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Fast Spherical Collocation: A General Implementation.

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Abstract: Spherical harmonic analysis of gridded (and noisy) data on a sphere (with uniform error for a fixed latitude) gives rise to simple systems of equations. This has for the method of least-squares collocation (using an isotropic covariance function or reproducing kernel) been implemented with as much generality as the theory allows. The data only needs to be at the same altitude and of the same kind for each latitude. This permits for example the combination of gravity data at the surface of the Earth and data at satellite altitude.

Suppose that data are associated with the points of a grid with N values in latitude and M values in longitude. The latitudes do not need to be spaced uniformly. Also suppose we want to determine the spherical harmonic coefficients to a maximal degree and order K. Then the method will require that we solve K systems of equations each having a symmetric positive definite matrix of size N * N, only.

Results of three simulation studies using the method are described.

Keywords: Collocation, gravity, EGM96, spherical approximation, polar gaps.

1. Introduction.

The gravity potential of the Earth, W, may be split into two parts: a reference potential, U, and an anomalous potential, T, so that W = U + T. The reference potential may for example be a spherical harmonic expansion to degree N (360) plus the centrifugal potential. T may be approximated ("modelled") using least-squares collocation, i.e. as a linear combination of representers of observation functionals or covariance functions, see e.g. Moritz (1980),

$$\mathbf{T(P)} = \sum_{i=1}^{n} \mathbf{a}_i \cdot \mathbf{cov(T(P), Q_i)}$$

where P is a point in space and the functionals Q_i associate the observations and T, $Q_i\,(T) = Q_{io}$ the i'th observation. The coefficients a_i are determined as solutions of systems of linear equations where σ_{ij} are the error covariances,

$$\{a_i\} = \{\mathrm{cov}(Q_i, Q_j) + \sigma_{ij}\}\{Q_{0j}\} = \overline{C}\cdot\{Q_{0j}\}$$

From the estimate T' we may compute the coefficients of a spherical harmonic series using covariance propagation, cf. Tscherning (2001),

$$\mathbf{T_{lm}} = \{\mathbf{cov(T_{lm}, Q_i)}\}^{t}\,\overline{\mathbf{C}}^{-1}\{\mathbf{Q_{oj}}\} = \{\Lambda_{ij}^{lm}\}\{\mathbf{Q_{oj}}\}$$

so that

$$T(\theta,\lambda,r) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} T_{lm}\left(\frac{a}{r}\right)^{l+1} Y_{lm}(\theta,\lambda)$$

where θ is the co-latitude, λ the longitude, r the distance to the origin, a the semi-major axis and bb

$$Y_{lm}(\theta,\lambda) \equiv \overline{P}_{lm}(\cos\theta)e_m(\lambda)$$

where $\overline{P}_{lm}$ are the fully normalized Legendre functions of degree l and order m,

$e_m(\lambda) = \cos(m\lambda)$ when m > 0 and

$e_m(\lambda) = \sin(m\lambda)$ when m < 0.

If the observations form a grid in longitude when projected onto the unit-sphere large savings can be made when determining the coefficients T_{lm}. Systems of equations which have to be solved may have matrices with a nice repetitive (Toplitz) structure. One may take advantage of this structure and reduce the numerical effort considerably, see e.g. Colombo (1979). In the following we will implement Colombo's results to a fairly general estend. A summary of the theoretical developments are given in section 2. In section 3 we show some numerical results.

2. Theory.

Suppose we have a grid formed by the intersection of N parallels (some may occur several times) and M meridians spaced equidistantly in longitude. We will suppose M is even so that M=2*L, with L being any integer, e.g. 180. The observations are associated with functionals Q_{ik} which when applied in *T* give the observations Q_{oik}. For the sake of simplicity we shall use systematically l, m as indexes of degree and order, i, j as indexes of latitude and k, n as indexes of longitude.

The observations will contain noise, but we will not consider this here. Furthermore we will only consider isotropic covariance functions, i.e. the function only depends on the spherical distance and the distance of the two points of evaluation from the origin.

We will only consider functionals which has a simple relationship to the spherical harmonic coefficients, so that their covariance with a coefficient of degree *lm* becomes

$$\mathrm{cov}(L_{lm}, Q_{ik}) = b_i^{lm} e_m(\lambda_k)$$

where L_{lm} is the functional which when applied on T gives the l,m'th coefficient.

The functionals may be geoid heights, radial derivatives of T, vertical gravity disturbance gradients or gravity anomalies associated with the radial derivative,

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2T}{r}$$

(The usual derivatives with respect to the vertical must be transformed into derivatives of this kind).

Mean values of the above quantities also belong to the permitted data types. However the means must be represented as means over parallels having the same ellipsoidal height.

It can be shown that the solutions are given by (again not considering observation errors)

$$\{\Lambda_{in}^{lm}\} = \{\Gamma_{ij}^{m}\}^{-1}\{b_j^{lm}\}\frac{2}{M\varepsilon_m} e_m(\lambda_n)$$

where ϵ_m=1 for 0<m<L and ϵ_m=2 for m=0 or m=L.

Γ_{ij}^{m} is the m'th element of the cosine transform of the M-vector $\{\mathrm{cov}(\mathbf{Q}_{i0}, \mathbf{Q}_{jn})\}$, the covariance between the first value on parallel no. i and the values of parallel no. j.

We see that we have to solve a system of equations with N unknowns. However if we use Cholesky decomposition we can use the reduced matrix when determining all solution vectors for fixed order m. The computational effort may be further reduced by considering various symmetries.

Note that the parallels may be associated with different altitudes, but the data on one parallel must all be of the same kind. Errors associated with the data also have to be the same for each parallel. Also error estimates may be computed taking advantage of the Cholesky reduced matrix.

3. Examples.

A FORTRAN program "sphgrid.f" has been written following the principles described in Chapter 2. Mean gravity anomalies are treated as the sum of values (here n x n), not at the same radial distance, but at distances corresponding to the same ellipsoidal height. This corresponds approximately to how mean gravity anomalies are computed in practice.

The program has been tested for small grids, by verifying that the "full" collocation solution and the "fast" solution were identical.

Covariance functions must be given by providing a set of degree-variances. This means that the variance will be too large at the poles and too small at Equator, see 3.3. (This issue will be treated in a forthcoming paper).

The program may use spherical approximation or no approximation, (see 3.1) and may combine data at different altitudes and of different kinds (see 3.2). The program may handle very large data sets. Solutions to degree 720 have been computed using simulated data. In section 3.3 we describe the computation of a 360 solution using real data. The EGM96 (Lemoine et al., 1998) gravity data have been used.

3.1. Errors due to spherical approximation.

Spherical approximation is in general use in gravity field related computations. The error is supposed to be of the order of the flattening, i.e. 1/300, (see e.g. Moritz, 1980). Spherical approximation means here that the geodetic latitude is put equal to the geocentric latitude and that the distance from the origin, r, is put equal to the mean radius of the Earth, R = 6371 km, plus the ellipsoidal or even orthometric height. In the program sphgrid we have implemented both spherical approximation and no-approximation.

Pertubations of the EGM96 coefficients were calculated so that they had a standard deviation equal to the EGM96 error-degree-variances, see Figure 1. A covariance function consistent with this was defined using the error-degree-variances of EGM96 up to degree 180 as degree variances, and the GPM98 (Wenzel, 1998) degree-variances from degree 181 to degree 720. Gravity anomalies in an equidistant grid with spacing 1 degree were generated at 300 km altitude. No noise was added to the data, but in the calculations a noise standard deviation of 0.03 mgal was used.

Fig. 1 shows the resulting standard deviations of the differences between the predicted coefficients and the "true" ones per degree. The error obtained using spherical approximation is not of the order of the flattening, but between 0.1 and 0.05. The error obtained not using any approximation is 100 times smaller. The formal error estimate is nearly the same in both cases, since it mainly depends on the covariance function. The error obtained when not using spherical approximation is somewhat smaller than the formal error estimate due to the fact that no noise was added to the data. However for spherical approximation the error was much larger than the estimated error.

3.2 Data combinations.

In order to illustrate the potential of the method, it has been used to study the influence of the polar gaps on a spherical harmonic solution based on anomalous vertical gravity gradient data.. Data was simulated as pertubations of EGM96, but in fact not needed, since what we wanted to study was the error estimates of the spherical harmonic coefficients. Five of the many experiments conducted are described here. An equidistant grid in longitude with 2 degree spacing was used in all 5 cases:

1. Anomalous gravity gradient data on parallels spaced 2 degree apart from latitude –82 degrees to 82 degrees, altitude 300 km. Noise standard deviation equal to 0.005 EU was assigned to the data.
2. The same kind of data was added at the parallels with latitude 84, 86 and 88 degrees.
3. And at parallels with latitude –84, -86 and –88.
4. To the dataset used in (1), anomalous gravity was added at the parallels between 83 and 89 with 1 degree spacing. The altitude used was 5 km and the data noise was 0.2 mgal.
5. Data with the same spacing was added on Antarctica.

In Figure 2 the effect of the polar gap is clearly seen, and also the effect of covering the gaps with gradient data. Experiments with gravity anomalies added in different altitudes and with different noise assigned were conducted. By "iteration" it was found that rather high resolution gravity anomalies (5 km altitude corresponds to 0.5 degree means) , spaced with the double density as compared to the gravity gradients, were needed to approximately obtain the same effect as when gravity gradient values were added, see Figure 3.

The experiments corresponds quite well to the situation we will have using GOCE (ESA, 1999) data, but after we have improved the spherical harmonics using CHAMP (GFZ, 2001). It does not correspond to the present situation, because an isotropic covariance function was used in the calculations. The use of a uniform noise variance for the gradient data corresponds very well to what we expect from the GOCE mission.

3.3 Solution using EGM96 0.5 degree equal-angular mean free air anomalies.

EGM96 has been computed using satellite orbit pertubations and ground gravity. The gravity data was 0.5 degree mean values, downward continued to the ellipsoid. Error estimates had been assigned so that the gravity data would not deteriorate the satellite data, and the gravity data were used to improve on a satellite-only solution to degree 72. Consequently in this study we subtracted the contribution of EGM96 to degree 72, and used as degree-variances in the covariance function the error-degree variances of EGM96 to degree 72. This gives a strong a-priori constraint on the coefficients up to degree 72, the effect of which can be seen in Figure 4 and 5.

For the degree 73 – 720 we used degree variances calculated from Wenzels GPM98 coefficients, see Figure 5 , where the square-root of the degree variances are shown. It should be noted that a consequence of using this kind of isotropic model is that the signal variance will vary with latitude. The factor R/r, where R is the Earth mean radius, will be small at the poles and large at Equator. (In degree-variance models which have infinite number of coefficients, a Bjerhammar sphere with a radius smaller than the semi-minor axis of the Earth must be used. Such models have not been included in the program in order to keep the program simple).

A number of numerical experiments were conducted, treating the mean values as point values or as mean values and assigning different error-estimates to the data. In the results of the numerical experiments shown here, we have used the error-estimates of EGM96 to calculate mean values of error-estimates for each parallel, which then were assigned to all data on the same parallel. In Figure 4 and 5 are shown the results treating the data as point values. Similar results have been obtained using mean values.

We see that rather large differences with respect to the EGM96 coefficients exist. This is (probably) because we use isotropic covariances having a too large correlation distance in areas of strong gravity variations. This will result in a forcing of the short wavelength information present in large, strongly varying, gravity anomalies, e.g. close to sea-mounts, into the higher degrees.

The "corrections" to the coefficients were added to the EGM96 coefficients, and the coefficients were used to calculate mean gravity anomalies which were compared with the original EGM96 values. Using the unperturbed EGM96 coefficients gave a considerable better fit to the data compared to when using the "corrected" coefficients.

Here we have to keep in mind, that a collocation solution will give an exact fit to noise-free data, and a fit within the "noise-band" for noisy data. Hence had we computed estimates for all coefficients up to degree 720, a perfect fit would have been obtained – especially if we had assigned a small noise to the data. We want to study these findings in depth, because they could help explaining some of the long-wavelength errors found in geoid calculations where EGM96 is used in a remove-restore procedure. In some way, it could seem that methods which are used to determine coefficients, and a-priori projects the solution to a space of low dimension, will "press" too much of the information into these coefficients.

4. Conclusion.

The results of Colombo(1979) have been generalized so that data on grids with parallels in varying altitude and with different data-types can be processed very efficiently using least-squares collocation. No spherical approximation is necessary. However the use of an isotropic covariance implies that signal variances generally will be too low at Equator and too large at the poles. Solutions to this problem is being studied.

The Fast Spherical Collocation method has been used to demonstrate the problems associated with the use of spherical approximation. It has been used to simulate results where satellite gravity gradient data were combined with ground gravity. Furthermore the method has been used to compute a set of spherical harmonic coefficients and associated error-estimates to degree and order 360 using the same ground gravity data as used when determining EGM96. The differences with respect to EGM96 need to be further analyzed in order to understand better the differences.

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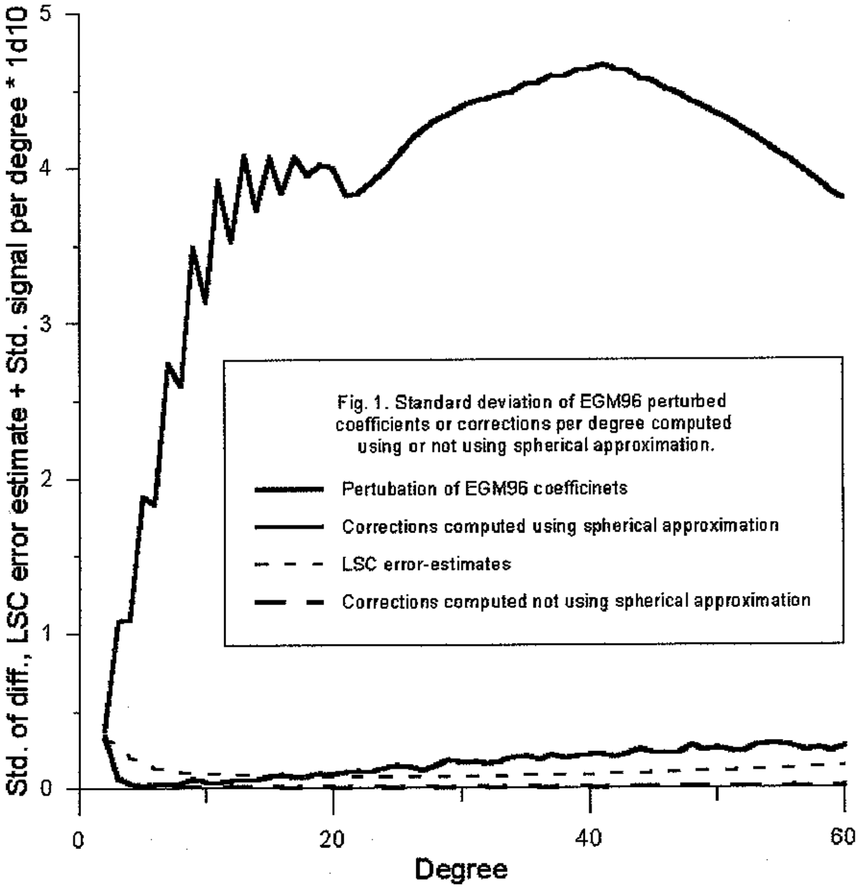

Fig. 1. Standard deviation of EGM96 perturbed coefficients or corrections per degree computed using or not using spherical approximation.

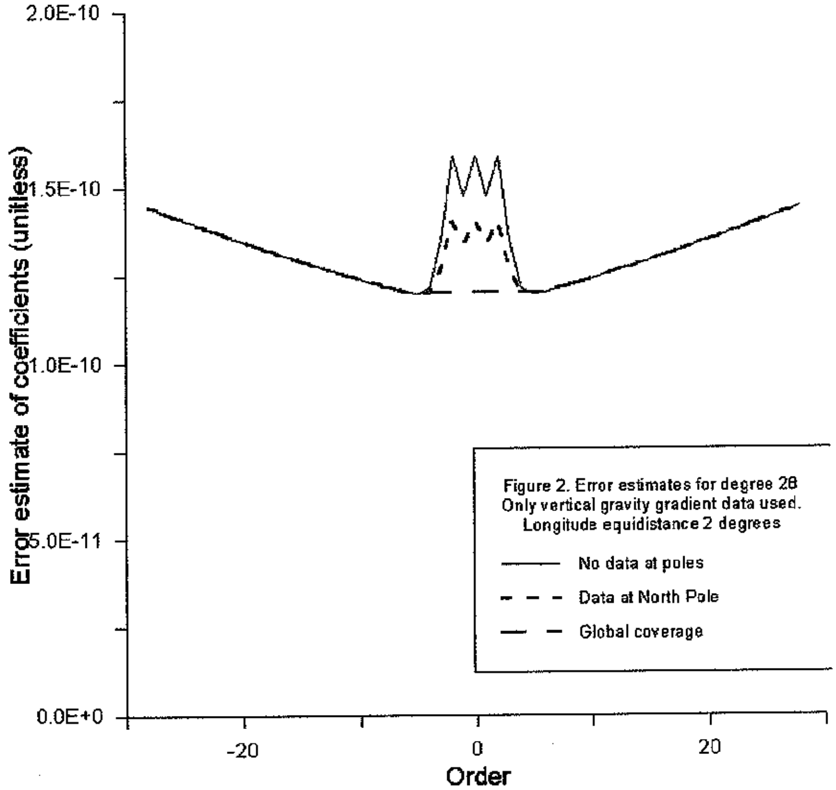

Figure 2. Error estimates for degree 28
Only vertical gravity gradient data used.
Longitude equidistance 2 degrees

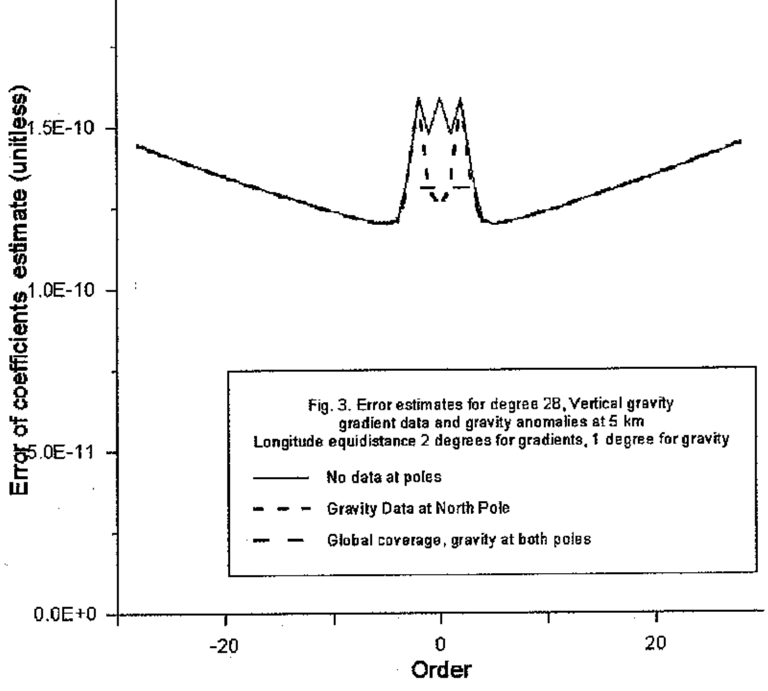

Fig. 3. Error estimates for degree 28, Vertical gravity
gradient data and gravity anomalies at 5 km
Longitude equidistance 2 degrees for gradients, 1 degree for gravity

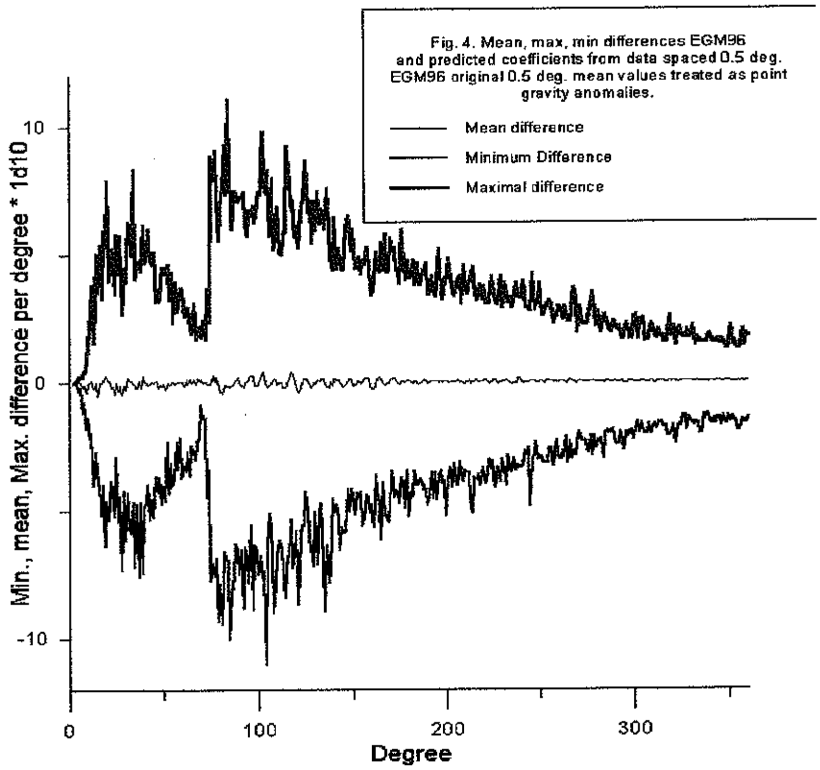

Fig. 4. Mean, max, min differences EGM96 and predicted coefficients from data spaced 0.5 deg. EGM96 original 0.5 deg. mean values treated as point gravity anomalies.

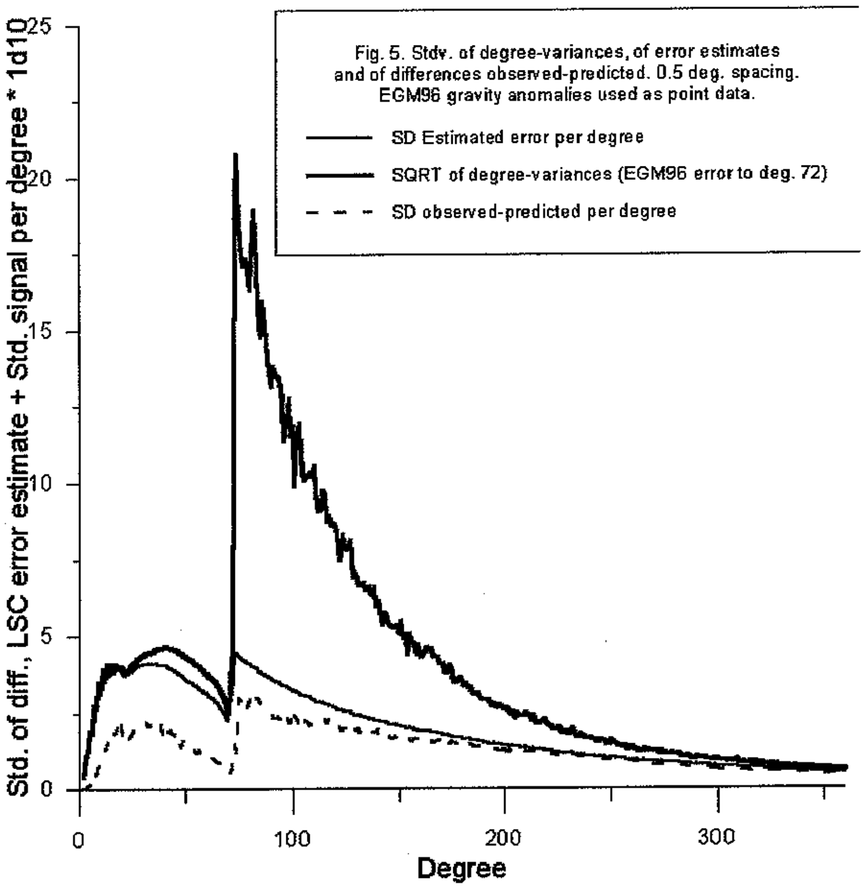

Fig. 5. Stdv. of degree-variances, of error estimates and of differences observed-predicted. 0.5 deg. spacing. EGM96 gravity anomalies used as point data.