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Comments and reply regarding Heck (2003) "On Helmert's method of condensation"

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Editorial note: As the IAG's official journal, the *Journal of Geodesy* offers a forum for scientific publications on fundamental or application-oriented research that cover the broad spectrum of geodetic science. As a service to our readers, well-documented comments on and corrections to material published in this Journal will also be considered for publication. These contributions should be submitted in accordance with the Instructions to Authors, which are printed in each edition of the Journal. This particular comment and reply concerns some of the alternative methods for computing a regional gravimetric geoid versus quasigeoid model.

1 Introduction

Helmert's methods of condensation (HCM) aim at removing (and later restoring) the gravitational potential of the topographic masses so that the conditions for using the Stokes integral for geoid determination are fulfilled. In Heck (2003), herein called 'the paper', the methods are analysed critically from a theoretical standpoint. It is pointed out that the method retains the high-frequency part of the gravity anomalies, which may cause problems when performing downward continuation of the anomalies to the geoid. Unfortunately, the paper disregards alternative procedures for (quasi-) geoid determination as discussed in, for example, Torge (2001). It also contains a number of other misconceptions, which will be discussed below.

2 Use of a global gravity field model

In all modern (quasi-)geoid determinations, the contribution from a global gravity field model (GGM) is removed and later restored. Such models also contain the potential of the masses corresponding approximately to the same wavelength as the spherical harmonic expansion used to represent the GGM. If the maximum degree is 360, geophysical features, such as isostatic (or non-isostatic) compensation, are also nearly all removed, since isostatic compensation requires a mass support with an extent larger than 25 km (see e.g. Sünkel 1985). Consequently, Sect. 4 of the paper, where different condensation schemes are discussed, becomes meaningless in this context, since very little of the compensation remains after the removal of a high-degree GGM.

The remaining topographic effects will correspond to positive and negative density anomalies, so the whole concept of Helmert condensation falls apart, since the condensed layer was introduced to keep the total mass change to zero and have a small indirect effect (Heiskanen and Moritz 1967, p. 145). There is no longer a need for a condensed layer, since the zero-order term in the expansion now is zero and the indirect effect is small. However, there is a need to smooth the residual anomalous gravity field, taking into account the residual topography, if downward continuation is still needed in order to apply Stokes's integral.

3 Use of the residual topography for smoothing

The gravitational attraction of the (positive and negative mass anomalies) may be calculated by numerical integration (see e.g. Forsberg and Tscherning 1981). This also assures a stable downward continuation (Tscherning and Forsberg 1990). There is no restriction on the point of evaluation using this procedure. In the HCM (see p. 160 of the paper), airborne data have to be located outside the Brillouin sphere if the gravitational potential is to be represented by a spherical harmonic series.

4 Spectral properties of the topography

In Appendix C of the paper, so-called spectral properties are discussed. This term is used wrongly, since a spectral representation of a quantity defined on the surface of the Earth is associated with the surface spherical harmonics.

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Heck (2003) introduces something denoted as the surface spherical harmonics of the powers of the topographic heights, $h'(\varphi,\lambda)$, where φ is the latitude and λ the longitude

$$H_n^k(\varphi,\lambda) = \frac{2n+1}{4\pi} \iint_\sigma \left(\frac{h'(\varphi',\lambda')}{R} \right)^k \cdot P_n(\cos\psi) \cdot \mathrm{d}\sigma \quad (1)$$

in which the integration is performed over the surface of the mean Earth sphere, R is the mean radius of the Earth and $P_n(\cos\psi)$ are the Legendre polynomials of degree n as functions of the cosine of the spherical distance ψ. Surface spherical harmonics have to be functions of both a degree and an order, and not of latitude and longitude. Therefore, the paper describes something different from what are called spectral properties.

5 Helmert and Molodensky

On page 160 (top) of the paper, it is claimed, "Helmert's second method of condensation has entered Molodensky's theory, which originally was completely free of any topographical reductions and density hypothesis". There is a misunderstanding here. Naturally, we may use information about the topography and density in order to smooth the gravity anomalies. This must be done so that the harmonicity is preserved and the various effects are calculated at the points where the quantities are observed. Nothing needs to be 'moved' anywhere, except if we want to use Stokes's integral. This 'move' may be achieved using harmonic downward continuation.

It is also claimed (p. 160 of the paper) that in quasi-geoid computation "a harmonic downward-continuation has to be performed". This is incorrect, since the quasi-geoidal heights (the height anomalies) may be calculated directly at the surface of the Earth. Only if Stokes's integral is used, is some kind of harmonic downward-continuation needed. For this purpose, least-squares collocation may be used (see e.g. Forsberg and Tscherning 1981), or the downward-continued values (Δg^*) may be computed using values of the vertical gravity anomaly gradient computed using Fourier techniques (see e.g., Schwarz et al. 1990).

6 Concluding remarks

In Sect. 7 of the paper, severe problems in the use of HCM are pointed out. However, as described above, the method contains many other problems when considering today's alternative approaches to (quasi-)geoid determination.

The problems created using the HCM have been solved using the remove–compute–restore (RCR) and residual terrain modelling (RTM) techniques. Further study of the HCM will probably reveal many more problems, especially if numerical investigations are performed, as announced in the paper. Additional theory will then not help the survival of the HCM. Only consistent solutions, like the combined RCR and RTM method, should be used in geodetic practice when Stokes' integral is subsequently to be used.

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1 Introduction

In addition to other methods of gravity field modelling, such as residual terrain modelling (RTM; see e.g. Omang and Forsberg 2000), the Stokes–Helmert approach to geoid determination is commonly used today (see e.g. Vanicek et al. 1999; Nahavandchi 2000; Novàk 2000; Sjöberg 2000; Jekeli and Serpas 2003; Tenzer et al. 2003). In practically all investigations, Helmert's second condensation method is applied, aiming at condensing the topographical masses outside the geoid onto a single layer coinciding with the geoidal surface. The 'pure' Stokes–Helmert method is often complemented by the remove–compute–restore (RCR) technique, removing—and later restoring—a higher - degree reference field such as the EGM96 global gravity field model (Lemoine et al. 1998).

Many publications dealing with practical and numerical aspects of Helmert's (second) method of condensation contain a number of fundamental errors, ambiguities and misconceptions [see e.g. discussions in Wang and Rapp (1990); Heck 1993a]; Martinec et al. (1993) These deficiencies gave rise to a basic discussion and a thorough treatment of the procedures in my paper (Heck 2003). Deliberately, the argumentation in this paper has been kept at the theoretical level in order to remain as general as possible and to be free of any reasoning on data quality and numerical errors. Nevertheless, a number of far-reaching results could be obtained, including the substantial statement that Helmert's second model of condensation produces unreliable results in geoid computation—in particular in high-resolution applications in mountainous regions.

Furthermore, it was pointed out that different sets of formulae have to be applied in geoid and quasigeoid determination, respectively. Finally, it was concluded that an extension of Helmert's condensation method—considering the condensation layer to be situated at a depth of 20–30 km below the geoid—will provide more reasonable results. Importantly, a comparison of this approach with other existing procedures for gravity field modelling and a judgement of these alternatives was not the intention of my paper.

2 Use of a global gravity field model

It is certainly true that removing a global gravity field model (GGM) in spherical harmonic representation provides a residual gravity field on and outside the Earth's masses, where the features due to large-scale geological mass anomalies (such as isostatic compensation masses) have been eliminated or strongly reduced. Nevertheless, I doubt that the removal of a high-degree GGM is a remedy in any respect, due to the arguments explained below.

First of all, we have to discriminate between applications in geoid determination, related to the solution of the Stokes–Helmert problem, and in the determination of height anomalies or quasigeoid heights, related to Molodensky's problem. It is evident that the geoid in continental regions is situated beneath the Earth's topographical masses. Considering the gravitational potential, this region of space is governed by Poisson's equation rather than by Laplace's differential equation. On the other hand, (solid) spherical harmonic expansions are solutions of the Laplace equation and thus are not able to describe the potential inside the topography.

Geoid determination according to Stokes's concept always requires the 'removal' of the masses outside the (co-)geoid in some way in order to make the disturbing potential harmonic in the whole domain outside the (co-)geoid. 'Shifting' the topographical masses to infinity corresponds to the use of Bouguer anomalies producing a very large indirect effect (1 km and more in the geoid height; Heiskanen and Moritz 1967). The physical reason for these strong effects lies in the fact that the topographical masses are mostly counterbalanced by isostatic mass anomalies, for example near the Mohorovicic discontinuity. Neglecting the balancing isostatic mass anomalies thus produces a strong signal in the disturbing potential. For geoid determination, isostatic gravity anomalies should be preferred, guaranteeing both a mass-free space outside the (co-)geoid and a small indirect effect. In this context, the Helmert condensation model can be considered as a simple alternative to the computationally more demanding isostatic model of, for example, Airy–Heiskanen type, but also fulfilling both above-mentioned postulates.

Mathematically, this procedure—figuratively interpreted as 'shifting' of masses—can be described in terms of the solution of Poisson's differential equation, namely by superposing the general solution of the respective homogeneous (i.e. Laplace) differential equation with a special solution of the inhomogeneous Poisson equation (i.e. Newton's integral over the topographic and isostatic masses). For further details of this concept see, for example, Heck (1993b). It is also well known that any spherical harmonic expansion, as a solution of Laplace's equation outside the masses, represents the effect of a harmonic density function (see e.g. Moritz, 1990, p. 188), while the non-harmonic constituents of the density function produce a zero potential outside the masses. For this reason, the inverse gravimetric problem is not uniquely solvable, since the external gravity field only reflects the effects of the harmonic part of the density distribution. Of course, this property is not valid in the space covered by masses, where a GGM never represents the complete potential function.

As a conclusion, it is not sufficient to subtract a GGM in the context of geoid determination, since this task always requires a 'mass-free' space outside the (co-)geoid. These basic physical properties of the gravity field cannot be ignored when useful results are desired.

The situation is very different when considering the determination of quasi-geoid heights or height anomalies in Molodensky's concept. This approach is completely related to the determination of the *external* gravity field, and thus the use of harmonic downward and upward continuation, as well as the application of high-degree GGMs, is fully justified. In this context, the effect of non-harmonic density constituents is irrelevant; the mathematical background of upward/downward continuation is provided by the Runge–Krarup theorem (see e.g. Moritz 1980, p. 67). Nevertheless, mass reductions can significantly improve the process of harmonic downward continuation, which is known to be ill conditioned. Mass reductions in Molodensky's concept can be introduced in order to smooth the gravity field by reducing the high-frequency terms, and thus to stabilize the results, in particular for high-resolution applications. Of course, the stabilization aspect is also valid for geoid determination, but there the second argument of creating a 'mass-free' space outside the geoid has to be put forward in addition, this is irrelevant in Molodensky's concept.

A second aspect of using high-degree GGMs concerns the significance and reliability of the harmonic coefficients. While the lower-degree coefficients have been derived from independent satellite observations, the higher-degree coefficients are determined from satellite altimetry data over the oceans and terrestrial gravity observations on the continents. However, large gaps still exist in the gravity data coverage over continental regions, and the available data are not always reliable, causing medium-and short-wavelength errors in the gravity field related to existing GGMs. For this reason some authors in the pre-CHAMP/GRACE era decided to restrict the maximum degree of GGMs in this context to about $N = 20$ (see e.g. Vanicek et al. 1996), taking into consideration the significance of the then-available harmonic coefficients derived from satellite tracking. Although the reliability of the harmonic coefficients will improve dramatically as a result of the CHAMP, GRACE and GOCE satellite gravity missions, the upper limit for N cannot be increased above 100 to 150, corresponding to a spatial resolution of about 130 to 200 km. It is therefore evident that medium-scale geophysical features of tectonic or isostatic origin are only partly eliminated when a GGM is applied, even for the high-degree GGMs expected to be released in the near future.

3 Use of the residual topography for smoothing

According to the arguments presented in Sect. 2 of this response, procedures such as RTM and RCR techniques are reasonable in the context of Molodensky's concept, but have to be considered more carefully if geoid heights are to be determined. Although it is evident that, due to an insufficient knowledge of the topographic mass–density distribution, the true geoid can never be absolutely determined, the concepts of geoid and quasigeoid should be clearly discriminated, in particular for high-precision and high-resolution applications. Any mixture of these concepts may not only cause confusion but also cause erroneous numerical results.

In order to be completely rigorous, it should be stressed that airborne data have to be located outside a Brillouin sphere surrounding the Earth's masses; only in this region of space is it guaranteed that the spherical harmonic expansion of the gravitational potential will converge to the correct values. For regional applications of airborne gravimetry this prerequisite should not be very critical, especially for larger flight heights.

4 Spectral properties of the topography

The functions $H_n^k(\varphi, \lambda)$ in Heck (2003) represent the (Laplace) surface spherical harmonics of degree n, based on the function $(h(\varphi\lambda)/R)^k$, where h is the topographic height depending on the geocentric latitude φ and longitude λ, R is the radius of the mean-Earth sphere and $k \in \aleph_0$ is an integer-valued exponent. In the general framework of potential theory (see e.g. Heiskanen and Moritz 1967, pp. 20, 29–30), a function $f(\Theta, \lambda)$ can be developed in surface spherical harmonics

$$f(\Theta, \lambda) = \sum_{n=0}^{\infty} Y_n(\Theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (a_{nm} \cdot R_{nm}(\Theta, \lambda) + b_{nm} \cdot S_{nm}(\Theta, \lambda)) \quad (1)$$

involving the surface spherical harmonics $Y_n(\Theta, \lambda)$ of degree n, related to the function $f(\Theta, \lambda)$. The functions $Y_n(\Theta, \lambda)$ are solutions of the Laplace–Beltrami differential equation and can be presented in the form (Heiskanen and Moritz 1967, p. 30)

$$Y_n(\Theta, \lambda) = \frac{2n+1}{4\pi} \iint_\sigma f(\Theta', \lambda') \cdot P_n(\cos\psi) \cdot d\sigma \quad (2)$$

where

$$\cos\psi = \cos\Theta \cdot \cos\Theta' + \sin\Theta \cdot \sin\Theta' \cdot \cos(\lambda' - \lambda) \quad (3)$$

and $P_n(\cos\psi)$ is the Legendre polynomial of degree n. It should be noted that (Θ, λ) and (Θ', λ') denote the fixed point of evaluation and the variable integration point, respectively. Equation (1) results from Eq. (2) by application of the well-known decomposition formula for spherical harmonics (see e.g. Heiskanen and Moritz 1967, p. 33).

This general concept can be applied to the kth power of the 'normalized' topographical height, i.e. $f(\Theta, \lambda) = (h(\Theta, \lambda)/R)^k$, considering that $\varphi = \pi/2 - \Theta$. In this way, it is proved that

$$H_n^k(\varphi, \lambda) := \frac{2n+1}{4\pi} \iint_\sigma \left(\frac{h(\varphi', \lambda')}{R}\right)^k \cdot P_n(\cos\psi) \cdot d\sigma \quad (4)$$

in fact corresponds to the nth degree surface spherical harmonic of the function $(h(\varphi, \lambda)/R)^k$; of course, it can be further split up in a sum over m of type

$$H_n^k(\varphi, \lambda) = \sum_{m=0}^{n} (a_{nm} \cdot R_{nm}(\varphi, \lambda) + b_{nm} \cdot S_{nm}(\varphi, \lambda)) \quad (5)$$

with fixed coefficients a_{nm} and b_{nm}.

5 Helmert and Molodensky

In the traditional understanding, presented nicely in Moritz (1980), Molodensky's theory aims at the simultaneous determination of the external gravity field and the figure of the Earth, based on terrestrial gravity measurements and geopotential numbers ('geodetic variant of Molodensky's problem' or 'scalar free geodetic boundary value problem'). After linearization of this non-linear problem by introducing the normal potential of a level ellipsoid and the concept of the telluroid, a solution for non-spherical boundary surfaces can be found in series form, for example by the procedures of Molodensky's shrinking or Marych–Moritz's analytical (upward and) downward continuation; these approaches produce equivalent results (Moritz 1980, p. 388; Otero and Auz 2004). Originally, the Molodensky problem had been related to the Earth's surface (or the telluroid, respectively), avoiding any mass reductions and density hypotheses. On the other hand, it is evident that Molodensky's concept can still be applied in the RTM approach, after a transition from the topographical surface to a smoothed boundary and consideration of the respective changes in the gravity field. Therefore, there is no contradiction if the RTM approach is understood in this sense.

Due to the equivalence of Molodensky's shrinking with Marych–Moritz's analytical continuation, any numerical procedure based on the resulting spherical integral formulae (including Fourier techniques) can be implicitly related to harmonic (upward and) downward continuation, although the height anomalies are calculated at the surface of the Earth. Whether the analytical continuation is numerically performed either by evaluation of spherical integrals (resulting from the L_n operators; see Moritz 1980, p. 384) or by least-squares collocation is of secondary importance. As a consequence, the comments made by C.C. Tscherning are correct.

On the other hand, it should be stressed again that the interpretation of the first-order correction term g_1, or correspondingly the terrain correction C, in terms of the second Helmert condensation model (Moritz 1968) is purely artificial or even misleading. The formal similarity of the respective integrals only exists under the postulate that the effect of the topographic masses is evaluated at the telluroid, while the effect of the condensed masses is calculated at sea level. Although the change of the evaluation point from the Earth's surface down to sea level may be insignificant for low-accuracy applications, there is no logical reason for this transition. A rigorous proof for this concept, sometimes called the 'Moritz–Pellinen approach' (see e.g. Jekeli and Serpas 2003) is still missing (at least, I am not aware of any reference in this respect).

6 Concluding remarks

The primary purpose of my paper (Heck 2003) was to point out a number of fundamental misconceptions and basic inconsistencies in the application of Helmert's methods of condensation, for example by mixing the concepts of geoid and quasigeoid determination, or by using Helmert's second condensation method. If properly applied, Helmert's (generalized) method is still useful in the context of gravity field modelling, in particular for smoothing the residual gravity field. A detailed investigation of the analytical and numerical properties of this approach versus the RTM technique, as proposed by C.C. Tscherning in his comment, should provide further insight into the relative behaviour of these methods and thus lead to further improvements in gravity field modelling from terrestrial and satellite observations.

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