A Comparison of Methods for Computing Gravimetric Quantities from High Degree Spherical Harmonic Expansions

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Summary
In the past few years geopotential coefficients have become available to
degree 180.   These coefficients can be used to calculate various gravimetric
quantities such as height anomalies, gravity anomalies and disturbances,
deflections of the vertical etc.   In doing so special care is needed to
insure computational stability in the generation of the associated Legendre
functions and their derivatives.

This paper describes four different programs that can be used for selected
calculations.   A program by Rapp and one by Tscherning/Goad are designed
to compute a number of different quantities at one time.   The programs
by Rizos and Colombo are designed to calculate one quantity on a grid
which may be regional or global (the Rizos program) or global only (the
Colombo program).

A single point calculation with five gravimetric quantities takes approx-
imately 0.5 seconds of computer time with a field to degree 180.   The
computation of a global $1° \times 1°$ grid takes 47   seconds using a 180 field
and the Colombo program.

The Fortran program designed by Tscherning and Goad is given in the ap-
pendix of the paper.

1.   Introduction

The computations of gravimetric quantities from spherical harmonic expan-
sions has grown in importance with the development in the past few years
of high degree spherical harmonic expansions of the earth's gravitational
potential.   Specifically of interest are the disturbing potential (T),
the height anomaly (ζ), the gravity anomaly (Δg), the gravity disturbance
(δg), and deflections of the vertical (ξ, η), and higher derivatives.

The evaluation of these quantities now can involve the use of published spherical harmonic expansions to degree 180 with higher degree expansions in the realm of reality. Because of the large number of coefficients in these high degree expansions, it is important to seek efficient algorithms for evaluation of the quantities of interest. In addition, algorithm formulations must be chosen to assure computational stability.

The purpose of this paper is to examine several computer programs in terms of their basic technique and applicability, and to compare the programs in terms of computer time for the same computational effort. The algorithms used here will also be of value in other disciplines such as geomagnetism.

Included with this paper is a Fortran program that can be used for the computation described in this report. The main program was originally written in Algol by Tscherning and described in Tscherning and Poder (1981). It was then translated and implemented in Fortran by Goad.

2. Fundamental Equations

The basic equation for representing the earth's exterior harmonic potential V at a point whose coordinates are r (geocentric distance), ψ (geocentric latitude), and λ (longitude) is:

$$V(r,\psi,\lambda) = \frac{kM}{r} \left[1 + \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} (\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda) \, \bar{P}_{nm}(\sin \psi) \right] \qquad (1)$$

where:

kM	is the geocentric gravitational constant;
$\bar{C}_{nm}, \bar{S}_{nm}$	are the fully normalized potential coefficients
a	is the scaling factor associated with the coefficients;
$\bar{P}_{nm}(\sin \psi)$	are the fully normalized associated Legendre functions.

One can define the disturbing potential, T, by removing from V a normal potential implied by an equipotential reference ellipsoid or some defined set of potential coefficients. We have:

$$T(r,\psi,\lambda) = V(r,\psi,\lambda) - U(r,\psi,\lambda). \tag{2}$$

Given values of T the various quantities of interest can be computed. For example we have:

$$\zeta = \frac{T}{\gamma}$$

$$\Delta g = \frac{-\partial T}{\partial r} - \frac{2}{r} T$$

$$\delta g = \frac{-\partial T}{\partial r} \tag{3}$$

$$\xi = \frac{-1}{\gamma r} \frac{\partial T}{\partial \psi}$$

$$\eta = -\frac{1}{\gamma r \cos \psi} \frac{\partial T}{\partial \lambda} .$$

These equations represent spherical approximation in some cases, or components in a radial direction in other cases. Specific equations for each of these quantities may be found, for example in Rapp (1982).

In reality the expansion in equation (1) is not taken to infinity but to some finite degree. If this degree is 180 there will be 32761 potential coefficients involved in the calculation. The evaluation of (1) or any of the other equations can be done under various circumstances. In the following sections we briefly discuss published procedures or computer programs.

3.   The Rizos Program   (SIMDAT)

Rizos (1979) described a computer technique for the evaluation of the height or gravity anomaly from potential coefficients for points defined on a two dimensional evenly spaced geographic grid. The original program used at The Ohio State University for this type of calculation was received from NASA in 1978. The program was called SIMDAT. This program was designed

to generate values given a specified latitude and longitude interval in an area defined by latitude and longitude limits. This area could be a local region or it could be the whole sphere. For each latitude the fully normalized associated Legendre functions are evaluated by recursive formulae that are described in detail by Singh (1982). The longitude dependent computations are first started by the "rotation" of the potential coefficients so that they refer to the longitude origin at the west edge of the area of interest. The calculation for a given anomaly is then based on simplification in the sin mx or cos mx computation resulting from having points equally spaced in longitude.

4. The Colombo Program (SSYNTH)

Colombo (1981) has described the theory and computer program that can be used for the efficient computation of height or gravity anomalies from potential coefficients. The program of interest is called SSYNTH. This program can be modified to incorporate fully normalized Legendre function calculations using a recursive algorithm described by Colombo and in more detail by Singh (1982). The calculation is designed for a global grid at a specified latitude and longitude interval $(\Delta\theta°)$. The original SSYNTH has been modified slightly, for the test described, by the introduction of the $(a/r)^n$ term in the disturbing potential, and several other changes.

The SSYNTH subroutine was designed for the calculation of area means or point values. The latter are the quantities of interest in this discussion. The Legendre functions are first computed for the grid interval recognizing that there is a grid symmetry with respect to the equator so that values are computed for latitudes above the equator. Colombo then applies a Fast Fourier Transform (FFT) technique to compute sums of series along the N $(N=180°/\Delta\theta°)$ latitude rows. Two latitude rows are formed at the same time to take advantage of the Legendre function odd or even symmetry with respect to the equator. Colombo points out that usual FFT are most efficient when the grid is such that N (the number of intervals in the grid) is an integer power of 2. Since most of our grids are based on $360°/\Delta\theta°$ which contains factors other than 2 a special FFT routine known

as the mixed radix FFT must be used (Singleton, 1967). In the revised version of SSYNTH this procedure is implemented by subroutine FFTCC of the IMSL subroutine library.

5.   The Rapp Program

This program (Rapp, 1982) is designed for the calculation of the five quantities listed in equation (3) on a point by point basis and is representative of the many programs that use similar recursion procedures. The normalized Legendre functions and their first derivatives are generated for a specific latitude using the recursive algorithm described by Colombo (1981). This subroutine (LEGFDN) was checked for stability by Colombo using double and quadruple precision for order (m) = 350 and degrees (n) from 350 to 400, for $2.5° \leq \theta \leq 90°$ where θ is the co-latitude. This subroutine does not compute the first derivative at the pole.

This subroutine is written such that the needed functions for a given order m and all degrees to the highest maximum degree are computed in one call to the subroutine. The subroutine is repeatedly called for $0 \leq m \leq N$ where N is the maximum degree being used in the expansion.

For discussion purposes visualize the associated Legendre functions in a lower triangular matrix where the rows correspond to degree n and the columns correspond to order m.

For a given m , the subroutine first calculates for $0 \leq n \leq m$ the diagonal elements corresponding to the diagonal passing through the $n = m$ location. We have:

$$\overline{P}_{nm}(\cos \theta) = \sqrt{\frac{2n+1}{2n}} \quad \sin\theta \, \overline{P}_{n-1,n-1}(\cos \theta)$$

$$\overline{P}_{00}(\cos a) = 1.0$$

$$\overline{P}_{11}(\cos \theta) = \sqrt{3} \, \sin\theta . \tag{4}$$

Then the following element is computed (with n = the given m):

$$\overline{P}_{n+1,n}(\cos \theta) = \sqrt{2n+3} \; \cos\theta \; \overline{P}_{n,n}(\cos \theta).$$ (5)

Then the following recursive relationship is used to calculate the remaining values of $\overline{P}_{nm}$ for $m+2 \leq n \leq N$:

$$\overline{P}_{nm}(\cos \theta) = \sqrt{\frac{(2n-1)(2n+1)}{(n-m)(m+n)}} \; \cos \theta \; \overline{P}_{n-1,m}(\cos \theta)$$

$$- \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(2n-3)(n+m)(n-m)}} \; \overline{P}_{n-2,m}(\cos \theta)$$ (6)

For the derivative of $\overline{P}_{nm}$ we first compute $\frac{d\overline{P}_{nm}(\cos \theta)}{d\theta}$ for n equal to m using the following equation recursively:

$$\frac{d\overline{P}_{nn}(\cos \theta)}{d\theta} = \sqrt{\frac{2n+1}{2n}} \left(\sin \theta \; \frac{d\overline{P}_{n-1,n-1}}{d\theta} + \cos\theta \; \overline{P}_{n-1,n-1}(\cos \theta) \right)$$ (7)

After this value is computed for the given m, the remaining derivatives for n from $m+1$ to N are computed from the following expression:

$$\frac{d\overline{P}_{nm}}{d\theta} = \frac{1}{\sin \theta} \left[n \; \overline{P}_{nm}(\cos \theta)\cos \theta - \left[\frac{(n^2-m^2)(2n+1)}{(2n-1)} \right]^{-\frac{1}{2}} \overline{P}_{n-1,m}(\cos \theta) \right]$$ (8)

The starting value is:

$$\frac{d\overline{P}_{00}}{d\theta} = 0$$ (9)

This program generates the $\cos m\lambda$ and $\sin m\lambda$ terms using recursive relation-ships. But the program does not implement efficiencies that exist if a uniform longitude grid were being computed.

6.   The Tscherning/Goad Program

We first discuss an alternative form of equation (1). Let us put $q = a/r$ and $t = \sin \psi$. Then (with N the maximum degree of the expansion):

$$V(r,\psi,\lambda) = \sum_{m=0}^{N} \frac{kM}{r} \left[\sum_{n=m}^{N} P_{nm}(t)q^n C_{nm} \cos m\lambda + \right.$$

$$\left. \sum_{n=m}^{N} P_{nm}(t)q^n S_{nm} \sin m\lambda \right] \qquad (10)$$

$$= \sum_{m=0}^{N} \frac{kM}{r} \left[V_m^1 \cos m\lambda + V_m^2 \sin m\lambda \right.$$

where

$$V_m^i = \sum_{n=m}^{N} P_{nm}(t)q^n \quad , \quad \begin{cases} C_{nm}, & i = 1 \\ S_{nm}, & i = 2 \end{cases} \qquad (11)$$

If the quantities V_m^i are computed for one set of values of ψ and r (and subsequently stored), then **it** is possible to compute the value of .
V for varying values of A, by simply evaluating the factors $\cos (m\lambda)$ and $\sin (mx)$ and then evaluating the product sum (10). This fact is used in the subroutine GPOTDR given in the Appendix. Further computational savings are possible, **if** V is evaluated at a set of equidistant points in the interval from 0° to 360°, but this requires a small change of the subroutine.

At this point we describe the concept of the Clenshaw summation, (Clenshaw, 1955). Suppose that we, for a set of functions $P_n(x)$, n = m,...,N of one or more variables $x = (x_1, x_2, \ldots x,)$ have:

$$P_{m+1}(x) + a_{m+1}(x)P_m(x) = 0 \qquad (12)$$

$$P_n(x) + a_n(x)P_{n-1}(x) + b_n P_{n-2}(x) = 0, \qquad n = m+2, \ldots , N \qquad (13)$$

where $a_n(x)$ is a function depending on x and b_n a constant. We may express this in matrix form as:

$$Ap = p_0 \tag{14}$$

where

$$A = \left\{ \begin{matrix} 1 & 0 & 0\ldots\ldots\ldots\ldots0 \\ a_{m+1} & 1 & 0\ldots\ldots\ldots\ldots0 \\ b_{m+1} & a_{m+2} & 1\ldots\ldots\ldots\ldots0 \\ \cdot & \cdot & \cdot \quad\quad\quad\quad \cdot \\ \cdot & \cdot & \cdot \quad\quad\quad \cdot \\ \cdot & \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \\ 0 & 0 & 0\ldots\ldots b_N a_N 1 \end{matrix} \right\}, \tag{15}$$

$$p = \left\{ \begin{matrix} P_m(x) \\ P_{m+1}(x) \\ \cdot \\ \cdot \\ \cdot \\ P_N(x) \end{matrix} \right\} \quad, \quad p_0 = \left\{ \begin{matrix} P_m(x) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{matrix} \right\}$$

The sum $S(x)$ of the series

$$S(x) = \sum_{n=m}^{N} y_n P_n(x) \tag{16}$$

may then be expressed as

$$S(x) = y^T p = y^T A^{-1} p_0 = p_0^T (A^T)^{-1} y = p_0^T s \tag{17}$$

where s is a vector with elements $s_m, \ldots, s_N$.

A^T is an upper triangular matrix with determinant equal to 1, so the solution vector $s = (A^T)^{-1}y$ may be found easily. If we put $s_{N+2} = s_{N+1} = 0$, then

$$s_n = -a_{n+1}s_{n+1} - b_{n+2}s_{n+2} + y_n, \quad n = N, \ldots, m \tag{18}$$

so

$$S(x) = s_m P_m(x) \tag{19}$$

This (equation (18) and (19)) is the Clenshaw algorithm for the summation of series which fulfill equations (12) and (13). (The algorithm is modified slightly, if only equation (13) is fulfilled). Algorithm for the computation of the partial derivatives of $S(x)$ with respect to any x_i can be found by taking the derivatives of the equations (17) - (19). Also the definite integral with respect to any x_i over an interval $[a,b]$ may be computed, if the integrals of the functions $P_n(x)$ are related to the values of $P_n(x)$ for $x_i = a$ and $x_i = b$ in a simple manner.

The advantage of using Clenshaw summation as compared to the usual method of finally evaluating $P_n(x)$ and then forming the scalar product $(y^T y)$ is of numerical character, as discussed in Clenshaw (1955) and Gerstl (1978). The number of operations (multiplications and additions) are nearly the same.

The numerical advantage consists in a decrease in the loss of significant digits during the summation. This is caused by the use of the identity, equation (13), in the evaluation of the sum $S(x)$.

Using this algorithm, it was possible to evaluate the sum *of* a spherical harmonic series with maximal degree and order 180 representing the Earth's gravity potential on a computer using only $10\frac{1}{2}$ significant digits, whereas the method described by Tscherning (1976 a,b) did not work.

When used for a spherical harmonic series, the Clenshaw-summation must be done in two steps, because different recursion formulae (corresponding to equation (13)) are valid for varying degree and varying order. For details see Tscherning and Poder (1982, equation (30) - (31b)).

Furthermore, in practice the coefficients C_{nm} and S_{nm} will not be given, but the corresponding fully normalized values $\bar{C}_{nm}$, $\bar{S}_{nm}$ will be available. This makes the constants $a_m(x)$ and b_m slightly more complicated, because a number of square-roots must be evaluated as in (4) through (9). These square-roots may be evaluated and stored once and for all, (which is also supposed to have been done before calling the subroutine GPOTDR). A small further computational saving is made by using quasi-normalized coefficients and associated Legendre functions:

$$\bar{\bar{P}}_{nm}(t) = P_{nm}(t) \left[\frac{(n-m)!}{(n+m)!} \right]^{\frac{1}{2}} \tag{20}$$

$$\left\{ \begin{array}{c} \bar{\bar{C}}_{nm} \\ \bar{\bar{S}}_{nm} \end{array} \right\} = \left\{ \begin{array}{c} C_{nm} \\ S_{nm} \end{array} \right\} \cdot \left[\frac{(n+m)!}{(n-m)!} \right]^{\frac{1}{2}} =$$

$$\left\{ \begin{array}{c} \bar{C}_{nm} \\ \bar{S}_{nm} \end{array} \right\} \left\{ \begin{array}{ll} (2n+1)^{\frac{1}{2}} & \text{for } m=0 \\ \\ (2(2n+1))^{\frac{1}{2}} & \text{for } m \neq 0 \end{array} \right. \tag{21}$$

The Clenshaw algorithm may, as mentioned above, be used for the computation of the partial derivatives (of any order) of the sum of the series. Derivatives with respect to longitude become undefined at the z-axis. One way to circumvent this problem is to compute the derivatives with respect to the coordinates (x,y,z) of a local Cartesian coordinate system having its origin in the point of evaluation, the x-axis pointing North, the y-axis pointing East and the z-axis pointing in the direction of the radius-vector. At the z-axis this coordinate system will have its axes parallel to those of the original coordinate system.

In this coordinate system, the Clenshaw algorithm is easily modified as to permit the evaluation of any partial derivation for points at the z-axis, (except for the point (0,0,0)).

The implementation of the primary equations of this section using the relationships in equation (3) was originally done in an Algol program described in Tscherning and Poder (1982). This program was translated into Fortran at the National Geodetic Survey with additional subroutines, needed for the actual computations, added. The resultant program is given in the Appendix.

7. Timing Tests

The four programs described above have been run at The Ohio State University's Amdahl 470 V/8 computer using the Fortran H extended (enhanced) compiler. Computations were first made to make sure that the programs yielded the same results for the common quantities being computed. The programs were then run under several circumstances to obtain timing comparisons using potential coefficient sets complete to degree 180. The computer times to be quoted exclude the time for inputting the potential coefficients and removing the reference field.

The first test was for a set of points having different latitudes and longitudes. The results are shown in Table 1.

Table 1

Computer Time for Computation of the Quantities Given in
Equation (3) at One Point

Program*	Time
Rapp	0.46 seconds
Tscherning/Goad	0.48 seconds

*(Timing estimates for the Rizos/Colombo programs are not included in Table 1, because these programs are not suited for single point csrnputations).

In the next test the Rapp, and Tscherning/Goad programs were modified
so that only the height anomaly would be computed. This test also involved
the Rizos program for calculations at a given latitude with 360 longitude
values at a 1° interval. The results are given in Table 2.

Table 2

Computer Time for Height Anomaly Computations
at One Latitude and 360 Equidistant Longitudes

Program*	Time (seconds)
Rapp	15.59
Rizos	0.39
Tscherning/Goad	1.91

*(Timing estimates for Colombo programarenot included in Table 2 because
this program is not suited for this type of computation).

The next test was designed to find the computer time needed to calculate
the height anomalies on a global 1°x1° grid. For this test the results
in Table 2 for the Rapp and Tscherning/Goad program were extrapolated
by multiplying by 180. The Rizos and Colombo times were determined from
actual runs. The results are given in Table 3.

Table 3

Computer Time for the Generation
of a Global 1°x1° Grid of Height Anomalies

Program	Time (seconds)
Rapp	2806*
Tscherning/Goad	344*
Rizos	66
Colombo	47

*extrapolation

8. Discussion of Timing Estimates

From Table 1 we see that the Rapp and the Tscherning/Goad programs have comparable point to point calculation times which was expected.

From Table 2 we see that the Rizos procedure is the fastest method followed by the Tscherning/Goad program followed by the Rapp procedure, a distant last. The Rapp procedure performs poorly in this type of test as no efficiencies are taken into account when the data is given on a uniform longitude grid.

For the computation on a global grid the program by Colombo is the fastest followed by the Rizos program.

9. Numerical Examples

When testing any computer programs it is convenient to have test values. We give in Table 4 such values (taken from Rapp (1982)) for five cases using three different sets of potential coefficients to degree 180. These sets are those of GEM10C (Lerch et al., 1981), Rapp (1981), and Rapp (1978). The reference field was that associated with the Geodetic Reference System 1980 (see e.g. (Rapp, 1982, p. 11)). The latitude (ϕ) given is geodetic.

Table 4

Sample Computed Values
(reference flattening = 1/298.257222)

$\phi°$	$\lambda°$	h(m)		ζ(m)	Δg(mgals)	δg(mgals)	ξ''	η''
21°	1°	0	Rapp78	34.46	20.07	30.65	0.68	-0.25
			Rapp81	30.56	7.73	17.11	0.60	0.40
			GEMIOC	28.37	4.12	12.83	-0.10	0.21
21"	45"	0	Rapp78	-9.76	-2.82	-5.81	-5.25	11.63
			Rapp81	-9.58	-5.55	-8.49	-4.24	10.63
			GEMIOC	-9.68	-8.46	-11.43	-2.23	8.98
5"	79"	0	Rapp78	-104.42	-84.60	-116.62	-1.43	0.64
			Rapp81	-107.48	-91.84	-124.81	0.02	0.65
			GEMIOC	-106.20	-87.66	-120.23	-1.13	-1.04
5"	79"	10000	Rapp78	-103.58	-78.90	-110.52	-1.63	0.35
			Rapp81	-106.58	-85.49	-118.02	-0.22	0.50
			GEMIOC	-105.35	-80.51	-112.66	-1.12	-0.93
87"	21"	0	Rapp78	15.43	-1.46	3.32	1.32	2.37
			Rapp81	20.23	8.86	15.12	0.81	1.86
			GEMIOC	18.38	3.58	9.26	2.59	4.05

10.   Summary and Conclusion

This paper has discussed four computer programs that can be used for the calculation of gravimetric quantities from high degree spherical harmonic expansions.   The programs of Rapp and Tscherning/Goad take about the same amount of computer time for point by point calculations.   The Tscherning/ Goad program has an advantage in that coding exists for the evaluation of the second order derivatives of the disturbing potential.

If computations of one quantity (such as the height or gravity anomaly) are to be done in a grid of limited geographic extent the Rizos program is most efficient.   If the calculation of all primary quantities are of interest in such a grid the Tscherning/Goad program is most efficient, since the Rizos program may only compute one quantity.

If a global grid of a single quantity is needed the Colombo procedure is fastest.   However the Rizos program is still relatively efficient and has the advantage of also being used in a limited geographic grid.

At times in the past some authors have expressed concern about the efficiency of using high degree spherical harmonic expansions.   The computer programs discussed in this paper demonstrate that efficient procedures do exist for working with such expansions when the programs are chosen considering the application., The proper choices can be made using the results discussed in this paper.

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00001 C
00002 C    PROGRAM TO ILLUSTRATE THE USE OF SUBROUTINE GPOTDR(CLENSHAW SUMMATION
00003 C    OF ANOMALOUS POTENTIAL IN SPHERICAL HARMONIC FORM WITH FIRST AND/OR
00004 C    SECOND DERIVATIVES IF DESIRED) AND THE ASSOCIATED SUPPORT ROUTINES.
00005 C    SUBROUTINE POT1 IS THE DRIVER FOR GPOTDR WHICH INITIALLY SETS
00006 C    THE VALUES OF RAPP'S (1978) 180-TH DEGREE SET OF COEFFICIENTS.
00007 C
00008 C    SINCE GEODETIC QUANTITIES WERE DESIRED WITH RESPECT TO GRS80, EVEN
00009 C    NORMAL ZONAL HARMONIC COEFFICIENTS ARE SUBTRACTED FROM THE TOTAL
00310 C    POTENTIAL IN SUBROUTINE POT1.  SINCE THE ANOMALOUS POTENTIAL
00011 C    AND ITS 3 FIRST DERIVATIVES ARE DESIRED, IORDER IS SET TO 1
00012 C    IN COMMON POT1CM AND THUS THIS EXAMPLE DOES NOT EXERCISE THE SECOND-
00013 C    DERIVATIVE CAPABILITY.
00014 C
00015        IMPLICIT REAL*8 (A-H,O-Z)
09316    10 READ(5,20,END=100)PHI,DLON,HT
00017    20 FORMAT(3D24.16)
00018        CALL POT1(PHI,DLON,HT,UN,XI,ETA,DIST)
00019        WRITE(6,30)PHI,DLON,HT
C0020        WRITE(6,30)UN,DIST,XI,ETA
03021    30 FORMAT(1X,4F12.2)
GC022        WRITE(6,30)
00323        CO TO 10
00024   100 STOP
03025        END
00026        BLOCK DATA
00527        IMPLICIT REAL*8 (A-H,O-Z)
00028    ·   LOGICAL INIT
03029        LOGICAL FIRST
00030        INTEGER OLDORD
00031        REAL*4 C,CO
00032        COMMON/GPOTCM/OLDT,OLDR,IZ,FIRST,OLDORD,I1,I2,I3,I4,
00033    .   I5,I6,I7,I8,I9,NMAXSV
00034        COMMON/POT1CM/SU(1810),DJN(20),GM,FLAT,AE,OMEGA,INIT,IORDER,NMAX,
00035    .   NEGN
00036        COMMON/PIDTR/PI,DTR
00037        COMMON/TRANCM/TOL,MAXIT
00038        COMMON/CM/C2OIN,G1(3),G2(3,3),CM3,CM2,CM1,CO,C(32760)
00039        DATA IZ/0/
00040        GATA FIRST/.FALSE./
00041        DATA OLDT/0.D0/,OLDR/0.D0/
00042        DATA OLDORD,I1,I2,I3,I4,I5,I6,I7,I8,I9/10*0/
00043        DATA NMAXSV/0/
00044        DATA DJN/20*0.D0/
00345        DATA INIT/.TRUE. /
00346        DATA IORDER/1/
00047        DATA NMAX/180/
00048        DATA PI/3.1415926535889793D0/
00049        DATA DTR /.17453292519943330D-1/
G0050        DATA TOL/1.D-14/,MAXIT/10/
00051        END
0C052        FUNCTION GPOTDR(PO,NMAX,ORDER,SU)
00053 C
03054 C
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00055 C  GI REG.NO. 81013 AUTHOR -C.C.TSCHERNING, DANISH GEODETIC INSTITUTE
00056 C                                       JULY 1981 IN ALGOL REF.(2)
00057 C                          -C.C.GOAD, NOAA/NOS/NATIONAL GEODETIC SURVEY
00058 C                                    MAY 1982 TRANSLATED TO FORTRAN
00059 C
00060 C REFERENCES:
00061 C (1) TSCHERNING, C.C.:ON THE CHAIN-RULE METHOD FOR COMPUTING
00062 C     POTENTIAL DERIVATIVES. MANUSCRIPTA GEODAETICA, VOL.1,
00063 C     PP. 125-141, 1976
00064 C
00065 C (2) TSCHERNING, C.C., AND PODER, K.: SOME APPLICATIONS OF CLENSHAW
00066 C     SUMMATION, PRESENTED AT VIII SYMPOSIUM ON MATHEMATICAL GEODESY,
00067 C     COMO, ITALY, SEPT 7-9, 1981
00068 C
00069 C  THE PROCEDURE COMPUTES THE VALUE AND UP TO THE SECOND-ORDER
00070 C DERIVATIVES OF THE POTENTIAL OF THE EARTH (W) OR OF ITS
00071 C CORRESPONDING ANOMALOUS POTENTIAL(T).
00072 C
00073 C  THE POTENTIAL IS REPRESENTED BY A SERIES OF SOLID SPHERICAL
00074 C HARMONICS, WITH UN-NORMALIZED OR QUASI-NORMLIZED COEFFICIENTS.
00075 C THE CHAIN-RULE IS USED ALONG WITH THE CLWSHAW ALGORITHM.
00076 C THE ARRAY C MUST HOLD THE COEFFICIENTS C(1)=C(1,0),C(2)=C(1,1),
00077 C C(3)=S(1,1), ETC. UP TO C((N+1)**2-1=S(N,N). C(0,0) IS STORED IN CO
00078 C WHICH IMPLICITLY ACTS AS C(0) (SEE THE COMMON BLOCK CH).
00079 C
00080 C
00081 C PARAMETERS:
00082 C
00083 C (A) INPUT VALUES:
00084 C
00085 C NMAX
00086 C    THE ABSOLUTE VALUE OF NMAX IS EQUAL TO THE MAXIMAL DEGREE AND
00087 C    ORDER OF THE SERIES. NEGATIVE NMAX INDICATES THAT THE COEFFICIENTS
00088 C    ARE QUASI-NORMALIZED.
00089 C
00090 C ORDER
00091 C    ORDER OF DERIVATIVES
00092 C    0 FOR POTENTIAL ONLY
00093 C    1 FOR POTENTIAL AND FIRJT DERIVATIVES
00094 C    2 FOR POTENTIAL, FIRST DERIVATIVES, AND SECOND DERIVATIVES
00095 C
00096 C PO
00097 C    ARRAY HOLDING POSITION INFORMATION. PO(6)
00098 C    PO(1)=P, THE DISTANCE FROM THE Z (ROTATION) AXIS,
00099 C    PO(2)=R, THE DISTANCE FROM THE ORIGIN,
00100 C    PO(3),PO(4) COS AND SIN OF THE GEOCENTRIC POLAR ANGLE(COLATITUDE),
00101 C    PO(5),PO(6) SIN AND COS OF THE LONGITUDE.
00102 C
00103 C C
00104 C    C((ABS(NMAX)+1)**2-1)     ARRAY OF C'S AND S'S DESCRIBED ABOVE
00105 C    CM3=GM
00106 C    CM2=A THE SEMI-MAJOR AXIS OF THE REFERENCE ELLIPSOID
00107 C    CM1=THE ANGULAR VELOCITY (=0,WHEN DEALING WITH T)
00108 C    CO=1.DO FOR W AND =0.DO FOR T
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00109 C
00110 C
00111 C     ROOT(K)=SQRT(K), O.LE.K.LE.2(ABS(N)+1)-1  WHEN NMAX.LT.0
00112 C
00113 C
00114 C (B)  RETURN VALUES:
03115 C
00116 C G1 AND G2
00117 C     THE RESULT IS STORED IN G1 AND G2 AS FOLLOWS:
00118 C
00119 C     G1(1)=DW/DX, G1(2)=DW/DY, G1(3)=DW/DZ
00120 C     G2(1,1)=DDW/DXX, G2(1,2)=G2(2,1)=DDW/DXDY,
00121 C     G2(1,3)=G2(3,1)=DDW/DXDZ, G2(2,2)=DDW/DYY,
00122 C     G2(2,3)=G2(3,2)=DDW/DYDZ AND G2(3,3)=DDW/DZZ
00123 C     WHERE W MAY BE INTERCHANGED WITH T AND
00124 C     VARIABLES X, Y, Z ARE THE CARTESIAN COORDINATES
00125 C     IN A LOCAL (FIXED) FRAME WITH ORIGIN IN THE POINT
00126 C     OF EVALUATION, X POSITIVE NORTH, Y POSITIVE EAST,
00127 C     AND Z POSITIVE IN THE DIRECTION OF THE RADIUS
00128 C     VECTOR, (CF.  REF.(1),EQ (4) AND (5)).
00129 C     THE VALUES OF W OR T WILL BE RETURNED IN GPOTDR.
00130 C
00131 C (C)  PASSED AND RETURNED VALUES:
00132 C
00133 C SU
00134 C     ARRAY OF DIMENSION K*(N+1), WHERE K-2 FOR NO DERIVATIVES,
00135 C     =6 FOR O-TH AND FIRST DERIVATIVES, =10 FOR O-TH, FIRST AND
00136 C     SECOND DERIVATIVES.  HERE ARE STORED THE PARTIAL SUM.  CF.
00137 C     REF.(2), EQ. (29), OF P(N,M)*(A/R)**(N+1-M)/P(M,M)*(C(N,M) OR
00138 C     S(N,M)) FROM N=M TO N=N, AND THE DERIVATIVES OF THESE SUM.
00139 C     THIS MAKES IT UNNECESSARY TO RECALCULATE THESE QUANTITIES, IF
00140 C     THE PROCEDURE IS CALLED SUBSEQUENTLY WITH THE SAME VALUE OF T
00141 C     AND R, AND THE SAME ORDER.
00142 C
00143       IMPLICIT REAL*8 (A-H, O-Z)
00144       INTEGER CAPN,CRDER,CAPN21,OLDORD
G0145       LOGICAL QUASI,DERIV1,DERIV2,POLE
00146       LOGICAL FIRST,NEW,OLD,NPOLE
00147       REAL*4 C,CO
00148       REAL*8 M21,M21T,M21U,M21UO
00149       DIMENSION SML(181),CML(181),SMLP1(182),CMLP1(182),PO(6)
00150       DIMENSION SU(1810)
00151       COMMON/SQROOT/DZERO,ROOT( 362)
00152       COMMON/GPOTCM/OLDT,OLDR,IZ,FIRST,OLDORD,I1,I2,I3,I4,
00153      .  I5,I6,I7,I8,I9,NMAXSV
00154       COMMON/CM/C2OIN,G1(3),G2(3,3),CM3,CM2,CM1,CO,C(32760)
00155       EQUIVALENCE(SML(1),SMLP1(2)),(CML(1),CMLP1(2))
00156       IF(NMAXSV.NE.NMAX)FIRST=.FALSE.
00157       NMAXSV=NMAX
00158       IF(FIRST)GO TO 100
C0159       FIRST-.TRUE.
00160       OLDT=2.D0
00161       J=IABS(NMAX)
00162       I=J+1
00163       I1=I+1
00164       I2=I1+I
00165       I3=I2+I
00166       I4=I3+I
00167       I5=I4+I
00168       I6=I5+I
00169       I7=I6+I
00170       I8=I7+I
00171       I9=I8+I
00172  100 CAPN=NMAX
00173 C DISTANCE FROM ROTATION AXIS
00174       P=PO(1)
00175 C DISTANCE FROM ORIGIN
00176       R=PO(2)
00177 C COSINE OF COLATITUDE
00178       T=PO(3)
00179 C SINE OF COLATITUDE
00180       U=PO(4)
00181 C SINE OF LONGITUDE
00182       SL=PO(5)   .
00183 C COSINE OF LONGITUDE
00184       CL=PO(6)
00185       T2=T+T
00186       POLE=DABS(U).LE.1.D-9
00187       NEW=DABS(OLDR-R).GT.1.D-3 .OR.  DABS(OLDT-T).GT.1.D-9 .OR.
00188      .  OLDORD.NE.ORDER .OR.  POLE
00189       OLD=.NOT.NEW
00190       NPOLE=.NOT.POLE
00191       IF(OLD)GO TO 200
00192       OLDR-R
00193       OLDT=T
00194       OLDORD=ORDER
00195  200 QUASI=.FALSE.
00196       IF(CAPN.LT.0)QUASI=.TRUE.
00197       IF(QUASI)CAPN=-CAPN
00198 C COMPUTE AE/R
00199       S=CM2/R
00200       S2=S**2
00201       CMLP1(1)=1.D0
00202 C     CML(0)=1.D0
00203       SMLP1(1)=0.D0
00204 C     SML(0)=0.D0
00205       DERIV1=.FALSE.
00206       IF(ORDER.GT.0)DERIV1=.TRUE.
00207       DERIV2=.FALSE.
00208       IF(ORDER.GT.1)  DERIV2=.TRUE.
00209 C
00210 C     SML(M) AND CML(M) ARE THE SINE AND COSINE OF M*LONGITUDE
00211 C
00212       SML(1)=SL
00213       CML(1)=CL
00214 C
00215       H1=1
00216       DO 300 M=2,CAPN
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