

## **A fast study of the aliasing of spherical harmonic coefficients.**

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**Abstract:** The aliasing of spherical harmonic coefficients computed from grids of height anomalies, gravity anomalies and radial gravity gradients,  $T_{rr}$  is studied using Fast Spherical Collocation. The coefficients of EGM96 from degree 25 to  $n_{max}$ , where  $n_{max}$  is 72, 144 or 288 has been used to generate point or block mean average height anomalies, gravity anomalies and  $T_{rr}$  in a  $2.5^\circ$  grid at zero altitude and at "satellite" altitude equal to 300 km. The numerical experiment shows a large effects of aliasing for  $n_{max}= 144$  or 288 dependent of how much power is left in the spherical harmonic coefficients above degree 72. For gravity anomalies and  $T_{rr}$  the square-root of the power is  $(n-1)$  and  $(n+1)(n+2)$  times larger per degree  $(n)$  than for height anomalies.

The magnitude of the effect is for data at satellite altitude considerably lower than at zero level, where all powers are damped due to the altitude. However, the aliasing increases up to 50 % for the highest degrees (67-72) for  $T_{rr}$  and to 30 % for height anomalies at satellite altitude. For mean height anomalies and gravity anomalies at ground level the maximal values in the same range are between 50% and 65 %.

### **Introduction.**

When computing spherical harmonic coefficients of the gravity potential  $V$  of the Earth, using least-squares or quadrature methods aliasing has always been a problem. One have tried to minimize its influence by removing as much as possible of the high degree information. This may e.g. be done by subtracting the attraction of the topography from the data which are used .

Also aliasing may be reduced using mean values (block averages) or upward continuation due to the damping of higher order frequencies. In the following are reported results using 3 different quantities, height anomalies ( $\zeta$ ), gravity anomalies ( $\Delta g$ ) and the radial second order derivative,  $T_{rr}$ . They each have a different relationship to the spherical harmonic coefficients, i.e. no change, multiplication with  $(n-1)$  or multiplication with  $(n+1)*(n+2)$ , where  $n$  is the degree.

### **Fast Spherical Collocation, FSC.**

In FSC (Sansò and Tscherning, 2002) the method of least-squares collocation, LSC, (Moritz, 1980) is used on data associated with a grid equidistant in longitude. Data may be of different types, do not need to be at the same altitude and the parallels do not need to be equidistantly spaced. In the present implementation (Feb. 2003) only the prediction of spherical harmonic coefficients is possible.

The general LSC method requires that as many equations as there are observations are solved. FSC takes advantage of the repetitive structure of the normal-equations which arise due to the association of the data with an equidistant grid in longitude. Only systems of equations with the number of unknowns equal to the number of parallels used has to be solved. One drawback, however, is that the noise variance has to be constant for each parallel.

A basic element in collocation is the use of a covariance function or equivalent reproducing kernel. For the covariance function used in this study we have used the error-degree-variances  $\varepsilon_n^2$  of EGM96 (Lemoine et al., 1998) below degree 25 and the degree variances  $\sigma_n^2$  of EGM96 from 25 to 360. From degree 361 to 720 we have used the degree-variances associated with GPM98a (Wenzel, 1998). In all experiments the same degree-variance model was used. Hence the covariance function for two values of the anomalous potential with spherical distance  $\psi$  and distances  $r$  and  $r'$  from the origin becomes with  $P_n$  the Legendre polynomials

$$\text{cov}(\psi, r, r') = \sum_{n=2}^{24} \varepsilon_n^2 \left(\frac{a}{rr'}\right)^{n+1} P_n(\cos \psi) + \sum_{n=25}^{720} \sigma_n^2 \left(\frac{a}{rr'}\right)^{n+1} P_n(\cos \psi)$$

Note that the semi-major axis  $a$  occurs in the equation, i.e. no Bjerhammar sphere, and no spherical approximation is used. The resulting variances at height zero will then be too small.

We have put the EGM96 coefficients used to generate the data equal to zero for degrees below 25, and used the degrees from 25 to  $n_{\text{max}}=72, 144, 288$  to generate the data.

In Table 1 is shown the signal variance associated with the different data types used, as well as the variance of the calculated signal for  $n_{\text{max}}=72, 144$  and  $288$ .

Type	$\zeta$	$\Delta g$	$T_{rr}$	$\zeta$	$\Delta g$	$T_{rr}$	$M\zeta$	$M\Delta g$	$M\zeta$
Units	$\text{m}^2$	$\text{mgal}^2$	$\text{E}^2$	$\text{m}^2$	$\text{mgal}^2$	$\text{E}^2$	$\text{m}^2$	$\text{mgal}^2$	$\text{m}^2$
Height, km	0	0	0	300	300	300	0	0	300
$n_{\text{max}}=72$	3.37	131.5	0.99	0.12	2.63	0.0096	2.80	102.7	0.11
$n_{\text{max}}=144$	4.19	313.0	6.38	0.12	2.66	0.0102	3.04	149.7	0.11
$n_{\text{max}}=288$	4.36	462.2	22.40	0.13	2.66	0.0102	3.06	160.0	0.11
$n_{\text{max}}=720$	4.70	611.8	80.64	0.13	2.80	0.0110	3.79	307.7	0.12

Table 1. Variances of generated data for altitudes 0 and 300 km as well as for mean values (M). Bottom row shows the variance according to the covariance function model. From these numbers one may calculate how much power is contained in the degrees between  $n_{\text{max}}+1$  and 720. At 300 km altitude very little power is left even for  $n_{\text{max}}=72$ .

## Numerical experiments.

The data which can be used with the present implementation of FSC are point or mean values of the anomalous potential,  $T$ , height anomalies, gravity anomalies or radial second order derivatives  $T_{rr}$ . For gravity anomalies we use in the simulations the following relationship without spherical approximation

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r}T$$

i.e. the data are generated using this formula.

The generated data were associated with a global grid with spacing  $2.5^\circ$  in both longitude and in geodetic latitude. Altitudes equal to 0 and to typical satellite altitude of 300 km have been used. Furthermore values of noise below 1 % of the signal standard deviation was used. (The influence of the noise could also have been studied using the software, but will be done in a subsequent study).

The method was then used calculate “predicted” spherical harmonic coefficients up to and inclusive degree 72, as well as the error of prediction. A comparison with the input coefficients from degree 25 to 72 will then give a picture of the effect of aliasing. This comparison is below expressed in terms of standard deviations of observed minus predicted value per degree.

The error-estimates shows also the effect of aliasing. They only depend on the data distribution, on the altitude and on the type of observation used. Hence the error do not depend on whether data has been generated using coefficients up to a specific degree. The error would have been **zero** if we had used a covariance function which has degree-variances equal to zero for degrees below 25 and above 72 and noise free data.

Consequently the error-estimate express the full effect of aliasing from the coefficients from degree 73 to 720. Figure 1 and 2 show the error-estimates for mean height anomalies and mean gravity anomalies at zero altitude and for point anomalies of all 3 quantities at 300 km altitude.

## **Results.**

There are several interesting results as illustrated in the figures. They confirm, however, what we should expect: The aliasing depends on how much power is left in the degrees above the Nyquist frequency. An the aliasing is well described by the FSC error-estimates, Fig. 1 and 2.

The difference between observed and predicted coefficients is however not zero when data has been generated using coefficients up to the Nyquist frequency  $n=72$ . In fact, since the space in which we are determining the predictions contain harmonics from degree 2 to 720, all the “power” will be spread over all degrees. This spreading is not even, but depends on the degree-variances associated with the individual quantity height anomaly, gravity anomaly or radial second order derivative).

### **Height anomalies at zero level, Fig. 3 and 4.**

This example is related to the use of satellite radar altimetry either as gridded data or as mean values. We see that the aliasing is large for point data, but much reduced as soon as mean values are used.

### **Gravity anomalies at zero level, Fig. 5 and 6.**

This example is related to the use of gravity data similar to what has been used for EGM96. We see that the result for point data is completely dominated by aliasing. When mean values are used the result becomes much better, but the effect of alizing is still large. From Fig. 6 on may conclude that there is a lot of power between degree 289 and 720 which causes aliasing.

### **Gravity Gradients at zero level., Fig. 7.**

This example would correspond to an airborne gradiometer mission. It would not work at all. The measurements would have to be done at a high altitude aircraft

### **Point or mean height anomalies at 300 km, Fig. 8 and 9.**

This example corresponds to the use of CHAMP or GRACE data converted to values of the anomalous potential. We see that the aliasing is very much reduced. But there is a rather large “spreading” effect due to the use of the space up to maximal degree 720.

### **Point gravity anomalies or second order radial derivatives at 300 km, Fig. 10 and 11.**

This example corresponds to GOCE (and somewhat also to CHAMP and GRACE). Satellite to satellite tracking may be thought of as giving gravity, while obviously GOCE will deliver second order derivatives. The results are close, and the aliasing is limited, but still considerable, see also Fig. 2. As seen from Table 1, some power is left between degree 288 and 720. It may be worthwhile to form mean values of the second order derivatives in order to reduce aliasing.

At ground level we see (Fig. 1) that if mean values are used, then the aliasing causes an error of 50% and 60% of the signal standard deviation for the highest degrees. At 300 km altitude (Fig. 2) the error is much lower, but still in the range between 30% and 40% of the signal standard deviation at degree 72.

## **Conclusion.**

The effect of aliasing is large at ground level, even if mean values are used. However it is smaller when height anomalies are used. Consequently one should consider using mean altimetric heights instead of converting these quantities to gravity anomalies.

At satellite level the aliasing is not at all negligible. It is smallest for data derived using the energy principle and a factor 2 larger for gravity gradients like  $T_{rr}$ . Consequently procedures for reducing the aliasing should be used, such as reducing the power of the higher frequencies. This could e.g. be done using spherical harmonic expansions of the attraction of the residual topography, i.e. the topography referring to a smooth surface corresponding to the maximal degree and order equal to the Nyquist frequency.

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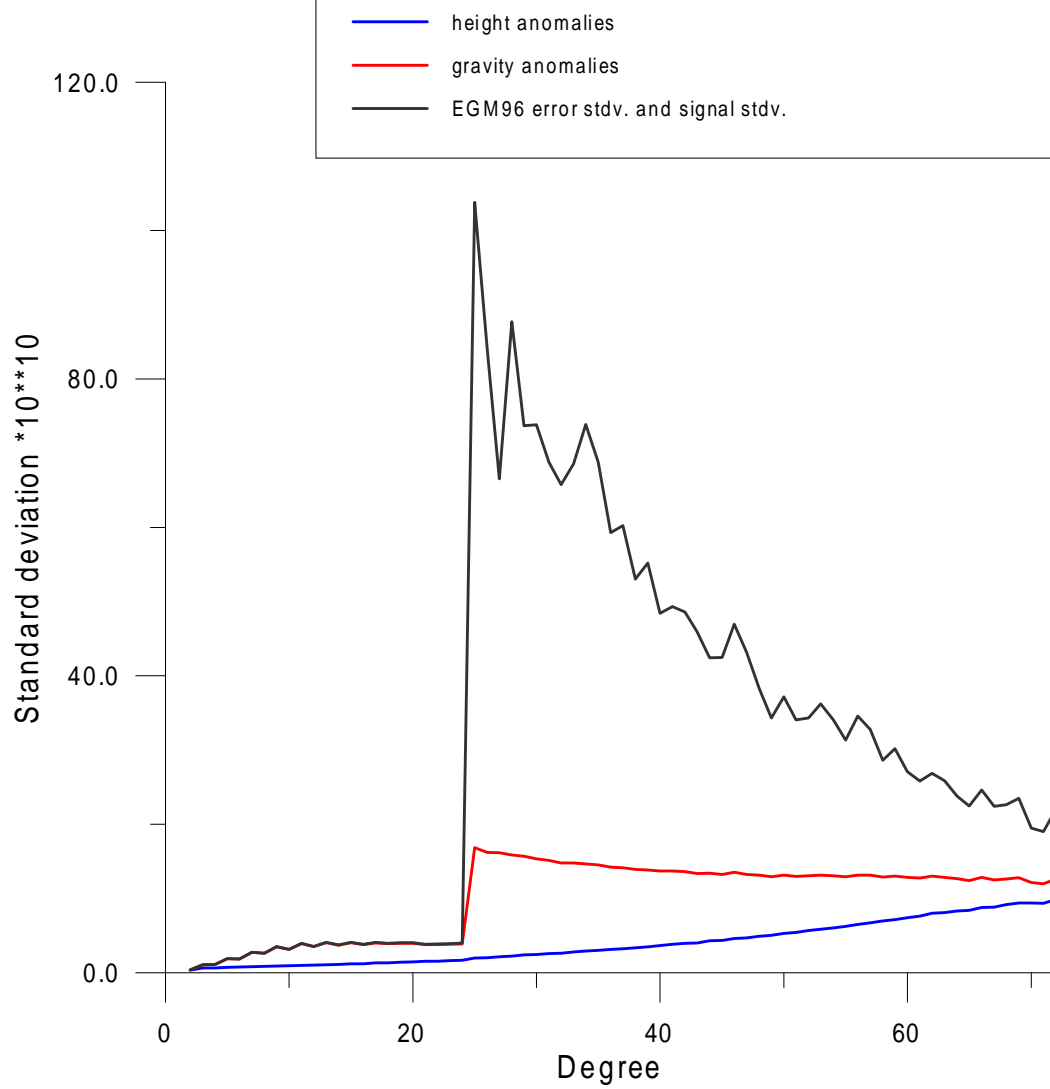
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Fig. 1. Square-root of EGM96 error-degree variances and of degree-variances compared to standar deviation (stdv.) of FSC error-estimate for block mean height anomalies, gravity anomalies at height 0.



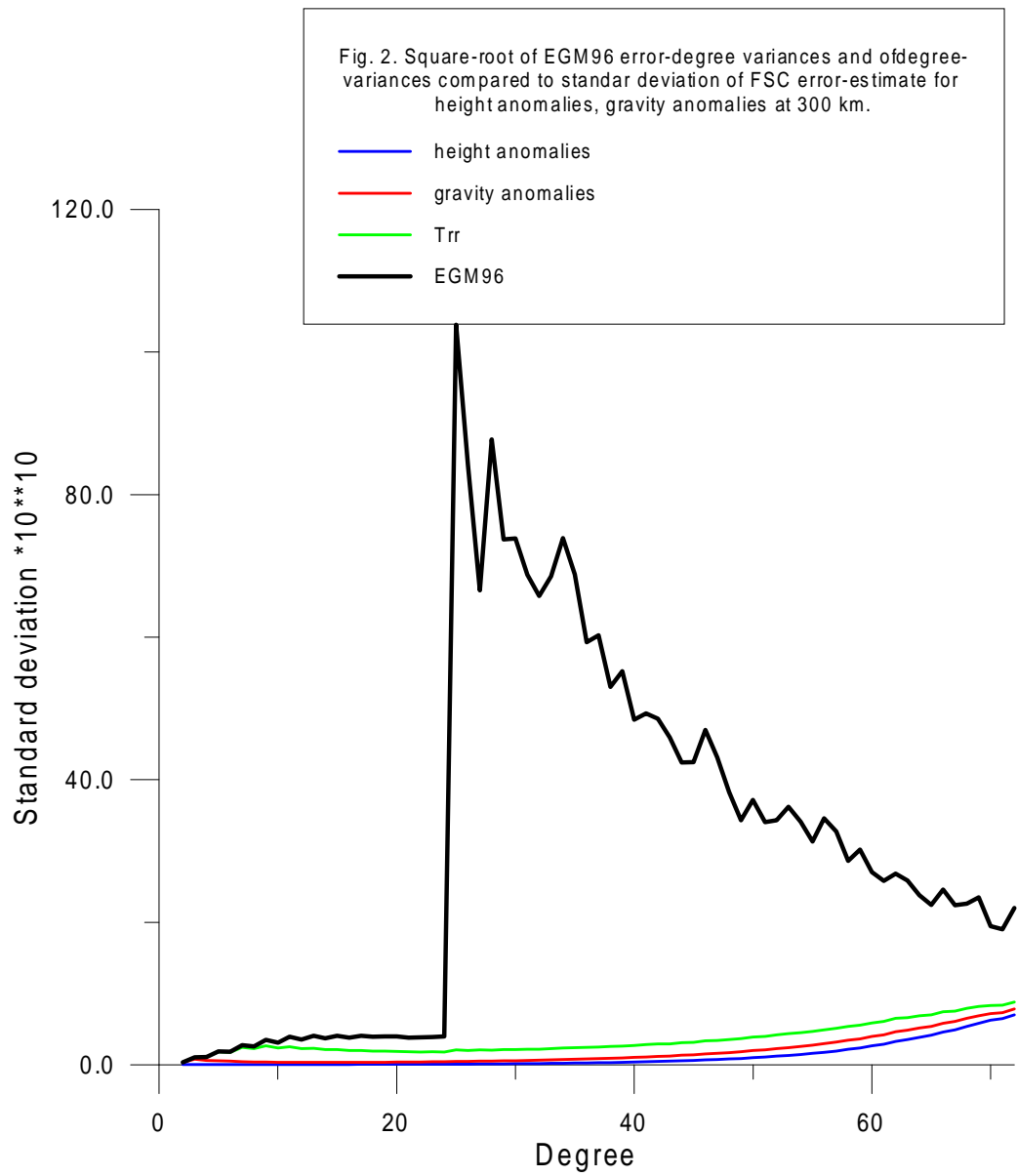


Fig. 3. Standard deviation of differences between EGM96 coefficients and coefficients computed using FSC with height anomaly data generated using EGM96 to degree  $n_{max}$ . Height = 0.

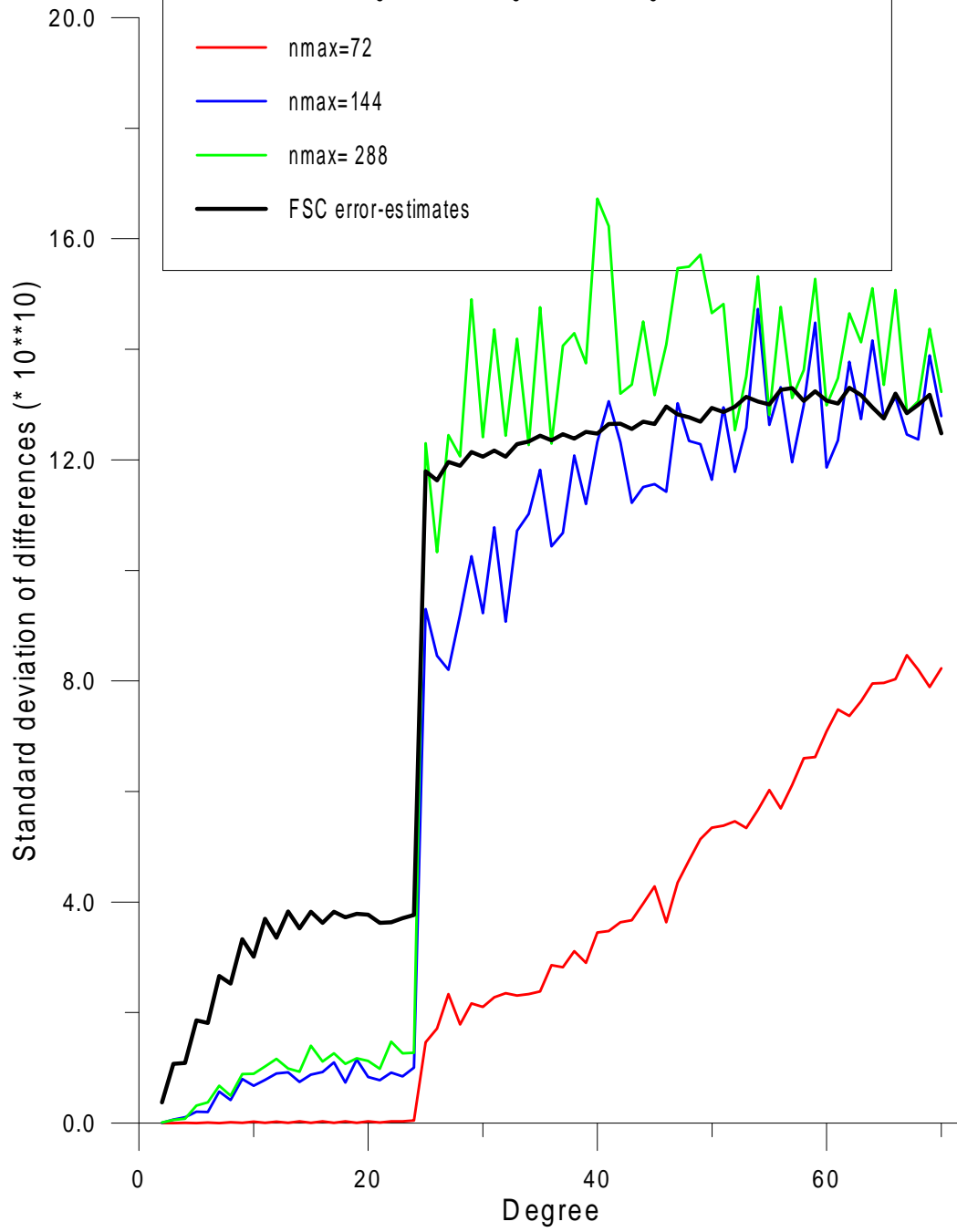
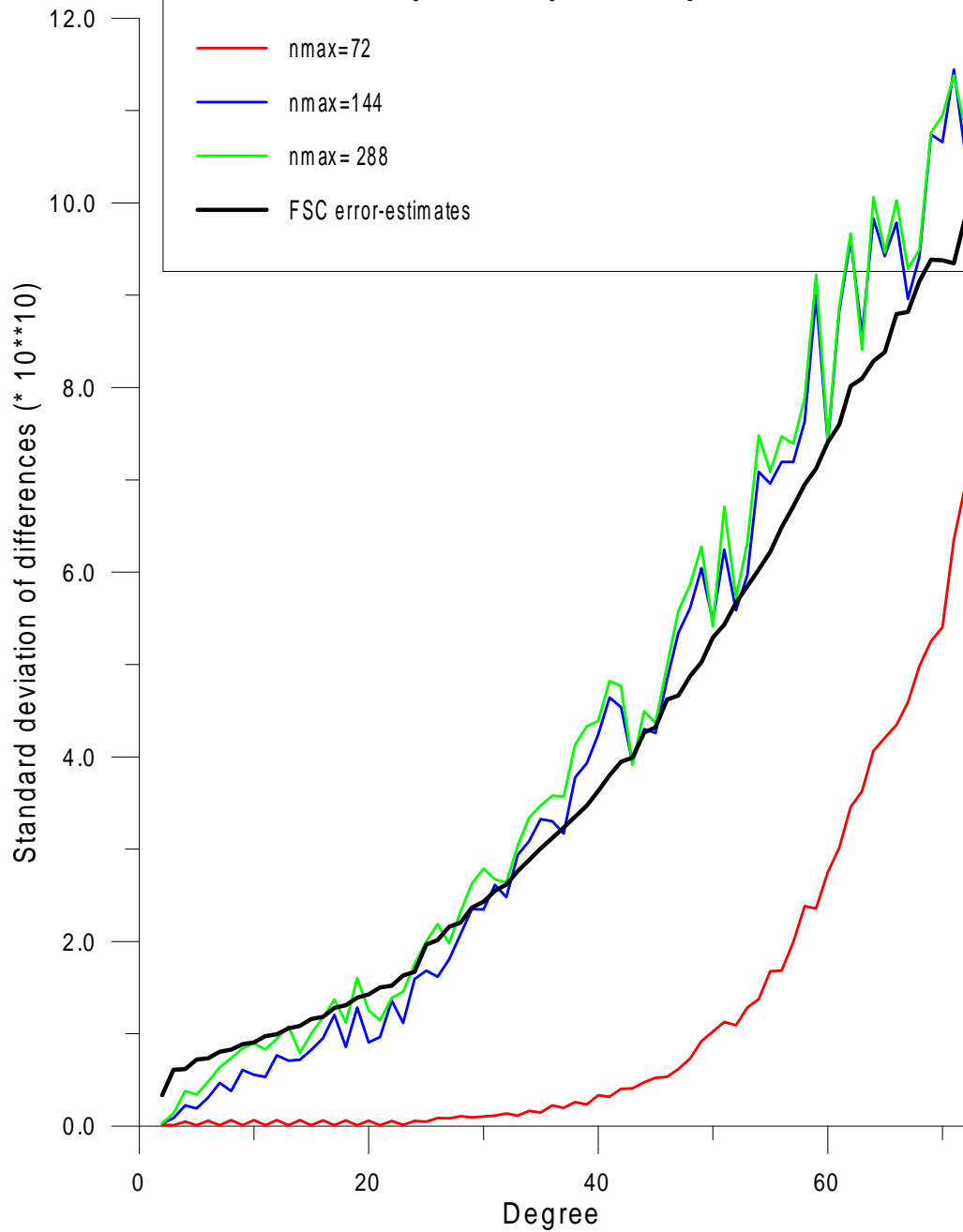
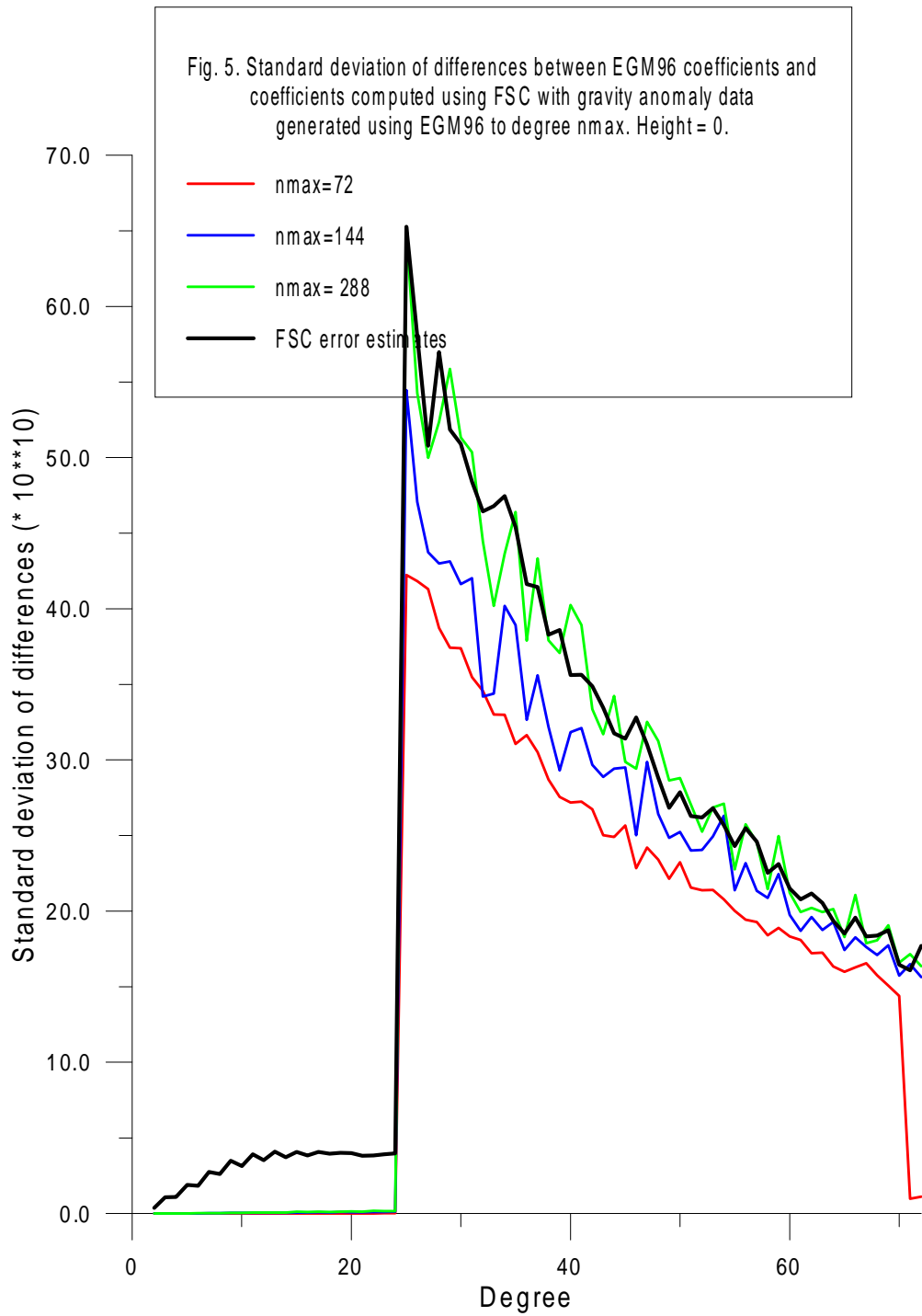
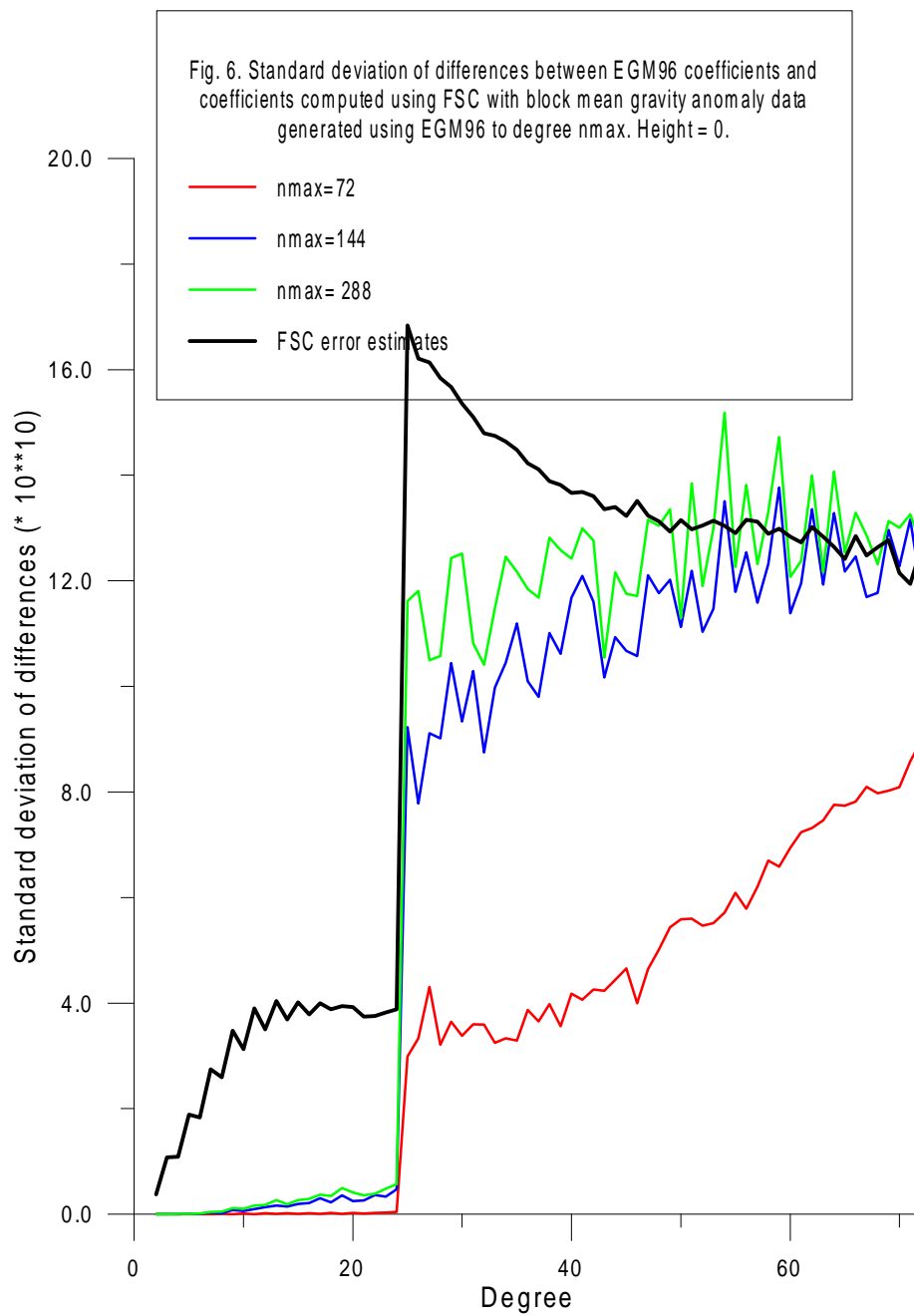


Fig. 4. Standard deviation of differences between EGM96 coefficients and coefficients computed using FSC with block mean height anomaly data generated using EGM96 to degree  $n_{max}$ . Height = 0.









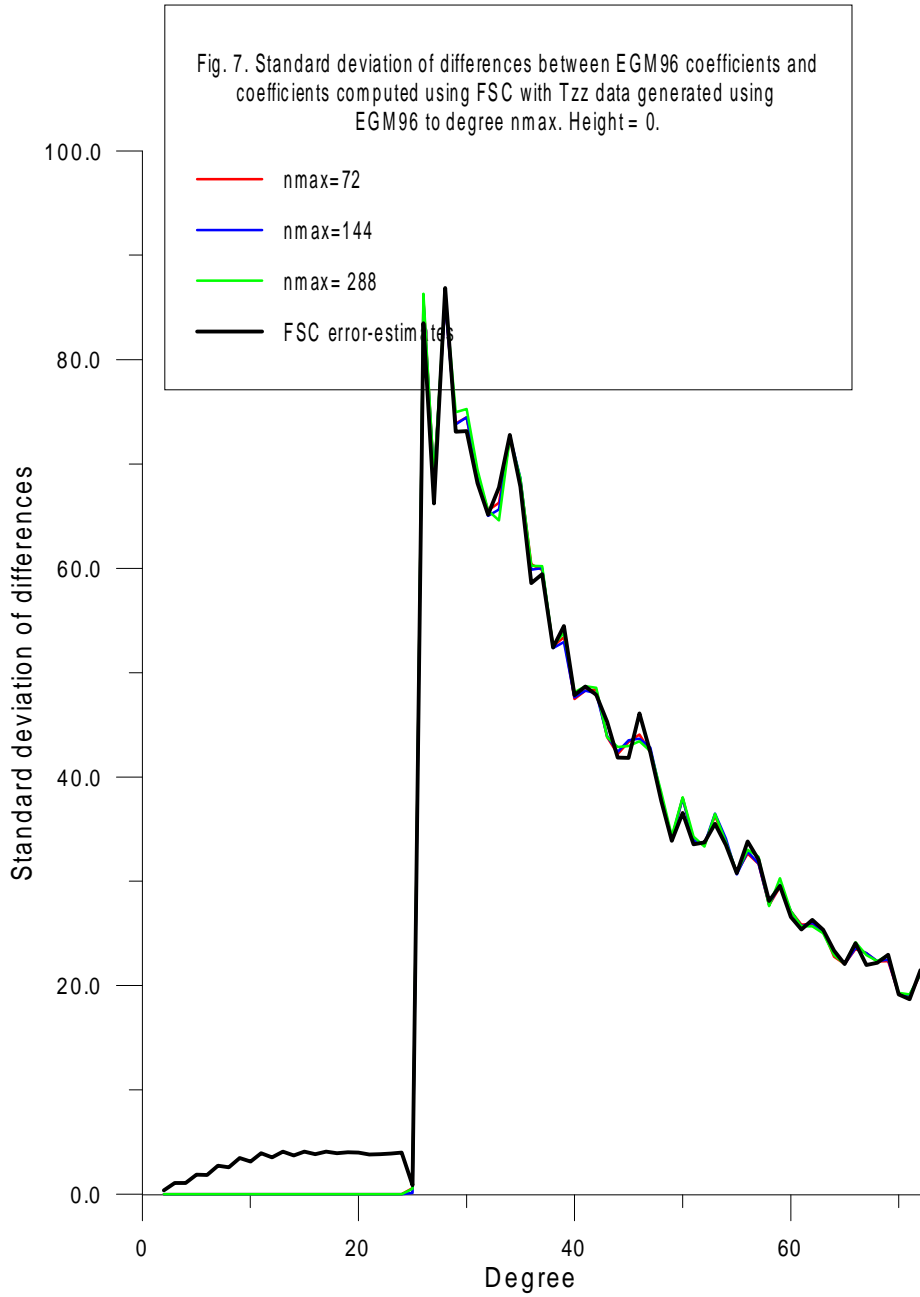


Fig. 8. Standard deviation of differences between EGM96 coefficients and coefficients computed using FSC with height anomaly data generated using EGM96 to degree nmax.  
Height= 300 km

