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Approximation of harmonic covariance functions on the sphere by non-harmonic locally supported functions

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Abstract. Three methods to construct positive definite functions with compact support for the approximation of general geophysical harmonic covariance functions are presented. The theoretical background is given and simulations carried out, for three types of covariance functions associated with the determination of the anomalous gravity potential from gravity anomalies. The results are compared with those of the finite covariance function of Sansò and Schuh (1987).

Key words. Positive definite functions · Harmonic covariance functions

1 Introduction

Spherical splines have been used in many branches of geodesy to interpolate and to predict data discretely given on the sphere (Freeden 1981, 1987; Wahba 1981, 1982; Schneider 1996). However, a spline problem requires the solution of a positive definite system where the left-hand-side matrix is a Gram matrix of a harmonic kernel (covariance function). Because this kernel is harmonic, its support covers the whole sphere and so the matrix has in general no zero entries. But at the same time and in most applications, there is an appropriate angle of separation beyond which the kernel values are negligibly small. Rygaard-Hjalsted et al. (1997) showed that the Gram matrix of the truncated covariance function is still positive definite in the case of a geomagnetic field and that the solution of this sparse system is a good approximation to that of the full system.

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Sansò and Schuh (1987) built a so-called finite covariance function to approximate the real one in order to substitute for the full positive definite linear system a sparse one. Arabelos and Tscherning (1996) used this finite covariance function for gravity modelling and noticed reasonable results depending on differentiation (gravity gradients from gravity) or integration (geoid and gravity from gravity gradients). Nevertheless, as regards the model of Sansò and Schuh, the results were encouraging enough for research to continue, i.e. to find a finite approximation of the covariance function.

Schreiner (1997) built locally supported basis functions for the spherical spline function, and found that the Gram matrix associated with the reproducing kernel is also sparse.

Both these methods have the main advantage of generating sparse positive definite matrices which for a grid with $N \times N$ values would contain about N^3 non-zero elements, while the full matrices would contain about $N^4/2$ entries. Moreover, efficient solvers can be successfully used for such large sparse symmetric and positive definite systems (George and Liu 1981).

In this paper, we present three techniques to approximate geophysical harmonic covariance functions by finite supported positive definite functions. Because each covariance function related to the anomalous potential of the earth (Tscherning 1972) can be seen as the spherical convolution of a so-called original function with itself and because the convolution of a finite supported function with itself gives a finite positive definite function with twice the support of the original one, we approximate the covariance functions by the self-spherical convolution of a finite supported approximation of the original function.

Our first method is based on the simplest way to approximate a function by a finite one, which is to cut it after a given distance; for the second method we approximate the original function by a piecewise polynomial which is also zero after a certain distance, whereas in the third technique we approximate the self-convolution of the truncated original function by a summation over the sphere.

In order to clarify the equations we decided to present only the case where the measured points are on the same sphere; nevertheless, our methods can be easily generalized to any distribution of points.

The article is organized as follows: after some preliminary facts given at the beginning of Sect. 2, we present in Sects. 2.1, 2.2 and 2.3 the theoretical aspects of our three methods; since the proofs of the different propositions of these sections are rather technical, they are given an Appendix. Then, in Sect. 3.1, we apply the three techniques to the approximation of three covariance functions which occur in the determination of the anomalous gravity potential from gravity anomalies, and compare our results with the ones obtained by the finite covariance function of Sansò and Schuh (1987). In Sect. 3.2 we show the impact of solving the sparse system instead of the full one for the third covariance function and with the first technique. Conclusions are drawn in Sect. 4.

2 Mathematical theory

For any point $P \in \mathbb{R}^3$ different from the origin we may write $P = r_P \xi_P$, where $\xi_P \in \Omega_1 = \{P \in \mathbb{R}^3 | r_P = 1\}$ (unit sphere). To be consistent we define $\Omega_R = \{P \in \mathbb{R}^3 | r_P = R\}$ to be the sphere of radius R centered at the origin and denote by $\Omega_R^e = \{P \in \mathbb{R}^3 | r_P > R\}$ the outer space of Ω_R.

As usual, Δ^* denotes the Beltrami operator and $Y_{nm} : \Omega_1 \to \mathbb{R}$ are the spherical harmonics which are the only eigenfunctions of Δ^* corresponding to the eigenvalue $\lambda_n = -n(n+1)$, i.e.

$$\Delta^* Y_{nm} = \lambda_n Y_{nm} \quad n = 0, 1, \ldots, \quad m = -n, \ldots, +n \ .$$

Furthermore, the set $\{Y_{nm}\}_{n=0,1,\ldots}^{m=-n,\ldots,n}$ is known to be orthonormal and complete in $L^2(\Omega_1)$ with respect to $\langle\cdot|\cdot\rangle_{L^2(\Omega_1)}$.

The Legendre polynomials $P_n : [-1;1] \to \mathbb{R}$ are the only everywhere on $[-1;1]$ infinitely differentiable eigenfunctions of the Legendre operator $(1-t^2)\,\mathrm{d}^2/\mathrm{d}t^2 - 2t\,\mathrm{d}/\mathrm{d}t$ corresponding to the eigenvalue λ_n which satisfy $P_n(1) = 1$. We give below some of the well-known properties of the Legendre polynomials which we are going to use repeatedly. They can be found in any standard mathematical handbook (e.g. Spiegel 1968; Heiskanen and Moritz 1967).

$$P_0(t) = 1 \tag{1}$$

$$P_1(t) = t \tag{2}$$

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t) \quad n \geq 1 \tag{3}$$

$$(2n+1)P_n(t) = P'_{n+1}(t) - P'_{n-1}(t) \quad n \geq 1 \tag{4}$$

$$\int_{-1}^{1} P_n(t)P_m(t)\mathrm{d}t = \frac{2}{2n+1}\delta_{nm} \quad n, m \geq 0 \ . \tag{5}$$

Moreover, the spherical harmonics and the Legendre polynomials are connected by the addition theorem (Müller 1966)

$$\sum_{m=-n}^{+n} Y_{nm}(\xi_P)Y_{nm}(\xi_Q) = \frac{2n+1}{4\pi}P_n(\xi_P \cdot \xi_Q) = \frac{2n+1}{4\pi}P_n(\cos\psi(P,Q)) \ , \tag{6}$$

where $\xi_P \cdot \xi_Q$ is the inner product of ξ_P and ξ_Q in $\mathbb{R}^3$ and where $\psi(P,Q)$ denotes the spherical distance between P and Q.

From the completeness of the spherical harmonics and the addition theorem it follows that the "Fourier" expansion of $F \in L^2(\Omega_1)$ can be written as

$$F(\xi) = \sum_{n=0}^{+\infty} \frac{2n+1}{4\pi} \int_{\Omega_1} F(\eta)P_n(\xi\cdot\eta)\mathrm{d}w(\eta) \ . \tag{7}$$

Definition. A function $F : \Omega_R^e \times \Omega_R^e \to \mathbb{R}$ is called strictly positive definite, if

$$\sum_{i=1}^{N}\sum_{j=1}^{N} v_i v_j F(P_i, P_j) > 0 \tag{8}$$

for all choices of pairwise distinct points $P_i (i = 1, 2, \ldots, N)$ and all non-zero vectors $(v_1, v_2, \ldots, v_N)^T \in \mathbb{R}^N$, and then its Gram matrix $\mathbf{F} = (F(P_i, P_j))_{i,j}$ is called positive definite. If equality with zero is also allowed in Eq. (8), function F is called positive definite and its Gram matrix $\mathbf{F}$ is called non-negative definite.

Let $\mathcal{N} \subset \mathbb{N} = \{0, 1, 2, 3, \ldots,\}$ and $\{\sigma_n\}_n$ be a sequence of real numbers such that for all $\rho \in]0;1[$

$$\sum_{n\in\mathcal{N}} \frac{2n+1}{4\pi}|\sigma_n|\rho^{n+1} < \infty \tag{9}$$

$$\sum_{n\in\mathcal{N}} \frac{2n+1}{4\pi}\sigma_n^2\rho^{n+1} < \infty \tag{10}$$

The subject of this study is the approximation of the positive definite function K defined of $\Omega_R^e \times \Omega_R^e$ by

$$K(P,Q) = \sum_{n\in\mathcal{N}}\sum_{m=-n}^{+n} \sigma_n^2 \left(\frac{R^2}{r_P r_Q}\right)^{n+1} Y_{nm}(\xi_P)Y_{nm}(\xi_Q) = \sum_{n\in\mathcal{N}} \frac{2n+1}{4\pi}\sigma_n^2 \rho(P,Q)^{n+1} P_n(\cos\psi(P,Q)) = K(\rho(P,Q), \psi(P,Q)) \tag{11}$$

The next proposition give the expression of the so-called *original function* G associated with the positive definite function K of Eq. (11). In terms of Freeden and Schreiner (1998), the function K is also called the iterated kernel of G.

Proposition 1. For $P, Q \in \Omega_R^e$, the function G defined on $\Omega_R^e \times \Omega_R$ by

$$G(P,M) = \frac{1}{R}\sum_{n\in\mathcal{N}} \frac{2n+1}{4\pi}\sigma_n \rho(P,M)^{n+1} P_n(\cos\psi(P,M)) = G(\rho(P,M), \psi(P,M)) \quad (12)$$

which satisfies

$$K(P,Q) = \int_{\Omega_R} G(P,M)G(Q,M)\mathrm{d}w(M) \quad (13)$$

is called the original function G associated with the function K.

In order to simplify the next equations, we will only consider the case where $\Omega_R^e = \Omega_{R+h}$ with $h > 0$ and thus, for convenience, we introduce the real parameter ρ:

$$\rho = \rho(P,M) = \frac{R}{R+h}$$

for $P \in \Omega_{R+h}$ and $M \in \Omega_R$.

Therefore, the functions K [Eq. (11)] and G [Eq. (12)] are now dependent on only the spherical distance, i.e.

$$K(P,Q) = K(\psi(P,Q)) = \sum_{n\in\mathcal{N}} \frac{2n+1}{4\pi}\sigma_n^2 \rho^{2(n+1)} P_n(\cos\psi(P,Q)) \quad (14)$$

$$G(P,M) = G(\psi(P,M)) = \frac{1}{R}\sum_{n\in\mathcal{N}} \frac{2n+1}{4\pi}\sigma_n \rho^{n+1} P_n(\cos\psi(P,M)) \quad (15)$$

2.1 Method 1

With this method, the so-called original function G [Eq. (15)] is approximated by the function G_1 which is its truncation after a given angular distance α, and then the iterated kernel K_1 of G_1, which is a positive definite approximation of K [Eq. (14)], is by construction zero for any angular distance larger than 2α.

Let us consider the new function $G_1 : \Omega_{R+h} \times \Omega_R \to \mathbb{R}$ defined by

$$G_1(P,M) = \begin{cases} G(P,M) & \text{for } \psi(P,M) \le \alpha \\ 0 & \text{else} \end{cases} \quad (16)$$

as well as its so-called iterated kernel $K_1 : \Omega_{R+h} \times \Omega_{R+h} \to \mathbb{R}$

$$K_1(P,Q) = \int_{\Omega_R} G_1(P,M)G_1(Q,M)\mathrm{d}w(M) = \int_{\Gamma(P)\cap\Gamma(Q)} G(P,M)G(Q,M)\mathrm{d}w(M) \ , \quad (17)$$

where

$$\Gamma(P) = \{M \in \Omega_R | \psi(P,M) \le \alpha\}$$

Proposition 2. The function K_1 is strictly positive definite on $\Omega_{R+h} \times \Omega_{R+h}$.

Expanding G_1 in terms of Legendre polynomials, we have

$$G_1(P,M) = \frac{1}{R}\sum_{n=0}^{+\infty} \frac{2n+1}{4\pi} g_{1,n}(\alpha) P_n(\cos\psi(P,M)) \quad (18)$$

and thus from Eq. (17) we arrive at

$$K_1(P,Q) = \sum_{n=0}^{+\infty} \frac{2n+1}{4\pi} g_{1,n}^2(\alpha) P_n(\cos\psi(P,Q)) \quad (19)$$

The Legendre coefficients $g_{1,n}(\alpha)$ of Eq. (18) are given by

$$g_{1,n}(\alpha) = \frac{2\pi}{R}\int_{\Omega_R} G_1(P,M)P_n(\cos\psi(P,M))\mathrm{d}w(M) = \sum_{k\in\mathcal{N}} \frac{2k+1}{2}\sigma_k \rho^{k+1} \int_{\cos\alpha}^{1} P_k(t)P_n(t)\mathrm{d}t = \sum_{k\in\mathcal{N}} \frac{2k+1}{2}\sigma_k \rho^{k+1} I_{k,n}(\cos\alpha, 1) \ , \quad (20)$$

where

$$I_{k,n}(t_1,t_2) = \int_{t_1}^{t_2} P_k(t)P_n(t)\mathrm{d}t$$

is defined for $t_1, t_2 \in [-1; 1]$.

Expressions of $I_{k,n}(-1, t_2)$ are available in Paul (1973b). Meanwhile, as for $k \neq n$ $I_{k,n}(-1, t_2)$ depends on $P_k(t_2)$ and $P_n(t_2)$, and as the computation $I_{n,n}(-1, t_2)$ requires the evaluation $I_{n+1,n-1}(-1, t_2)$, we decided to show some new relations on $I_{k,n}(t_1, t_2)$ involving only Legendre polynomial differences, which can be obtained very quickly and with very good accuracy by the Clenshaw technique (Tscherning and Poder 1982), and where only some $I_{l,p}(t_1, t_2)$ appear for $l \le k, p \le n$.

Because $I_{k,n} = I_{n,k}$ we only have to compute these quantities for $n \ge k \ge 0$.

From Eqs. (1) and (3) we obtain for $n \ge 1$

$$I_{0,0}(t_1,t_2) = t_2 - t_1 \quad (21)$$

$$(2n+1)I_{0,n}(t_1,t_2) = [P_{n+1}(t) - P_{n-1}(t)]_{t_1}^{t_2} \quad (22)$$

A straightforward integration of Eq. (4) yields for $n \ge 1$

$$(2n+1)I_{1,n}(t_1,t_2) = (n+1)\ I_{0,n+1}(t_1,t_2) + nI_{0,n-1}(t_1,t_2) \quad (23)$$

and for $k \ge 2, n \ge 1$ we find from Eq. (3) that

$$k(2n+1)I_{k,n}(t_1,t_2) = (2k-1)(n+1)I_{k-1,n+1}(t_1,t_2) + (2k-1)nI_{k-1,n-1}(t_1,t_2) - (k-1)(2n+1)I_{k-2,n}(t_1,t_2) \quad (24)$$

Furthermore, we also obtain from Eq. (4) for $k \ge 2$ and $n \ge 1$

$$(2k-1)I_{k-1,n+1}(t_1,t_2) = [(P_k(t)-P_{k-2}(t))(P_{n+1}(t)-P_{n-1}(t))]_{t_1}^{t_2} - (2n+1)I_{k,n}(t_1,t_2) + (2n+1)I_{k-2,n}(t_1,t_2) + (2k-1)I_{k-1,n-1}(t_1,t_2) \ . \tag{25}$$

Combining Eqs. (24) and (25) we arrive at $(k, n \geq 2)$

$$(k+n+1)I_{k,n}(t_1,t_2) = \frac{n+1}{2n+1}[(P_k(t)-P_{k-2}(t))(P_{n+1}(t)-P_{n-1}(t))]_{t_1}^{t_2} \tag{26}$$

$$+ (n-k+2)I_{k-2,n}(t_1,t_2) + (2k-1)I_{k-1,n-1}(t_1,t_2) \tag{27}$$

and for $t_2 = 1$, because $P_n(1) = 1, n \geq 0$, we have for $k \geq 2$ and $n \geq 1$

$$[P_{n+1}(t) - P_{n-1}(t)]_{t_1}^{1} = P_{n-1}(t_1) - P_{n+1}(t_1) \tag{28}$$

$$[(P_k(t)-P_{k-2}(t))(P_{n+1}(t)-P_{n-1}(t))]_{t_1}^{1} = -[P_k(t_1)-P_{k-2}(t_1)][P_{n+1}(t_1)-P_{n-1}(t_1)] = -(2k-1)(2n+1)I_{0,k-1}(t_1,1)I_{0,n}(t_1,1) \ . \tag{29}$$

The next proposition gives an estimation of the difference between the positive definite functions K and K_1.

Proposition 3. Let $\alpha \geq 0$ and $P, Q \in \Omega_{R+h}$. If $\psi(P,Q) > 2\alpha$, then

$$|K(P,Q) - K_1(P,Q)| = |K(P,Q)|$$

else

$$|K(P,Q) - K_1(P,Q)| \leq c(P,Q)G(0)\max_{\psi>\alpha}|G(\psi)| \ ,$$

where

$$c(P,Q) = \int_{\Omega_R \setminus (\Gamma(P)\cap\Gamma(Q))} \mathrm{d}w(M)$$

2.2 Method 2

In this section, we approximate the function G of Eq. (15) by a piecewise polynomial G_2 which is zero after a given angular distance α and build its iterated kernel K_2 to approximate the positive definite function K of Eq. (14).

For the following discretization of the interval $[0;\alpha]$:

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_l = \alpha$$

$$\alpha_{i-1} = \alpha_{i0} < \alpha_{i1} < \cdots < \alpha_{id} = \alpha_i$$

$$i = 1,2,\ldots,l, d \in \mathbb{N}\setminus\{0\}$$

we define the function p_d by

$$p_d(\cos\psi) = \begin{cases} p_{1d}(\cos\psi) & \text{for} \quad \alpha_0 \leq \psi \leq \alpha_1 \\ \vdots & \vdots \\ p_{ld}(\cos\psi) & \text{for} \quad \alpha_{l-1} \leq \psi \leq \alpha_l \ , \end{cases} \tag{30}$$

where the polynomials p_{id} defined on $[-1;1]$ given by $p_{id}(t) = \sum_{k=0}^{d} a_{ik}t^k (i = 1,2,\ldots,l)$ satisfy

$$p_{id}(\cos\alpha_{ij}) = G(\alpha_{ij}) \quad j = 0,\ldots,d \ .$$

Let us now introduce the new function $G_2 : \Omega_{R+h} \times \Omega_R \to \mathbb{R}$ defined by

$$G_2(P,M) = \begin{cases} p_d(\cos\psi(P,M)) & \text{for } \cos\psi(P,M) \leq \alpha \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

as well as its so-called iterated kernel $K_2 : \Omega_{R+h} \times \Omega_{R+h} \to \mathbb{R}$

$$K_2(P,Q) = \int_{\Omega_R} G_2(P,M)G_2(Q,M)\mathrm{d}w(M) \tag{32}$$

Lemma. The function K_2 is strictly positive definite on $\Omega_{R+h} \times \Omega_{R+h}$.

G_2 can be expressed in term of Legendre polynomials by

$$G_2(P,M) = \frac{1}{R}\sum_{n=0}^{+\infty}\frac{2n+1}{4\pi}g_{2,n}(\alpha)P_n(\cos\psi(P,M)) \tag{33}$$

and thus

$$K_2(P,Q) = \sum_{n=0}^{+\infty}\frac{2n+1}{4\pi}g_{2,n}^2(\alpha)P_n(\cos\psi(P,Q)) \ . \tag{34}$$

The Legendre coefficients $g_{2,n}(\alpha)$ of Eq. (33) are given by

$$g_{2,n}(\alpha) = \frac{2\pi}{R}\int_{\Omega_R} G_2(P,M)P_n(\cos\psi(P,M))\mathrm{d}w(M) = 2\pi R\sum_{i=1}^{l}\sum_{k=0}^{d} a_{ik}\int_{\cos\alpha_i}^{\cos\alpha_{i-1}} t^k P_n(t)\mathrm{d}t = 2\pi R\sum_{i=1}^{l}\sum_{k=0}^{d} a_{ik}J_{k,n}(\cos\alpha_i,\cos\alpha_{i-1}) \ , \tag{35}$$

where the integrals $J_{k,n}$ are defined by

$$J_{k,n}(t_1,t_2) = \int_{t_1}^{t_2} t^k P_n(t)\mathrm{d}t$$

for $t_1, t_2 \in [-1;1]$.

Straightforward integrations yield

$$(k+1)J_{k,0}(t_1,t_2) = t_2^{k+1} - t_1^{k+1} \quad k \geq 0 \tag{36}$$

$$(k+2)J_{k,1}(t_1,t_2) = t_2^{k+2} - t_1^{k+2} \quad k \geq 0 \tag{37}$$

$$(2n+1)J_{0,n}(t_1,t_2) = [P_{n+1}(t) - P_{n-1}(t)]_{t_1}^{t_2} \quad n \geq 1 \tag{38}$$

From the recurrence formula of Eq. (3) we have for $k \geq 0, n \geq 1$

$$(2n+1)J_{k+1,n}(t_1,t_2) = (n+1)J_{k,n+1}(t_1,t_2) + nJ_{k,n-1}(t_1,t_2) \ . \tag{39}$$

Furthermore, multiplying Eq. (4) by t^k and by partial integration, we obtain for $k, n \geq 1$

$$\begin{aligned}(2n+1)J_{k,n}(t_1,t_2) &= k[J_{k-1,n-1}(t_1,t_2) - J_{k-1,n+1}(t_1,t_2)] \\ &\quad + [t^k(P_{n+1}(t) - P_{n-1}(t))]_{t_1}^{t_2} \ . \end{aligned} \quad (40)$$

Combining Eqs. (39) and (40), we arrive at the following formulas ($k \geq 0, n \geq 1$):

$$\begin{aligned}(k+n+2)J_{k,n+1}(t_1,t_2) &= (k-n+1)J_{k,n-1}(t_1,t_2) + [t^{k+1}(P_{n+1}(t) - P_{n-1}(t))]_{t_1}^{t_2} \end{aligned} \quad (41)$$

$$\begin{aligned}(k+n+2)(2n+1)J_{k+1,n}(t_1,t_2) &= (k+1)(2n+1)J_{k,n-1}(t_1,t_2) \\ &\quad + (n+1)[t^{k+1}(P_{n+1}(t) - P_{n-1}(t))]_{t_1}^{t_2} \ . \end{aligned} \quad (42)$$

Proposition 4. Let $\alpha \geq 0$ and $P, Q \in \Omega_{R+h}$. If $\psi(P,Q) > 2\alpha$ then

$$|K(P,Q) - K_2(P,Q)| = |K(P,Q)|$$

else

$$|K(P,Q) - K_2(P,Q)| \leq \Psi\left[\Psi + \sqrt{K(0)}\right]$$

where

$$\begin{aligned}\Psi^2 &= \sum_{i=1}^{l}\left(\max_{\alpha_{i-1}<\theta<\alpha_i}\frac{|g^{(d+1)}(\cos\theta)|}{(d+1)!}\right)^2 \\ &\quad \times \int_{\alpha_{i-1}}^{\alpha_i}\prod_{k=0}^{d}(\cos\psi - \cos\alpha_{ik})^2 \sin\psi \, \mathrm{d}\psi \\ &\quad + \int_{\alpha}^{\pi} G^2(\psi)\sin\psi \, \mathrm{d}\psi \end{aligned}$$

$$g(\cos\psi) = G(\psi) \ .$$

2.3 Method 3

Because the simplest expression of a positive definite function F on $\Omega_R^e \times \Omega_R^e$ is given by

$$F(P,Q) = \sum_i F_i(P)F_i(Q)$$

in this section, we approximate the positive definite function K of Eq. (14) by a function K_3 which has the same expression as F where the equivalents of the function F_i are obtained by discretization of the integral of Eq. (13).

We introduce the subsets $\Omega_R^i \subset \Omega_R, i \in I$, such that

$$\bigcup_{i\in I}{\Omega_R}^i = \Omega_R$$

$$\dot{\Omega}_R{}^i \cap \dot{\Omega}_R^j = \emptyset \quad \text{for } i \neq j \ ,$$

where $\dot{\Omega}_R^i$ is the interior of Ω_R^i and we associate to the subsets Ω_R^i the points $M_i \in \Omega_R^i$. We denote the area of Ω_R^i by c_i^2.

For $i \in I$ we define the functions $G_{3,i} : \Omega_{R+h} \to \mathbb{R}$ by

$$G_{3,i}(P) = \begin{cases} G(P,M_i) & \text{for } \psi(P,M_i) \leq \alpha \\ 0 & \text{else} \end{cases} \quad (43)$$

and thus, the positive definite function K can be approximated by the function K_3, with finite support $[0; 2\alpha]$, defined on $\Omega_{R+h} \times \Omega_{R+h}$ by

$$\begin{aligned}K_3(P,Q) &= \sum_{i\in I} c_i^2 G_{3,i}(P)G_{3,i}(Q) \\ &= \sum_{i\in\Gamma_3(P)\cap\Gamma_3(Q)} c_i^2 G_{3,i}(P)G_{3,i}(Q) \end{aligned} \quad (44)$$

with

$$\Gamma_3(P) = \{i \in I | \psi(P,M_i) \leq \alpha\} \ .$$

Lemma. The function K_3 is strictly positive definite on $\Omega_{R+h} \times \Omega_{R+h}$.

Remarks. By construction, the function K_3 is non-isotropic.

Let $P_1, \ldots, P_n$ be n points in Ω_{R+h}, and denote by $\mathtt{K}_3$ and by $\mathtt{G}_3$ the following Gram matrices:

$$\mathtt{G}_3(i,j) = c_j G_{3,j}(P_i)$$
$$\mathtt{K}_3(i,j) = K_3(P_i,P_j)$$

Then we have

$$\mathtt{K}_3 = \mathtt{G}_3\mathtt{G}_3^T \ .$$

Because the matrix $\mathtt{G}_3$ is not necessarily lower triangular, it cannot necessarily be the Cholesky factor of the matrix $\mathtt{K}_3$ □

Proposition 5. Let $\alpha \geq 0$ and $P, Q \in \Omega_{R+h}$. If $\psi(P,Q) > 2\alpha$ then

$$|K(P,Q) - K_3(P,Q)| = |K(P,Q)|$$

else

$$\begin{aligned}&|K(P,Q) - K_3(P,Q)| \\ &\quad \leq \Psi(P)\Psi(Q) + \sqrt{K(0)}[\Psi(P) + \Psi(Q)]\end{aligned}$$

where

$$\begin{aligned}\Psi^2(P) &= \max_{0<\theta<\alpha}[g'(\cos\theta)]^2 \\ &\quad \times \sum_{i\in\Gamma_3(P)}\int_{\Omega_R^i}[\cos\psi(P,M) - \cos\psi(P,M_i)]^2 \mathrm{d}w(M) \\ &\quad + \sum_{i\in I\setminus\Gamma_3(P)}\int_{\Omega_R^i} G(P,M)^2 \mathrm{d}w(M)\end{aligned}$$

$$g(\cos\psi) = G(\psi) \ .$$

3 Numerical examples

In this section we will apply the three techniques described in Sect. 2 to the approximation of the covariance functions associated with the determination of the anomalous potential from gravity anomalies, for which $\mathcal{N} = \{2, 3, \ldots, 360\}$, $\rho = R/(R+h)$ with $R = 6371$ km and $h = 200$ km, i.e.

$$K(P,Q) = k^2\rho^2 \sum_{n=2}^{360} \frac{(2n+1)(n-1)^2}{4\pi} \times \sigma_n^2 \rho^{2(n+1)} P_n(\cos\psi(P,Q)) = K(\psi(P,Q)) \ , \tag{45}$$

where

1. $\sigma_n = 1$
2. $\sigma_n = n - 1$
3. $\sigma_n = (n-1)^2$,

and where the scale factor k is such that $K(0) = 100$.

Therefore, from Eq. (12), the original function G associated with K is given by

$$G(P,M) = \frac{k}{R} \sum_{n=2}^{360} \frac{(2n+1)(n-1)}{4\pi} \sigma_n \rho^{(n+2)} P_n(\cos\psi(P,M)) = G(\psi(P,M)) \ . \tag{46}$$

3.1 Functions K, K_1, K_2, K_3 and K_s

The finite covariance function K_s of Sansò and Schuh (1987) will be used as a reference and so our results will be compared with the ones obtain with this function given by

$$K_s(\psi) = k_s \begin{cases} \frac{1}{3}\beta^6\pi - \frac{1}{2}\beta^4\psi^2\pi + \frac{1}{3}\left(\beta^4\psi + \frac{4}{3}\beta^2\psi^3 - \frac{1}{12}\psi^5\right) \\ \sqrt{\beta^2 - \left(\frac{\psi}{2}\right)^2} + \left(\beta^4\psi^2 - \frac{2}{3}\beta^6\right)\arcsin\frac{\psi}{2\beta} & \text{for } \psi \le 2\beta \\ 0 & \text{otherwise} \ , \end{cases} \tag{47}$$

where the scale factor k_s is such that

$$K_s(0) = K(0) = 100 \ .$$

As in Arabelos and Tscherning (1996), the real parameter β is chosen such that

$$K_s(\psi_1) = \frac{1}{2}K_s(0) \ ,$$

where ψ_1 satisfies

$$K(\psi_1) = \frac{1}{2}K(0) \ .$$

The subroutines computing the correlation distance ψ_1, the value of β (Table 1) as well as the finite covariance function K_s itself were extracted from the program GEOCOL, version 12 (Tscherning 1997).

Table 1. Angles α, β and maximum in absolute value of K associated with the three functions σ

	σ_n		
	1	$n-1$	$(n-1)^2$
α	5.8227°	4.1730°	3.0265°
β	1.7401°	1.1640°	0.8770°
$\max_{\psi \ge 2\alpha} \lvert K(\rho,\psi)\rvert$	0.2228	0.1126	0.0837

For each function σ, we computed $K_j(P_1, P_i)$, $j = 1, 2, 3$, for the 500 points $P_i \in \Omega_{R+h}$, given by

$$r_i = 6571 \text{ km}$$
$$\theta_i = \pi/2$$
$$\phi_i = (i-1)/499\alpha$$

such that $\psi(P_1, P_i) \le \alpha$.

Only the first 361 ($n = 0, \ldots, 360$) terms of expression (17) of K_1 were computed.

We only have implemented the simplest application of the second method in using polynomials p_{id} of degree $d = 1$ for $i = 1, \ldots, 50$ to construct the positive definite function K_2.

For the computations of K_3 [Eq. (44)] we opted for:

$$M_i = (R, \theta(M_i), \phi(M_i))$$
$$\Omega_R^i = \{M \in \Omega_R \| \theta - \theta(M_i)| \le \pi/720, \ |\phi - \phi(M_i)| \le \pi/720\} \ ,$$

where

$$\theta(M_i) = (2[(i-1)/720] + 1) \times \pi/720$$
$$\phi(M_i) = (2(i - 720 \times ([(i-1)/720] + 1) - 1) \times \pi/720 \ .$$

The real variables α associated with these three functions K, taken such that

$$G(\psi) \sim -\frac{1}{100}G(0) \tag{48}$$

are shown in Table 1.

The positive definite function K_3 is not by definition isotropic, as are the functions K, K_1, K_2 and K_s, but nevertheless similar results as the ones which will be shown were obtained for different configurations of points P_i.

In Fig. 1a (resp. Fig. 1b, c, d) the covariance function K of gravity anomalies associated with $\sigma_n = 1$, the corresponding finite one K_1 (resp. K_2, K_3, K_s), as well as their differences, are shown. The statistics of $K - K_i$ ($i = 1, 2, 3, s$) are available in Table 2.

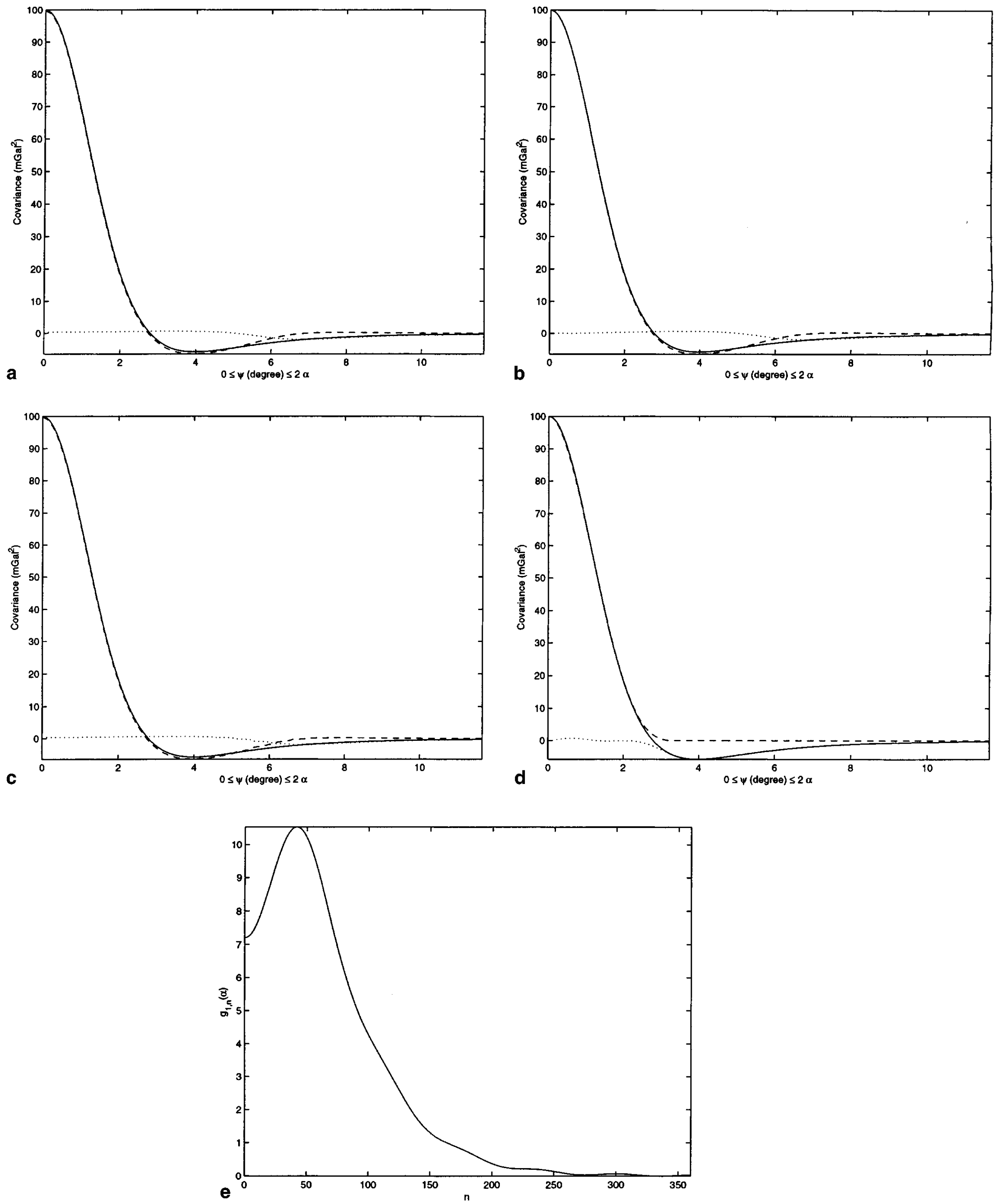

Fig. 1. **a** Functions K (*solid*), K_1 (*long dashes*) and $K - K_1$ (*dashed*) associated with $\sigma_n = 1$. **b** Functions K (*solid*), K_2 (*long dashes*) and $K - K_2$ (*dashed*) associated with $\sigma_n = 1$. **c** Functions K (*solid*), K_3 (*long dashes*) and $K - K_3$ (*dashed*) associated with $\sigma_n = 1$. **d** Functions K (*solid*), K_s (*long dashes*) and $K - K_s$ (*dashed*) associated with $\sigma_n = 1$. **e** Coefficients $g_{1,n}(\alpha)$ associated with $\sigma_n = 1$

Table 2. Statistics of the differences between K, K_1, K_2, K_3 and K_s associated with $\sigma_n = 1$ in (P_1, P_i). Unit: mGal^2

	$K - K_1$	$K - K_2$	$K - K_3$	$K - K_s$
Min	−1.9261	−1.9223	−1.9155	−5.7340
Max	0.6619	0.6663	0.6607	0.7344
Mean	−0.3316	−0.3936	−0.3119	−1.7180
SD	0.8668	0.8144	0.8494	1.9091

Figure 2 and Table 3 (resp. Fig. 3, Table 4) are the corresponding ones for $\sigma_n = n - 1$ [resp. $\sigma_n = (n-1)^2$].

Because the shapes of the curves representing the Legendre coefficients $g_{1,n}(\alpha)$ and $g_{2,n}(\alpha)$, for $n = 0, 1, \ldots, 360$, are analogous for each choice of σ, we show only the first ones in Figs. 1e, 2e and 3e.

3.2 System solutions involving K and K_1

In the last section, we saw that our finite positive definite functions K_i ($i = 1, 2, 3$) give good approximations of the positive definite functions K for the three choices of σ.

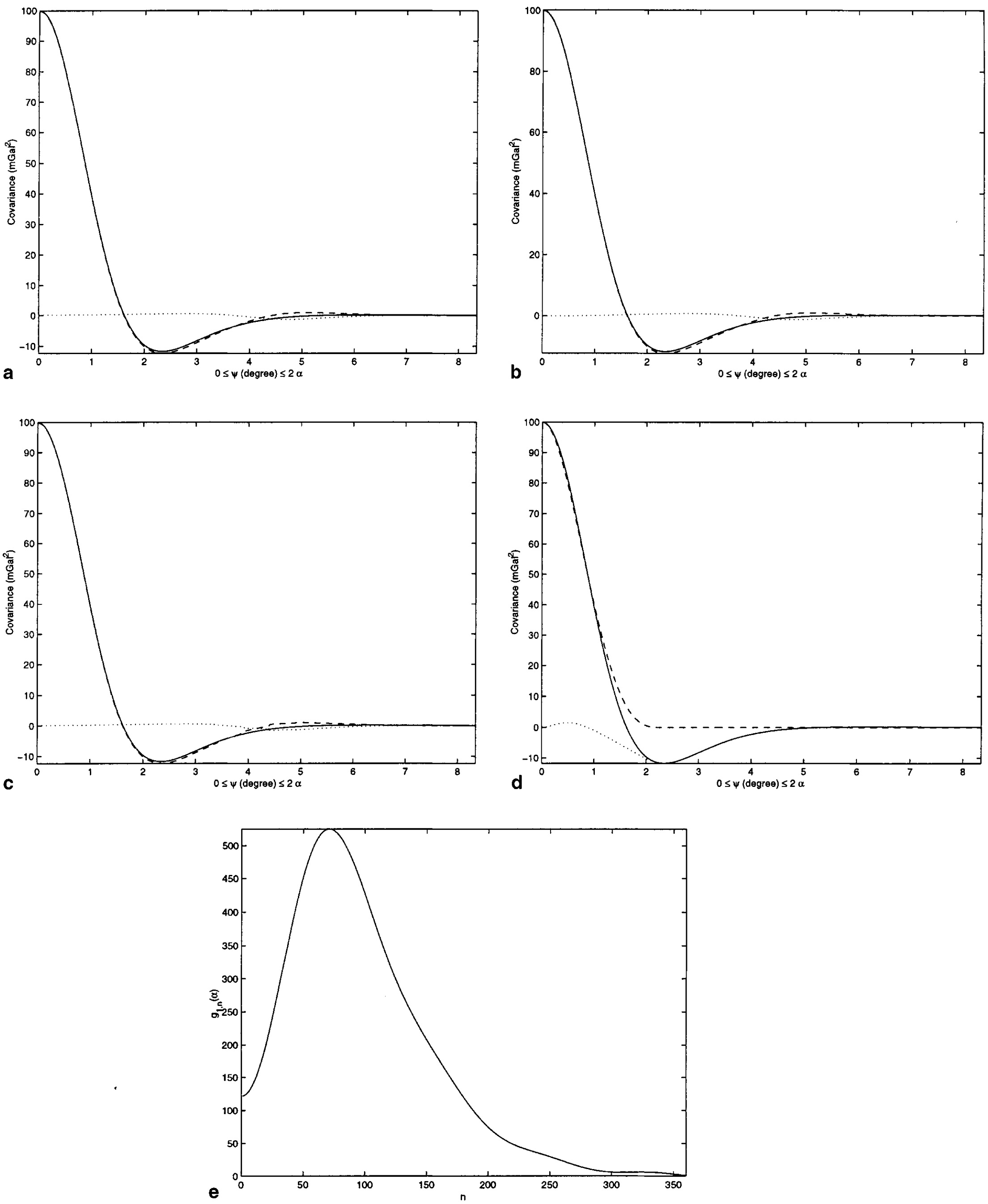

Fig. 2. **a** Functions K (*solid*), K_1 (*long dashes*) and $K - K_1$ (*dashed*) associated with $\sigma_n = n - 1$. **b** Functions K (*solid*), K_2 (*long dashes*) and $K - K_2$ (*dashed*) associated with $\sigma_n = n - 1$. **c** Functions K (*solid*), K_3 (*long dashes*) and $K - K_3$ (*dashed*) associated with $\sigma_n = n - 1$. **d** Functions K (*solid*), K_s (*long dashes*) and $K - K_s$ (*dashed*) associated with $\sigma_n = n - 1$. **e** Coefficients $g_{1,n}(\alpha)$ associated with $\sigma_n = n - 1$

However, as the main idea of building such finite functions is to replace a time-consuming system

$$\mathbf{K} \times x = b \tag{49}$$

by a sparse one, i.e.

$$\mathbf{K}_i \times x_i = b \ , \tag{50}$$

where $\mathbf{K}$ (resp. $\mathbf{K}_i$) is the Gram matrix associated with the function K (resp. K_i), it is now interesting to see the error generated by this substitution.

Because the results obtained in Sect. 3.1 by our three methods are very close, we decided to do this test with only the first method, where the upper limit of the summation is still 360, and for the last choice of σ ($\sigma_n = (n-1)^2$).

Let $b = (100, \ldots, 100) \in \mathbb{R}^{1681}$ and b_1 be given by

$$b_1 = \mathbf{K} \times (\mathbf{K}_1)^{-1} \times b \ . \tag{51}$$

For these tests we used the 1681 points of Ω_{R+h} shown by Fig. 4 for which the matrices $\mathbf{K}$ and $\mathbf{K}_1$ are positive definite.

Table 3. Statistics of the differences between K, K_1, K_2, K_3 and K_s associated with $\sigma_n = n - 1$ in (P_1, P_i). Unit: mGal^2

	$K - K_1$	$K - K_2$	$K - K_3$	$K - K_s$
Min	−1.2550	−1.2493	−1.5363	−11.7296
Max	0.6410	0.6835	0.6469	1.4968
Mean	−0.0314	−0.0751	−0.0601	−2.5124
SD	0.5458	0.5328	0.5863	3.9512

Table 4. Statistics of the diffferences between K, K_1, K_2, K_3 and K_s associated with $\sigma_n = (n-1)^2$ in (P_1, P_i). Unit: mGal^2

	$K - K_1$	$K - K_2$	$K - K_3$	$K - K_s$
Min	−0.7930	−0.7847	−1.1033	−16.1764
Max	0.4191	0.4618	0.4348	1.8896
Mean	0.0236	−0.0057	0.0054	−2.9910
SD	0.3423	0.3397	0.3479	5.5917

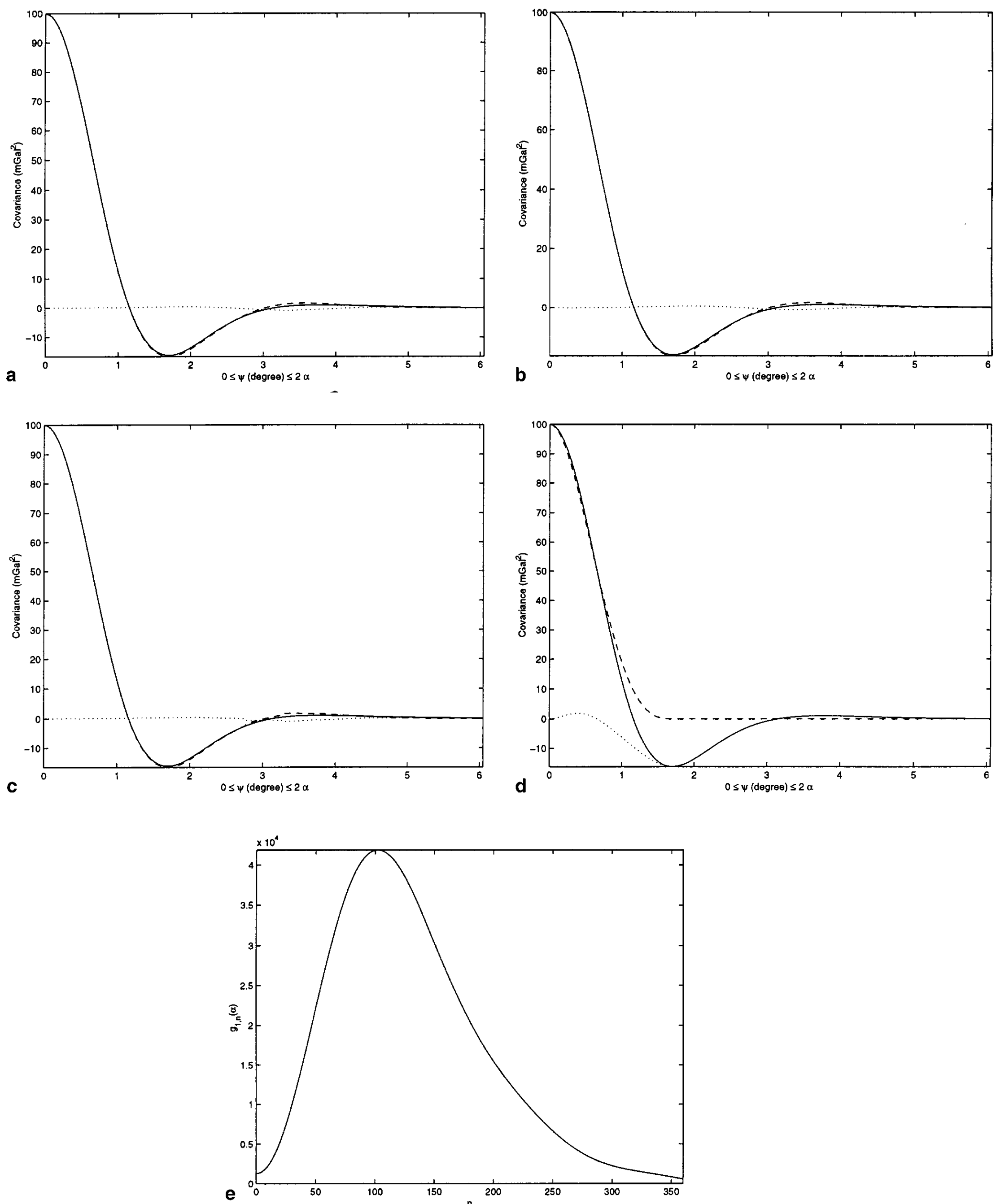

Fig. 3. **a** Functions K (*solid*), K_1 (*long dashes*) and $K - K_1$ (*dashed*) associated with $\sigma_n = (n-1)^2$. **b** Functions K (*solid*), K_2 (*long dashes*) and $K - K_2$ (*dashed*) associated with $\sigma_n = (n-1)^2$. **c** Functions K (*solid*), K_3 (*long dashes*) and $K - K_3$ (*dashed*) associated with $\sigma_n = (n-1)^2$. **d** Functions K (*solid*), K_s (*long dashes*) and $K - K_s$ (*dashed*) associated with $\sigma_n = (n-1)^2$. **e** Coefficients $g_{1,n}(\alpha)$ associated with $\sigma_n = (n-1)^2$

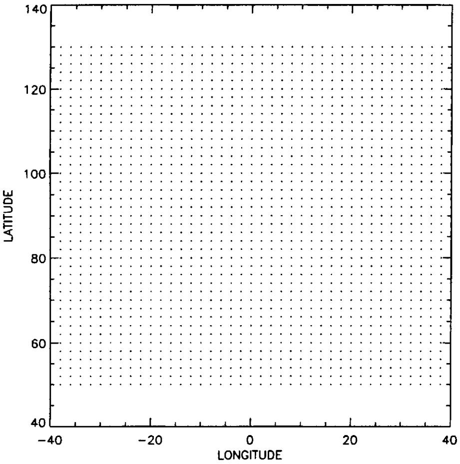

Fig. 4. Data coverage

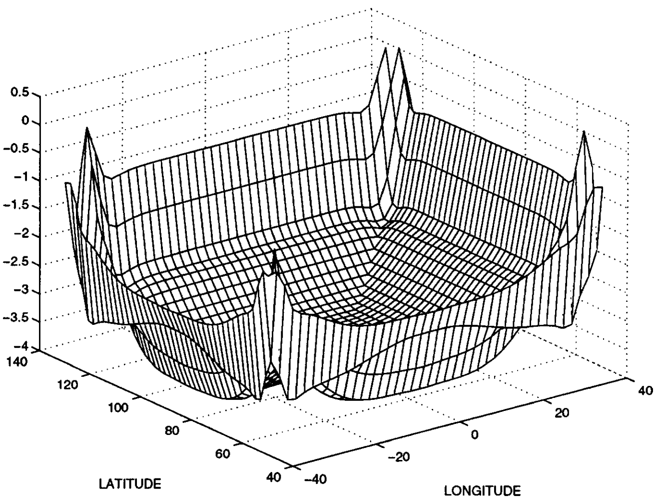

Fig. 5. Difference between the vectors b and b_1 (mGal)

Table 5. Statistics of the difference between b and b_1 associated with $\sigma_n = (n-1)^2$. Unit: mGal

	$b - b_1$
Min	−3.7521
Max	0.1259
Mean	−2.9160
SD	0.8623

Figure 5 represents a plot of the difference $b - b_1$ associated with the function $\sigma_n = (n-1)^2$ as a function of longitude and latitude; statistics are shown in Table 5.

The multi-banded matrix $\mathbf{K}_1$ (Fig. 6) with 49 177 non-zero elements is 1.74% full, while its banded (bandwidth = 124) Cholesky factor (Fig. 7.) with 196 990 non-zero elements is 6.97% full.

4 Conclusions

The first experiments (Sect. 3.1) proved that our three functions K_1, K_2 and K_3 are a better finite approximation

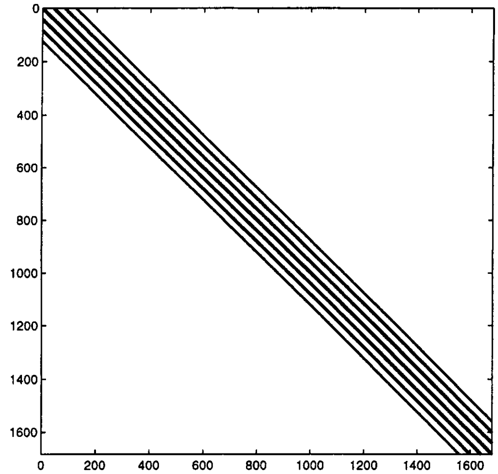

Fig. 6. Positions of non-zero entries in the Gram matrix $\mathbf{K}_1$

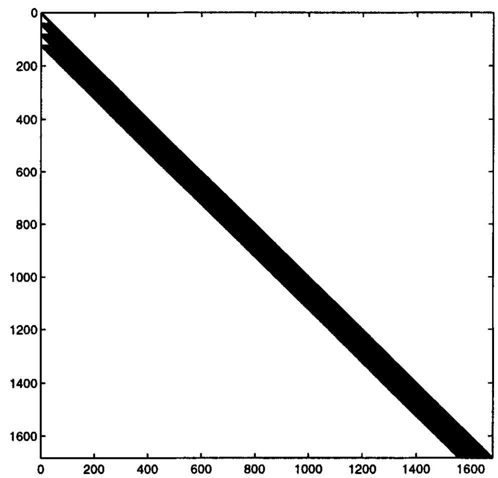

Fig. 7. Positions of non-zero entries in the Cholesky factor of $\mathbf{K}_1$

of the covariance function K than the finite covariance function K_s, independently of the choice of the function σ. We can also note that the results of K_1, K_2 and K_3 are very close, and the larger the absolute value of the minimum of K is the larger is the ratio between the means of $K - K_s$ and $K - K_i$ ($i = 1, 2, 3$), due to the fact that the finite covariance function K_s is always positive. Our results could be improved with a better choice of the parameter α; the best one should be the second root of the original function G, but as this root was close to 20° for the three samples, we only worked with an approximation of it [Eq. (48)]. We also think that it should be better to use a cubic spline for the approximation of the second method, with points M_i (Sect. 3) equally spaced with respect to the angular distance, other than the one we used for the numerical tests (Sect. 3.1) associated with the third technique or points M_i associated with equal-area blocks Ω_R^i (Paul 1973a).

Nevertheless, our three methods require both more storage and more CPU time than the method of Sansò and Schuh. There are two main reasons for this: the first one is that the functions K_1, K_2 and K_3 do not have a finite expression as K_s, and the second reason is that the supports of our three functions are bigger than those of K_s (Table 1) so our Gram matrices K_i ($i = 1, 2, 3$) are less sparse than K_s. Moreover, in the case of a covariance function K with finite expression, its Gram matrix is built in less time than the time used by our three methods, especially the third one, but this difference in time is largely compensated for in solving the system because the solution of a full positive definite linear system is attained in about $n^3/3 + 2n^2$ flops (floating point operations per second) for a Cholesky decomposition instead of the $np^2 + 7np + 2n$ flops for a sparse one with bandwidth p. The above flops ratings are from Golub and Van Loan (1996).

In Sect. 3.2 we showed the error generated by the use of a sparse system associated with our first technique instead of the full original one, and from Fig. 8 and Table 5 we see that this error is less than 4%. Because the points (Fig. 7) used for this test were separated by 2° and because the support of the finite covariance function K_s of Sansò and Schuh was about 1.8°, the results with this function were not good enough to be shown.

As a practical conclusion, our first method seems to be the most relevant method, compared to our other two and also to the method of Sansò and Schuh. Because we only worked with simple covariance functions and without noisy data, we will test this first method on more realistic cases as was done in Arabelos and Tscherning (1996) for the finite covariance function of Sansò and Schuh (1987), which will be the subject of a following study.

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Appendix

1 Proof of proposition 1

By the summation theorem of Eq. (6), we have

$$G(P,M) = \frac{1}{R}\sum_{n\in\mathcal{N}}\sum_{m=-n}^{+n}\sigma_n\rho(P,M)^{n+1}Y_{nm}(\xi_P)Y_{nm}(\xi_M) \quad .$$

Because the normalized spherical harmonics Y_{nm} satisfy

$$\int_{\Omega_R} Y_{nm}(\xi_M)Y_{n'm'}(\xi_M)\mathrm{d}w(M) = R^2\delta_{nn'}\delta_{mm'}$$

we obtain

$$\begin{aligned}
&\int_{\Omega_R} G(P,M)G(Q,M)\mathrm{d}w(M)\\
&= \sum_{n\in\mathcal{N}}\sum_{m=-n}^{+n}\sigma_n^2\rho(P,M)^{n+1}\rho(Q,M)^{n+1}Y_{nm}(\xi_P)Y_{nm}(\xi_Q)\\
&= \sum_{n\in\mathcal{N}}\frac{2n+1}{4\pi}\sigma_n^2\rho(P,Q)^{n+1}P_n(\cos\psi(P,Q))\\
&= K(P,Q) \quad .
\end{aligned}$$

2 Proof of proposition 2

Let $Q_1, \ldots, Q_I \in \overline{\Omega_R^e}$ be pairwisely distinct. It is obvious that the function K_1 is positive definite, so we shall prove that the functions $\widetilde{G}_i = \widetilde{G}_1(Q_i, \cdot)$ are linearly independent. Assume therefore that $v_1, \ldots, v_I \in \mathbb{R}$ satisfy $\hat{G} = \sum_{i=1}^{I} v_i\widetilde{G}_i = 0$. Since the function $\widetilde{G}_i$ is obtained from $G(Q_i, \cdot)$ by truncation, there exists an integer γ_i such that $\widetilde{G}_i^{(\gamma_i)}$ is not continuous. Suppose now that a v_i (without loss of generality, say v_1) is different from zero. In this case, the function $\hat{G}^{(\gamma_1)} = v_1\widetilde{G}_1^{(\gamma_1)} = 0$ is continuous, but not $\widetilde{G}_1^{(\gamma_1)}$, and so $v_1 = 0$.

3 Proof of proposition 3

The expression of the error is obvious in the case $\psi(P,Q) > 2\alpha$ because there exists no point M such that $\psi(P,M), \psi(Q,M) \leq \alpha$ and due to the fact that $G_1(P,M) = 0$ for $\psi > \alpha$.

For $P, Q \in \Omega_{R+h}$ such that $\psi(P,Q) \leq 2\alpha$, we have

$$\begin{aligned}
&|K(P,Q) - K_1(P,Q)|\\
&= \left|\int_{\Omega_R} G(P,M)G(Q,M)\mathrm{d}w(M) - \int_{\Omega_R} G_1(P,M)G_1(Q,M)\mathrm{d}w(M)\right|\\
&= \left|\int_{\Omega_R\setminus(\Gamma(P)\cap\Gamma(Q))} G(P,M)G(Q,M)\mathrm{d}w(M)\right|\\
&\leq G(0)\max_{\psi>\alpha}|G(\psi)|\int_{\Omega_R\setminus(\Gamma(P)\cap\Gamma(Q))}\mathrm{d}w(M) \quad .
\end{aligned}$$

4 Proof of proposition 4

Due to the fact that $G_2(\psi) = 0$ for $\alpha > \alpha$ and because for $\psi(P,Q) > 2\alpha$ there exists no point M such that $\psi(P,M), \psi(Q,M) \leq \alpha$, then for $\psi(P,Q) > 2\alpha$ we obviously have $K(P,Q) - K_2(P,Q) = K(P,Q)$.

By definition of p_{id} and by a well-known result on Taylor expansion, for all $\psi \in [\alpha_{i-1}, \alpha_i]$ there exists $\xi \in [\alpha_{i-1}, \alpha_i]$ such that

$$G(\psi) - p_{id}(\cos\psi) = \frac{g^{(d+1)}(\cos\xi)}{(d+1)!}\prod_{k=0}^{d}(\cos\psi - \cos\alpha_{ik})$$

and thus

$$\begin{aligned}
&|G(\psi) - G_2(\psi)|\\
&\leq \max_{\alpha_{i-1}<\theta<\alpha_i}\frac{|g^{(d+1)}(\cos\theta)|}{(d+1)!}\prod_{k=0}^{d}|\cos\psi - \cos\alpha_{ik}|
\end{aligned}$$

Because

$$G_2(P,M)G_2(Q,M) - G(P,M)G(Q,M) = [G(P,M) - G_2(P,M)][G(Q,M) - G_2(Q,M)] - G(P,M)[G(Q,M) - G_2(Q,M)] - G(Q,M)[G(P,M) - G_2(P,M)] \ , \quad \text{(A1)}$$

and with the triangle inequality, we have

$$|K(P,Q) - K_2(P,Q)| \leq \left| \int_{\Omega_R} [G(P,M) - G_2(P,M)][G(Q,M) - G_2(Q,M)] \mathrm{d}w(M) \right| + \left| \int_{\Omega_R} G(P,M)[G(Q,M) - G_2(Q,M)] \mathrm{d}w(M) \right| + \left| \int_{\Omega_R} G(Q,M)[G(P,M) - G_2(P,M)] \mathrm{d}w(M) \right| \ .$$

By the Cauchy–Schwarz inequality we obtain

$$|K(P,Q) - K_2(P,Q)| \leq \left(\int_{\Omega_R} [G(P,M) - G_2(P,M)]^2 \mathrm{d}w(M) \right)^{1/2} \times \left(\int_{\Omega_R} [G(Q,M) - G_2(Q,M)]^2 \mathrm{d}w(M) \right)^{1/2} + \left(\int_{\Omega_R} G(P,M)^2 \mathrm{d}w(M) \right)^{1/2} \times \left(\int_{\Omega_R} [G(Q,M) - G_2(Q,M)]^2 \mathrm{d}w(M) \right)^{1/2} + \left(\int_{\Omega_R} G(Q,M)^2 \mathrm{d}w(M) \right)^{1/2} \times \left(\int_{\Omega_R} [G(P,M) - G_2(P,M)]^2 \mathrm{d}w(M) \right)^{1/2} \ .$$

As the functions G and G_2 are isotropic and as by definition

$$\int_{\Omega_R} G(P,M)^2 \mathrm{d}w(M) = K(P,P) = K(0) \ ,$$

we obtain

$$|K(P,Q) - K_2(P,Q)| \leq \left(\int_{\Omega_R} [G(P,M) - G_2(P,M)]^2 \mathrm{d}w(M) \right)^{1/2} \times \left[\left(\int_{\Omega_R} [G(P,M) - G_2(P,M)]^2 \mathrm{d}w(M) \right)^{1/2} + 2\sqrt{K(0)} \right] \ .$$

The proof is completed by noting that

$$\int_{\Omega_R} [G(P,M) - G_2(P,M)]^2 \mathrm{d}w(M) = \int_{\Gamma(P)} [G(P,M) - G_2(P,M)]^2 \mathrm{d}w(M) + \int_{\Omega_R \setminus \Gamma(P)} G(P,M)^2 \mathrm{d}w(M) \leq \sum_{i=1}^{l} \left(\max_{\alpha_{i-1} < \theta < \alpha_i} \frac{|g^{(d+1)}(\cos\theta)|}{(d+1)!} \right)^2 \times \int_{\alpha_{i-1}}^{\alpha_i} \prod_{k=0}^{d} (\cos\psi - \cos\alpha_{ik})^2 \sin\psi \mathrm{d}\psi + \int_{\alpha}^{\pi} G^2(\psi) \sin\psi \mathrm{d}\psi \ .$$

5 Proof of proposition 5

Because for $\psi(P,Q) > 2\alpha$ there exists no point M such that $\psi(P,M), \psi(Q,M) \leq \alpha$, then for $\psi(P,Q) > 2\alpha$ we obviously have $K(P,Q) - K_3(P,Q) = K(P,Q)$.

From the Taylor theorem (Johnson and Riess 1982), for all $\psi \in [0;\alpha]$ there exists $\xi \in [0,\alpha]$ such that

$$G(\psi) - G(\psi_0) = g'(\cos\xi)(\cos\psi - \cos\psi_0)$$

and thus

$$|G(\rho,\psi) - G(\rho,\psi_0)| \leq \max_{0<\theta<\alpha} |g'(\cos\theta)| |\cos\psi - \cos\psi_0| \ .$$

By definition of K and K_3 we have

$$K(P,Q) = \int_{\Omega_R} G(P,M)G(Q,M) \mathrm{d}w(M) = \sum_{i\in I} \int_{\Omega_R^i} G(P,M)G(Q,M) \mathrm{d}w(M)$$

$$K_3(P,Q) = \sum_{i\in I} c_i^2 G_{3,i}(P) G_{3,i}(Q) = \sum_{i\in I} \int_{\Omega_R^i} G_{3,i}(P) G_{3,i}(Q) \mathrm{d}w(M) \ .$$

From Eq. (A1) and by the triangle inequality, we obtain

$$|K(P,Q) - K_3(P,Q)| \leq \left| \sum_{i\in I} \int_{\Omega_R^i} [G(P,M) - G_{3,i}(P)] \times [G(Q,M) - G_{3,i}(Q)] \mathrm{d}w(M) \right| + \left| \sum_{i\in I} \int_{\Omega_R^i} G(P,M)[G(Q,M) - G_{3,i}(Q)] \mathrm{d}w(M) \right| + \left| \sum_{i\in I} \int_{\Omega_R^i} G(Q,M)[G(P,M) - G_{3,i}(P)] \mathrm{d}w(M) \right| \ ,$$

and by the Cauchy–Schwarz inequality we obtain

$$
\begin{aligned}
&|K(P,Q) - K_3(P,Q)| \\
&\quad \le \left(\sum_{i\in I} \int_{\Omega_R^i} [G(P,M) - G_{3,i}(P)]^2 \mathrm{d}w(M) \right)^{1/2} \\
&\qquad \times \left(\sum_{i\in I} \int_{\Omega_R^i} [G(Q,M) - G_{3,i}(Q)]^2 \mathrm{d}w(M) \right)^{1/2} \\
&\qquad + \left(\sum_{i\in I} \int_{\Omega_R^i} G(P,M)^2 \mathrm{d}w(M) \right)^{1/2} \\
&\qquad \times \left(\sum_{i\in I} \int_{\Omega_R^i} [G(Q,M) - G_{3,i}(Q)]^2 \mathrm{d}w(M) \right)^{1/2} \\
&\qquad + \left(\sum_{i\in I} \int_{\Omega_R^i} G(Q,M)^2 \mathrm{d}w(M) \right)^{1/2} \\
&\qquad \times \left(\sum_{i\in I} \int_{\Omega_R^i} [G(P,M) - G_{3,i}(P)]^2 \mathrm{d}w(M) \right)^{1/2} \quad ,
\end{aligned}
$$

where

$$
\begin{aligned}
\sum_{i\in I} \int_{\Omega_R^i} G(P,M)^2 \mathrm{d}w(M) &= \int_{\Omega_R} G(P,M)^2 \mathrm{d}w(M) \\
&= K(P,P) \\
&= K(0) \quad .
\end{aligned}
$$

The proof is achieved with the next majorisation.

$$
\begin{aligned}
&\sum_{i\in I} \int_{\Omega_R^i} [G(P,M) - G_{3,i}(P)]^2 \mathrm{d}w(M) \\
&\quad = \sum_{i\in\Gamma_3(P)} \int_{\Omega_R^i} [G(P,M) - G(P,M_i)]^2 \mathrm{d}w(M) \\
&\qquad + \sum_{i\in I\setminus\Gamma_3(P)} \int_{\Omega_R^i} G(P,M)^2 \mathrm{d}w(M) \\
&\quad \le \sum_{i\in\Gamma_3(P)} \int_{\Omega_R^i} [\cos\psi(P,M) - \cos\psi(P,M_i)]^2 \mathrm{d}w(M) \\
&\qquad \times \max_{0<\theta<\alpha} [g'(\cos\theta)]^2 + \sum_{i\in I\setminus\Gamma_3(P)} \int_{\Omega_R^i} G(P,M)^2 \mathrm{d}w(M) \quad .
\end{aligned}
$$

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