sin\varphi_Q \ \cos\varphi_Q \ \sin^2\lambda\cos\lambda \cdot 3sL^5) d\lambda$$

$$= - 3\frac{s^3}{r_Q} \cos\varphi_Q \ \sin\varphi_Q (E_9 - E_{13}) \qquad (79)$$

from eq. (41), (56), (59), (63), (57), (60)

$$L'_y I_a^b(\cos\lambda\ L_y K(P,Q)) = \frac{s^2}{r_Q} \int_a^b \cos\lambda(\cos\lambda\ L^3 - \sin^2\lambda \cos\varphi_Q\ 3s\ L^5)d\lambda$$

$$= \frac{s^2}{r_Q} (E_6 - 3s\ \cos\varphi_Q(E_9 - E_{13})) \tag{80}$$

$$L'_y I_a^b(\sin\lambda\ L_y\ K(P,Q)) = \frac{s^2}{r_Q} \int_a^b \sin\lambda(\cos\lambda\ L^3 - \sin^2\lambda\ \cos\varphi_Q \cdot 3sL^5)d\lambda$$

$$= \frac{s^2}{r_Q} (E_7 - 3s\ \cos\varphi_Q(E_{10} - E_{14})) \tag{81}$$

from eq. (35), (52), (58), (59)

$$L'_z I_a^b(L_x K(P,Q)) = \frac{s^2}{r_Q} \int_a^b \sin\varphi_Q(L^3 - 3(1 - s\ \cos\varphi_Q \cos\lambda)L^5)d\lambda$$

$$= \frac{s^2}{r_Q} \sin\varphi_Q(E_3 - 3(E_8 - s\ \cos\varphi_Q\ E_9)) \tag{82}$$

from eq. (38), (59), (52),

$$L'_x I_a^b(L_x K(P,Q)) = \frac{s^2}{r_Q} \int_a^b (-\sin^2\varphi_Q \cos\lambda\ 3sL^5 + \cos\varphi_Q L^3)d\lambda$$

$$= \frac{s^2}{r_Q}(-3s\ \sin^2\varphi_Q E_9 + \cos\varphi_Q\ E_3) \tag{83}$$

from eq. (40), (60)

$$L'_y I_a^b(L_x K(P,Q)) = \frac{s^2}{r_Q} \int_a^b \sin\varphi_Q\ \cos\varphi_Q\ \sin\lambda \cdot 3s\ L^5 d\lambda$$

$$= \frac{s^2}{r_Q} \sin\varphi_Q\ \cos\varphi_Q\ 3s\ E_{10}\ . \tag{84}$$

Unfortunately we can not continue in the same manner with the computation of I_c^d applied on eq. (65) - (84), because the integrals of elliptic integrals can only be evaluated by numerical integration.

4. Application of the closed expressions

The simple kernel regarded in section 3 is not the only one for which the "orbit-integrals" can be expressed analytically. Reproducing kernels like the one related to the Poisson integral formula, $K(P,Q) \approx L^3$, cf. Krarup (1969, eq. (8)) or linear combinations of such kernels having varying Bjerhammar-sphere radii can also be used. This means as mentioned above that analytical expressions can be derived for the covariance function used in Jordan (1978). It is doubtful whether closed expressions can be formed for the covariance functions used in Tscherning and Rapp (1974) or in Tscherning (1976).

Alternatively the integrals $I_c^d(L_x'(I_a^b(L_x K(P,Q))))$, $I_c^d(L_x'(I_a^b(L_Y K(P,Q))))$ etc. can be evaluated using a numerical integration technique. The numerical properties of the chosen technique can then be checked by applying it on the simple kernel $s \cdot L$, for which analytic expressions have been derived.

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