SHORT NOTE: Computation of error-covariances of the estimates of spherical harmonic coefficients using fast spherical collocation

C.C. Tscherning

Department of Geophysics, University of Copenhagen, Juliane Maries Vej 30, DK-2100 Copenhagen Ø, Denmark

e-mail. cct@gfy.ku.dk, phone: +4535320582, fax: +4535365357

Abstract: Least-squares collocation (LSC) may be used for the estimation of spherical harmonic coefficients of the (anomalous) gravity potential from various kinds of gravity field data. LSC requires that as many equations as the number of observations are solved. However, the computational effort may be dramatically reduced if the data are associated with points located equi-distantly on parallels. An implementation of LSC, which takes advantage of this property, is called Fast Spherical Collocation (FSC). FSC may also be used for the computation of error-covariances of any two coefficients. Since sums over all meridians, m=0,..,M, of products such as cos(j($m\Delta\lambda$)*cos(k($m\Delta\lambda$), where $\Delta\lambda$ is the longitude spacing, are zero when $j \neq k$, will error-covariances only be non-zero for $j=k$. This gives large savings in storage space and computational speed. If the data are distributed on parallels situated symmetrically around the equator, error-covariances may, for odd j and even k (or the reverse), also become equal to zero.

Keywords: Gravity potential, fast spherical collocation, error-covariances

1. Introduction

The method of Least-Squares Collocation, LSC, (Moritz 1980, Tscherning 2001) may be used for the estimation of the spherical harmonic coefficients C_{ij} of the anomalous gravity potential,

$$T(\overline{\varphi},\lambda,r)=\frac{GM}{r}\sum_{i=2}^{\infty}\left(\frac{a}{r}\right)^{i}\sum_{j=-i}^{i} P_{ij}(\sin\overline{\varphi})\left[C_{ij}e_j(\lambda)\right] \quad e_j=\cos(j\lambda), j\geq 0, =\sin(j\lambda), j<0 \tag{1}$$

where $\overline{\varphi}$ is the geocentric latitude, λ the longitude, r the radial distance, a the semi-major axis (or similar scaling constant), P_{ij} the associated Legendre function of degree i and order j, and GM the product of the gravitational constant and the mass of the Earth.

LSC requires the knowledge of an estimate of the covariance function of the anomalous gravity potential,

$$\text{cov}(T(P),T(Q))=\sum_{i=2}^{\infty}\left(\frac{R^2}{rr'}\right)^{i+1}\sigma_i^2 P_i(\cos\psi) \tag{2}$$

where P,Q are points in space with geocentric radial distances r,r' respectively and separated by spherical distance ψ, R is the radius of a sphere inside the Earth, σ_i^2 are the so-called degree-variances and P_i are the Legendre polynomials. From Eq. (2), covariances of other quantities are obtained by covariance-propagation, i.e. the associated linear functionals are applied to the basic covariance function. For covariances between a coefficient and an observation such as point values of T, the radial derivative, the second-order radial derivative or the derivatives with respect to latitude, the covariance becomes (Tscherning 2001)

$$\text{cov}(C_{ij},obs)=\frac{c_m}{r}\left(\frac{a}{r}\right)^{i}P_{ij}^{n}(\sin\overline{\varphi})e_j(\lambda)=u(r,\overline{\varphi})e_j(\lambda) \tag{3}$$

where c_m is a constant depending on the type of observation, P_{ij}^n is the n-th derivative of the Legendre function with respect to the latitude, and u is equal to the first part of the second expression in Eq. (3).

The estimated coefficient is obtained using

$$\begin{aligned}\tilde{C}_{ij} &= \{\text{cov}(C_{ij}, obs_k)\}^T \{\text{cov}(obs_k, obs_m) + e_{km}\}^{-1} \{g_k\} \\ &= \{b_m^{ij}\}^T \{g_m\}\end{aligned} \tag{4}$$

where g_k are the observations and e_{km} are the data noise covariances.

The use of LSC requires, as seen from Eq. (4), that a system of equations with as many unknowns as the number of observations are solved. This may be avoided using the method of Fast Spherical Collocation, FSC, (Sansò and Tscherning 2003). FSC requires data to be associated with the points of a grid, equidistant in longitude for data on a parallel at a specific height. There are M meridians and N parallels.

The solutions b_m^{ij} are obtained by solving p systems of equations with as many unknowns as the number of parallels, N. The term $p=[M/2]$ is the Nyquist frequency with respect to longitude, i.e. equal to the integer part of half the number of points on a parallel. Therefore, one system of equations is solved for each order j from 0 to p, and for each degree i from j to p. Let such a system have the solutions s_k. The solution for degree i and order j is a NxM vector with dimension equal to the number of observations

$$\{b_m^{ij}\} = \{\cos(j(l\Delta\lambda)\}\{s_k\}, \quad m = (k-1) * M + l \tag{5}$$

where l is the subscript for longitude and k the subscript for latitude (parallel).

In this short note, we will show that the use of FSC also leads to a dramatic reduction in the computational effort required to compute error-covariances. We will prove analytically that most of the error-covariances become equal to zero. It is intended to use this method in the processing of data from the European Space Agency (ESA) Gravity Field and Steady-State Ocean Circulation Explorer (GOCE) mission (Johannesen et al. 2003).

2. Calculation of error-covariances

The error-covariances (e.g., Heiskanen and Moritz 1967, eq. 7-65) are equal to

$$\text{ecov}(ij,rt) = -\{b_m^{ij}\}^T \{\text{cov}(C_{rt}, g_m)\} \tag{6}$$

where we have assumed the original coefficients to be uncorrelated.

The covariances are equal to a constant $u(r_k, \overline{\varphi}_k)$ for each parallel, *k*, multiplied by $e_t(l\Delta\lambda)$, see Eq. (3), such that

$$\text{ecov}(ij,rt) = -\sum_k u_k(r,\overline{\varphi})\, s_k \sum_l e_j(l\Delta\lambda) e_t(l\Delta\lambda) \tag{7}$$

The last sum in Eq. (7) is zero except for $j=t$. It is equal to *M* for $i=0$ and otherwise equal to $M/2$ (Sansò and Tscherning 2003, Eqs. B9 and B10).

If the parallels are ordered symmetrically around equator, the error-covariances for odd or even degree *i* and even or odd degree *r* may be close to zero because the associated Legendre functions, or their derivatives in this case, have opposite sign in each hemisphere. If the error estimates associated with the data also are symmetric with respect to the equator, the error-covariances become identical to zero. In this case, we may write Eq. (6) as (with subscript *n* for the Northern Hemisphere and *s* for the Southern Hemisphere)

$$er\,\text{cov}(ij,rt) = \begin{Bmatrix} C_{n,rt} \\ C_{s,rt} \end{Bmatrix}^T \begin{Bmatrix} C_{nn} & C_{ns} \\ C_{ns} & C_{ss} \end{Bmatrix}^{-1} \begin{Bmatrix} C_{n,ij} \\ C_{s,ij} \end{Bmatrix} \tag{8}$$

where $C_{nn} = C_{ss}$, $C_{n,ij} = pC_{s,ij}$, $C_{n,rt} = kC_{s,rt}$, with $p=1$ for even *i* and -1 for odd *i*, $k=1$ for even *r* and -1 for odd *r* if the observations are not derivatives with respect to latitude. Evaluation of Eq. (8) in this case will show that the error-covariance is zero when *i* and *k* have opposite signs. If the observations are associated with derivatives with respect to latitude, a similar combination of even and odd orders *i* and *k* will also result in zero error-covariances.

3. Numerical example

The algorithm in Section 2 has been implemented in versions dated after 2004-04-23 of the program sphgric.f, which is available as electronic supplementary material (ESM) to this short note. This program has been used in order to compute the error-covariances of coefficients estimated from values of the anomalous gravity potential, T, and of the second-order radial derivative T_{rr} on M=500 meridians ($\Delta\lambda$=0.72°) and on N=464 parallels, symmetric around equator, see Migliaccio et al. (2004). The test data used were generated using EGM96 (Lemoine et al. 1998) from degree 25 to 300. Consequently. all degree-variances for degrees lower than 25 (cf. Eq. 2) were set equal to zero. Figures 2 to 4 below illustrate some typical error-correlations (error covariances divided by the square-root of the variances) for fixed order.

FIGURES 1, 2 and 3 NEAR HERE

Figures 1 to 3 illustrate three typical error correlation patterns obtained using the simulated data described above. Note that these data were distributed symmetrically with respect to the equator and the error-estimates of the observations were similar, but not identical, for all parallels. A different pattern is found if this is not the case.

4. Conclusion

The theoretical results presented here make it possible to compute the error-covariances of spherical harmonic coefficients of the gravity potential very quickly, and without using very much computer memory. For each degree i and order j, there will be at most j-i non-zero error covariances. If the parallels are situated symmetrically with respect to equator, more error-covariances will be equal to or close to zero.

It may be considered an advantage that so many error-covariances are zero, but is this really realistic? It is a consequence of using gridded data with uncorrelated errors and error-variances being constant for each parallel. However, this is not expected to be true for GOCE data. However, if the data are statistically homogenized, e.g. by subtracting (residual) topographic effects (Miggliaccio et al. 2004) we may come closer to this situation.

The values of the error-correlations show interesting patterns (Figs. 1-3), which warrant further investigations. Such investigations may aid in understanding and improving the FSC method. It may very well be that certain choices of grid-spacing may be better than others.

Acknowledgement: This short note is a contribution to the European GOCE Gravity Consortium (EGG-C) work.

References:

Heiskanen WA, Moritz H (1967) Physical Geodesy. W.H. Freeman & Co, San Francisco

Johannesen JA, Balmino G, Le Provost C, Rummel R, Sabadini R, Sünkel H, Tscherning CC, Visser PNAM, Woodworth P, Huges CH, Le Grand P, Sneeuw N, Perosanz F, Aguirre-Martinez M, Rebhan H, Drinkwater M (2003) The European Gravity Field and Steady-State Ocean Circulation Explorer Mission: Impact on Geophysics, Surveys of Geophysics 24(4): 339-386

Lemoine FG, Kenyon SC, Factor JK, Trimmer RG, Pavlis NK, Chinn DS, Cox CM, Klosko SM, Luthcke SB, Torrence MH, Wan YM, Williamson RG, Pavlis EC, Rapp RH, Olson TR (1998) The Development of the Joint NASA GSFC and the National Imagery and Mapping Agency (NIMA) Geopotential Model EGM96, NASA/TP-1998-206861, Goddard Space Flight Center, Greenbelt

Migliaccio F, Reguzzoni M, Sansò F, Tscherning CC (2004) An enhanced space-wise simulation for GOCE data reduction. Proc 2nd Int. GOCE user Workshop, ESRIN, March, 2004.

Migliaccio F, Reguzzoni M, Sansò F, Tscherning CC (2004) The performance of the space-wise approach to GOCE data analysis, when statistical homogenization is applied. Submitted Proceedings GGSM2004, Porto, 2004.

Moritz H (1980) Advanced Physical Geodesy, Wichmann, Karlsruhe

Sansò F, Tscherning CC (2003) Fast spherical collocation - theory and examples. J Geod 77: 101-112, DOI 10.1007/s00190-002-0310-5

Tscherning CC (2001) Computation of spherical harmonic coefficients and their error estimates using Least Squares Collocation. J Geod 75: 14-18

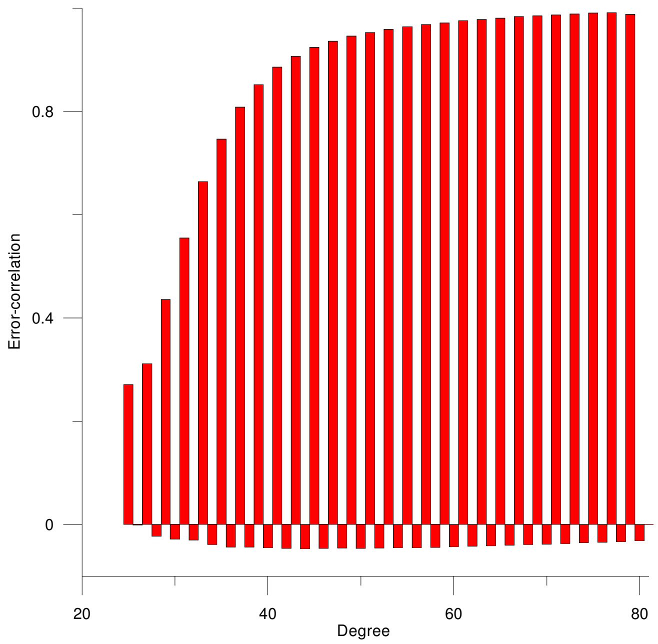

Figure 1. The error-correlations of degree 81 and order 2 with all non-zero coefficients between degree 25 and 81 and having also order 2. Note the large correlations for **odd** degrees.

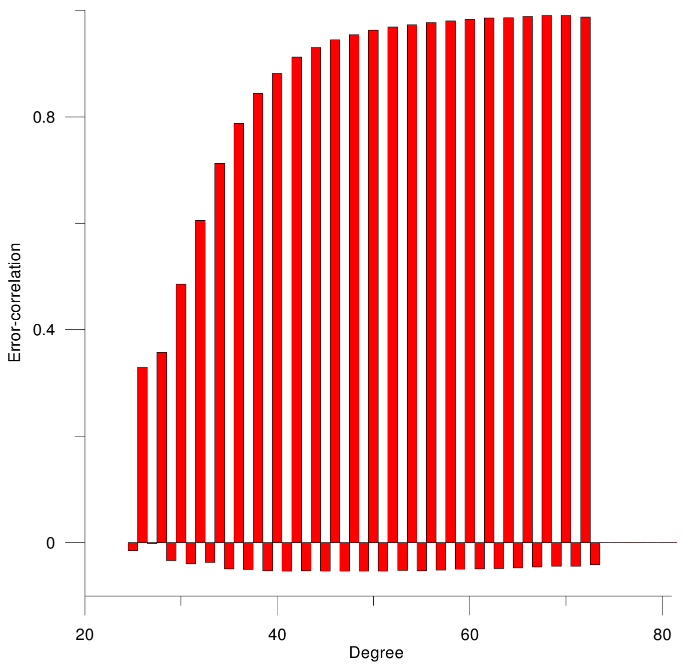

Figure 2. Error-correlations for degree 74 and order 2 with all coefficients of order 2 and degree between 25 and 74. Note the large correlations for **even** degree.

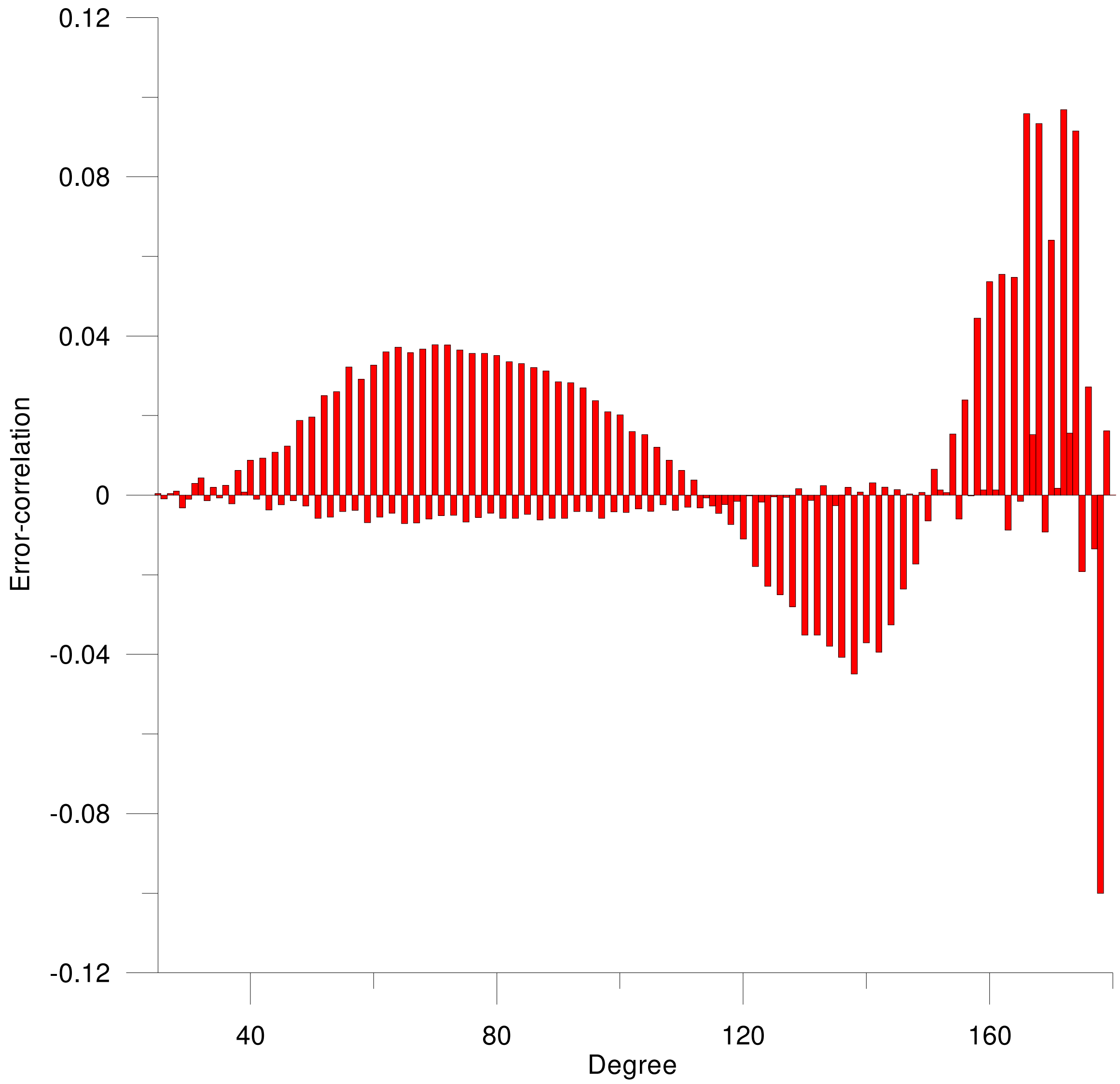

Figure 3. Error-correlation for degree 180 and order 6 with all coefficients of degree 6 and order from 25 to 180. The correlations are small but have a variation that needs further explanation.