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Topographic Effects in Gravity Field Modelling for BVP

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1. INTRODUCTION

The written form of these lecture notes is a modified version of the lecture notes "Terrain effects in geoid computations" by R.Forsberg, published in the lecture notes of the International School on the Determination and Use of the geoid, Milano, 1994. The text will therefore to a high degree refer to the treatment of topographic effects in relation to geoid computations. However, the solution of boundary value problems for the Earth's gravity field, including the determination of the (quasi-)geoid, is a part of the more general problem of modelling the gravity field. The main difference is that the data not necessarily will be associated with points at the surface of the earth, but may be points in space measured in an aircraft or a satellite. The problems associated with treating the topography, however, remains the same.

Geodetic gravity field modelling is, given some quantities - "observations" - of the earth's gravity field, what are other quantities - "predictions". The modelling

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procedures can be e.g. Stokes' or Vening-Meinesz' integrals, FFT methods, least-squares collocation, point-mass fitting, spherical or ellipsoidal harmonic expansions.

For geoid prediction the basic formula is the familiar Stokes' integral over the surface of the earth

$$N = \frac{R}{4\pi\gamma} \iint_{\sigma} \Delta g \; S(\psi) \; d\sigma \qquad (1)$$

The topography in a mountainous area affects gravity field modelling in two ways:

1) A strong gravity signal is due to the gravitational attraction of the topographic masses itself, a signal which dominates at shorter wavelengths, and therefore information of topography can be used to smooth the gravity field prior to any modelling process, and

2) The topography implies that the basic observation data - notably gravity anomalies -*are given on a non-level surface,* violating the basic requirements for Stokes' integral.

In the first case the gravity field may be smoothed by *terrain reductions*, in the second case *Molodensky* or *Helmert condensation* corrections are applied to offset the non-level surface. In the Molodensky theory the geoid is substituted by the quasi-geoid, and geoid undulations N by height anomalies ζ ("geoid heights at the surface of the topography"). Modified integral formulas of typical form

$$\zeta = \frac{R}{4\pi\gamma} \iint_{\sigma} (\Delta g + g_1) \; S(\psi) \; d\sigma \qquad (2)$$

must be substituted instead of Stokes' formula, where g_1 is the first term in the Molodensky expansions, cf. Heiskanen and Moritz (1967, sec. 8) or Moritz (1980).

In these lecture notes we will review both of these "topographic" effects, starting with some basic concepts of the spatial "modern" gravity field description. The main emphasis will, however, be put on the definition and practical use of proper *terrain reductions*, which are dominant and usually far bigger than the non-level surface ("Molodensky-style") corrections. When "operational" methods such as least-squares collocation and point-mass modelling are used, these corrections are in principle irrelevant, since the spatial operational methods may directly take the arying height of the observation points into account.

he non-level surface corrections are of course irrelevant on the *ocean* as well, here the geoid and quasi-geoid are identical, and gravity observations refer to the eoid. However, the bathymetry generally has a strong "terrain effect" in the marine ivironment. We will in these notes by the term "terrain" in general include both pography and bathymetry, the bathymetry typically represented as negative terrain ights in the various program packages. Generally the influence of the bathymetry comparable or even larger than the corresponding topography. In many cases the thymetric effects are, however, neglected, either due to lacking data (high-solution detailed bathymetric grids are not so common), or because the effects of thymetry are considered marginal in improving e.g. a continental geoid ediction.

ne of the impacts of local terrain reductions is to remove the correlation of free-anomalies with height, and avoid the *aliasing* which might appear when gravity tions are systematically observed at different levels than the average topographic vel, e.g. when gravity points have a tendency to be located in valleys in ountainous areas. Such aliasing errors can be very big and devastating for geoid ediction, but can be avoided through proper terrain reduction and gridding chniques. In the marine environment "non-random" gravity measurements are less ely to occur, as survey ship (or satellite altimeter) measurements are typically iependent of bathymetry, and therefore terrain reductions as an "anti-aliasing ter" are of less importance.

e will in these notes focus on the continental terrain effects, marine follows the ne principles except for changes of density etc. The next section provides a basic ory review, and in section 3 an overview of the practical computation principles given.

BASICS OF THE GRAVITY FIELD AND TERRAIN REDUCTIONS

e field we have access to through measurements is the total gravity field, e.g. ivatives of the physical potential W. By using a normal ellipsoidal gravity field we construct the anomalous potential $T = W\text{-}U$, and obtain the usual quantities h as gravity anomalies Δg, height anomalies ζ, and deflections of the vertical η) by the well-known linear operators

ere γ is normal gravity. These quantities should in principle be viewed as *spatial ressions* - i.e. as functions of both latitude, longitude *and* radial distance or ght. The basic linear functional relationships (3) may be viewed as the nitions of the corresponding spatial gravity field quantities. The anomalous ential $T(r,\phi,\lambda)$ is a spatial function, which outside the terrain masses will be monic, i.e. satisfy the Laplace equation

$$\begin{aligned} \Delta g &= -\frac{\partial T}{\partial r} - \frac{2}{r}T \\ \zeta &= \frac{T}{\gamma} \\ \xi &= -\frac{1}{r\gamma}\frac{\partial T}{\partial r} \\ \eta &= -\frac{1}{r\gamma\cos\phi}\frac{\partial T}{\partial r} \end{aligned} \qquad (3)$$

$$\Delta T = 0 \qquad (4)$$

Inside the topography T = W-U is still well-defined, but will now fulfil the Poisson equation

$$\Delta T = -4\pi G\rho \qquad (5)$$

where G is the gravitational constant and ρ the density.

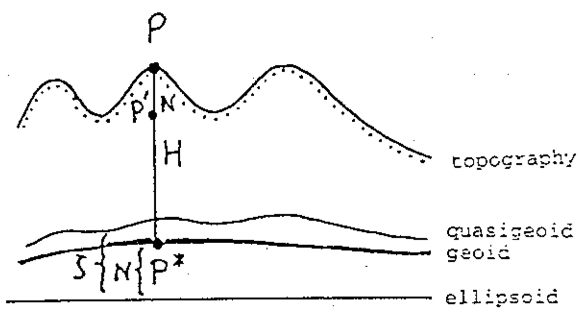

Fig. 1

For gravity anomalies the spatial concept means that the gravity anomaly refer to a point P at the earth surface, and not the corresponding geoid point P* (fig. 1). The gravity anomaly is defined as the difference between observed gravity at the surface, and normal gravity computed at the same height above the ellipsoid

$$\Delta g_P = g_P^{observed} - \gamma_{P'} \approx g_P^{observed} - \gamma_o + \frac{\partial\gamma}{\partial h}H \qquad (6)$$

Here γ_o is the normal gravity on the ellipsoid, and the term $\partial\gamma/\partial h \approx 0.3086$ mgal/m is the normal free-air gradient. This "spatial" view of the definition of the gravity anomaly is fundamental for "modern" physical geodesy, but is often not stressed with sufficient clarity in introductory classes - here the free-air anomaly is frequently interpreted as the "gravity on the geoid, downward continued through free-air" (hence the name). However, actual gravity is not readily downward continued because the gradient $\partial g/\partial h$ can vary quite substantially (typically 5-10%) from the normal value 0.3086 mgal/m in rough topography.

2.1. Geoid, quasigeoid and harmonic continuation

Because the gravity field quantities are viewed as functions in space, the use of the term *height anomaly* is preferable to *geoid heights*. In these notes the term *geoid height* will be used in a somewhat loose fashion, referring either to the geoid or the quasigeoid, especially when talking about terrain effects. However, when geoid heights are needed at the cm-level, the difference between the concepts becomes quite significant, and it is important to be aware of the difference.

Rigorously the term *geoid* is reserved to one particular equipotential surface of the earth - the one approximating the global mean sea level. The geoid can be imagined as the surface which the water would stand at inside a tunnel connected with the ocean. This surface will on the continents in general be located inside the terrain masses, and thus not be an equipotential surface of a harmonic function (because T is not harmonic inside the topography). The geoid height N obeys the same formula as the height anomaly

$$N = \frac{T}{\gamma} \tag{7}$$

(Bruns' formula), but T in this case is the anomalous potential value at the geoid, *inside* the mass.

The *height anomalies* may be viewed as "geoid heights in space". The same formula as (7) holds for height anomalies, but now T is the potential value at the surface or aloft. Height anomalies evaluated at the topographic surface are known as the *quasigeoid*. Bruns' formula for the quasigeoid thus reads

$$\zeta(\phi,\lambda) = \frac{T\,[\phi,\lambda,r_{topo}(\phi,\lambda)]}{\zeta} \tag{8}$$

When the purpose of a geoid determination is e.g. to replace levelling with GPS, it is important to use the right kind of "geoid". Dependent on the national height system in use (orthometric heights or normal heights), the ellipsoidal height of a GPS point is expressed as either

$$h_{ellipsoisdal} = H^{*} + \zeta \tag{9}$$

where H* is the normal height, or

$$h_{ellipsoidal} = H + N \tag{10}$$

where H is the classical orthometric height. Therefore a national "geoid" in a country using normal heights should be a quasi-geoid rather than a geoid.

The classical dilemma in determining the "physical" geoid height N is that a knowledge of the density of the topographic masses above the geoid is required. The same problem occurs in the definition of the orthometric heights (heights along the plumbline from the geoid to the surface point). The basic observable in geodetic levelling is the geopotential number C (i.e., the potential difference W_0-W) at a surface point P, with practical formulas for orthometric and normal heights

$$\begin{aligned} H_{Helmert} &= \frac{C}{\bar{g}} \approx \frac{C}{g_P + 0.0424[mgal/m]H} \\ H^{*} &= \frac{C}{\bar{\gamma}} \approx \frac{C}{\gamma_o - 0.1543[mgal/m]H^{*}} \end{aligned} \tag{11}$$

The Helmert orthometric heights build on an assumption of a constant density of 2.67 g/cm³, using Bouguer plates (the Prey reduction) to estimate the mean gravity value inside the earth. For a detailed discussion of this see Heiskanen and Moritz (1967, sec. 4).

The point to be stressed here is that both heights (11) involve the potential at the surface point P (not at the geoid). It is relatively simple and straightforward at the approximation level of (11) to convert between geoid and quasigeoid: For a point

P at the surface of the topography (11) results in the simple formula

$$\zeta - N = H_P - H_P^{*} \approx - \frac{g_P - \gamma_o + 0.1967[mgal/m]H}{\gamma_o} H = - \frac{\Delta g}{\gamma_o} \qquad (12)$$

where Δg_B is the Bouguer anomaly.

The conversion between geoid and quasigeoid is just a simple matter of applying a correction involving the Bouguer anomaly and the height. If gravity and heights are given on grids, this is straightforward, and - very important - if *Helmert* orthometric heights are used (what is nearly always done in practice), then the formula (12) may be considered as virtually exact. Trying to attempt a higher-order "downward continuation" of the quasigeoid is wrong in this case, because then corresponding higher order terms should also be applied to the computation of the mean gravity in the orthometric height formula (11).

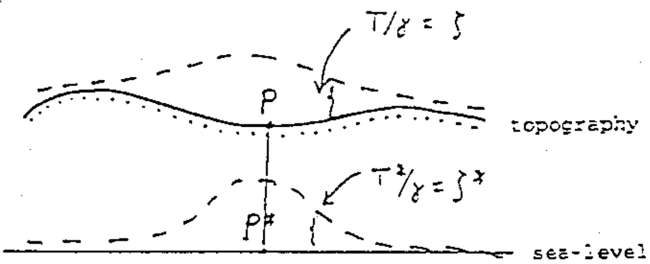

Fig. 2. Surface potential (T) and downward continued potential (T)*

In addition to the geoid height N and height anomaly ζ, another "geoid" term is also of relevance: the *height anomaly at sea level*, ζ^*. This quantity is directly related to the concept of *harmonic downward continuation*.

The harmonic downward continued anomalous potential T^* is a potential which fulfils $T^* = T$ outside all mass, but is harmonic ($\Delta T^* = 0$) overall down to some level, e.g. the geoid level or an internal level (a Bjerhammar sphere). The existence of the T^* solution inside the topography has been proven with the Runge-Krarup theorem, cf. Moritz (1980). The Runge-Krarup theorem says that it is always possible to find a T^*-potential, harmonic down to any Bjerhammar sphere inside the earth, which approximates the outer potential T arbitrarily well.

The height anomaly at sea level ζ^* is corresponding to T^*, and ζ^* (not N) will be the quantity obtained if e.g. least-squares collocation is rigorously spatially applied, and predictions are made at sea level. Downward continued and surface potential field values are related through the *Poisson integral*, which when written in planar approximation is of form

$$T_P = \frac{H_P}{2\pi} \iint_{-\infty}^{\infty} \frac{T^*}{(x^2+y^2+H_P^2)^{3/2}}\, dxdy \qquad (13)$$

where H is the height of the computation point (Heiskanen and Moritz, 1967, sec. 6).

To first order the relationship between ζ and ζ^* is simply obtained by noting that

$$\zeta - \zeta^* \approx \frac{\partial\zeta}{\partial h} H = -\frac{\Delta g}{\gamma_o} H \qquad (14)$$

where the Δg is the free-air anomaly (opposed to the Bouguer anomaly in (12)). It should thus be stressed that N and ζ^* are very different quantities. In typical mountainous areas with, say, 1 km changes in height and corresponding gravity changes in the 100 mgal range, the differences in N, ζ and ζ^* will be at the 10 cm level.

2.2. Density anomalies and conventional gravity terrain reductions

The Helmert condensation or Molodensky theory approach is one aspect of the role of topography on geoid determination, the second aspect is the direct potential of the topography itself. Here the concept of *density anomaly* is important.

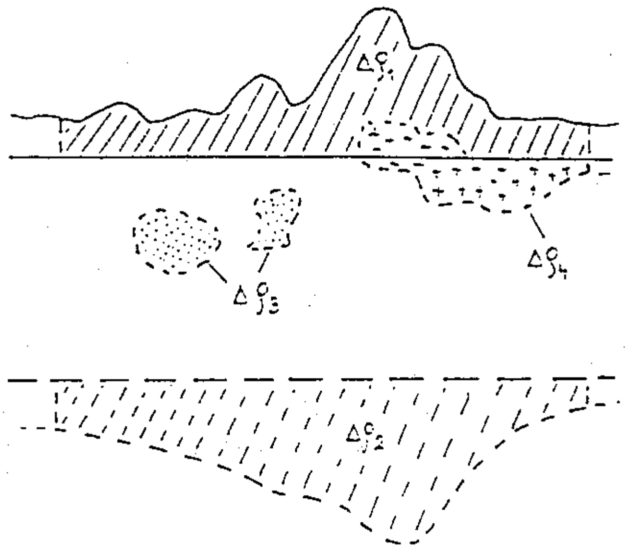

Fig. 3. Typical density anomalies ($\Delta\rho_1 \sim 2.67$, $\Delta\rho_2 \sim -0.4$, $\Delta\rho_{3-4} \sim \pm 0.1$-$0.3$)

We may view the anomalous potential both inside and outside the masses as being generated by an anomalous density distribution $\Delta\rho = \rho - \rho_{ref}$,

$$T(P) = G \iiint_{\Omega} \frac{\Delta\rho}{r_{PQ}} \, dV_Q \qquad (15)$$

where G is the gravitational constant, P the computation point (at, inside or outside the topography), and Ω the integration volume - in principle the whole earth.

The selection of a reference density function consistent with the ellipsoidal normal gravity field U is a complicated problem, but in *spherical approximation* (the approximation underlying (3)) the reference density distribution may be considered spherical, and in this case *any* reference density distribution with the correct total earth mass may be used - the potential of a spherical shell of constant density is equivalent to a point mass at the earth's center. It is therefore intuitively clear that the reference density distribution can be selected rather freely, and preferably in consistence with geophysical reality.

A schematic typical normal density model could be a crust of density ranging from 2.67 g/cm³ at the surface, smoothly increasing to 2.9 at the Moho interface around 30-35 km depth, with a density increase of around 0.4 across the Moho. With such a density model the major density anomalies would be the topography ($\Delta\rho = 2.67$), the ocean bathymetry ($\Delta\rho = -1.64$), and the isostatic compensation mass body ($\Delta\rho$

≈ -0.4). Geological mass bodies inside the crust would typically have much lower density anomalies (Δρ ≈ ±0.1-0.2, rarely larger).

In mountainous areas the topography itself produces a dominating signal in the anomalous gravity field, and the influence of the topography itself is removed by the *topographic* or *complete Bouguer* reduction

$$\Delta g_B = \Delta g - 2\pi G \rho H + c \tag{16}$$

where c is an auxillary integral - "the classical terrain correction", given by an integral over the irregularities of the topographic mass body relative to a Bouguer plate passing through the computation point P. In planar coordinates and for constant density

$$c_P = G\rho \int\int_{-\infty}^{\infty} \int_{z=H_P}^{z=H(x,y)} \frac{z-H_P}{[(x_Q-x_P)^2+(y_Q-y_P)^2+(z_Q-H_P)^2]^{3/2}} \, dx_Q dy_Q dz_Q \tag{17}$$

The classical terrain correction c is typically one order of magnitude smaller than the dominant term 2πGρh. Unfortunately, the terrain correction c also happens to be an approximation to the Molodensky term g_1, which often causes some confusion. Applying the terrain correction *alone* produces the socalled *Faye anomaly*, which is just as dependent on the local topographic height variations as the free-air anomaly itself (fig. 4).

The terrain correction c is an important quantity, and very dependent on the topography in the immediate surroundings of a gravity point. High-resolution digital terrain models are required for accurate computations in mountainous areas. Since the correction is always positive (a hill above the station level h_P and a valley below both yield the same sign of the contribution in (17)), insufficient resolution in the height models will yield terrain corrections which are systematically too low.

Another type of terrain-related density anomaly is the *isostatic compensation*. The isostatic compensation mass body is an idealized model for the earth lithosphere's general ability to provide a hydrostatic equilibirium of features of the crust. The Airy isostatic model assumes that all topography H is supported by a similar thicknening of a crustal "root" below the normal crust of thickness T = αH, where the constant α = ρ/Δρ is the root topography "magnification" factor.

The isostatic models serve as a useful model for providing smooth residual

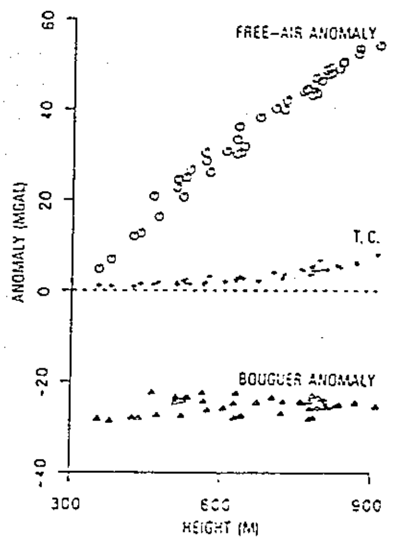

Fig. 4. Correlation of gravity with height in a local area in east Greenland

anomalies, but obviously the real earth does not follow such simple models. The strength of the earth's crust and dynamic forces may support topography without isostatic compensation (e.g. many islands), and the isostatic mass bodies might be deeper density anomalies in the upper mantle (e.g. mid-ocean ridges). However, in spite of the model oversimplifications, the practical applications of isostatic models do indeed produce the smoothest residual fields. For geoid modelling the Bouguer reduction must always be associated with some kind of isostatic or compensating reduction, because otherwise very large residual anomalies will occur. This is a because the Bouguer anomalies of the earth are systematically negative over the continents and positive over the oceans, a direct consequence of isostasy.

2.3. Residual density anomalies and RTM gravity anomalies

In practical geoid prediction schemes a global high-order spherical harmonic reference expansion is used as a reference field. Such a spherical harmonic expansion obviously includes the effects of the global topography. If a spherical reference field is used, i.e. gravity field modelling is applied on the residual

$$T' = T - T_{ref} \tag{18}$$

then the subtraction of a further topographic/isostatic effect may introduce long-wavelength effects into the residual potential. To avoid this a similar spherical harmonic expansion of global topographic/isostatic reference potential must be used, or - preferably - only the shorter wavelengths of the topographic/isostatic effect reference potential degree and order, then the topography will in principle be properly accounted for, but both too "short-wavelength" and "long-wavelength" reference surfaces may be used with advantage in practice. A short-wavelength surface is useful to minimize the magnitude of the RTM effects, and a more long-wavelength surface yields a better smoothing of the residual field. A mean height surface with resolution around 100 km will typically yield residual anomalies quite similar to isostatic anomalies in magnitude.

The topographic RTM density anomalies will make a "balanced set" of positive and negative density anomalies, representing areas where the topography is either above or below the reference topography. The effect of the RTM density anomalies will therefore in general cancel out in zones at larger distances from a computation point (say, e.g. a distance of 2-3 times the resolution of the mean height surface), which makes RTM reductions easy to work with in practice.

The RTM gravity terrain effect is in the planar approximation given by an integral of form

$$\Delta g_{RTM} = G\rho \int\int_{-\infty}^{\infty} \int_{z=h_{ref}(x,y)}^{z=h(x,y)} \frac{z-h_P}{[(x_Q-x_P)^2+(y_Q-y_P)^2+(z_Q-h_P)^2]^{3/2}} \, dx_Q dy_Q dz_Q \tag{19}$$

where h are the topographic heights, e.g. given by a digital terrain model. When the mean elevation surface is a sufficiently long-wavelength surface, the RTM reduction may be approximated by a Bouguer reduction to the reference level

$$\Delta g_{RTM} \approx 2\pi G\rho(h-h_{ref}) - c \tag{20}$$

This approximation again shows that the classical terrain correction is a key quantity, and it is therefore the basic "terrain" quantity, which should e.g. be stored in gravity data bases.

2.4. General remove-restore terrain reductions

For general gravity field modelling the terrain-reductions may be applied in a remove-restore fashion: Terrain effects are first removed from observations, then predictions are carried out, and finally the terrain effects are restored, symbolically written

$$L^c_{obs}(T) = L_{obs}(T) - L_{obs}(T_m)$$

$$\dots L_{obs} \rightarrow L_{pred} \dots \tag{21}$$

$$L_{pred}(T) = L_{pred}(T^c) + L_{pred}(T_m)$$

In the case of geoid computation from gravity, we would have the following observation and prediction functionals

$$L_{obs}(T) = L_{\Delta g}(T) = -\frac{\partial T}{\partial r} - \frac{2}{r}T$$

$$L_{pred}(T) = L_{\zeta}(T) = \frac{T}{\gamma} \tag{22}$$

The above scheme is only valid if the terrain reduction can be represented by a terrain potential T_m. This is the case for the complete Bouguer reduction, topographic/isostatic reduction and the RTM reduction, if 1) either a *fixed* area is taken into account (e.g. a square area between a given set of lat/lon limits), or 2) if - at least in principle - the corresponding density anomalies are taken into account globally. The terrain correction itself does *not* fit into this (the density anomalies associated with a c-computation are dependent on the actual computation point P), nor does terrain effects which are only computed inside a fixed spherical cap (e.g. the conventional computation of topographic/isostatic effects only out to a given radius, such as the Hayford 167-km zone).

Only RTM terrain effects may be computed in a spherical cap around the computation point, provided the cap is so big that the remote residual topography has a negligible effect, typically for a radius of computation 2-3 times the resolution of the mean height surface.

When the RTM reduction is used in a remove-restore fashion, special consideration is required for points under the reference topography level. If $h_P < h_{ref}$, and prisms are used to compute the residual density anomalies, i.e. "cutting away" mountains above the reference surface (density 2.67) and "filling" valleys below the surface (density -2.67), then the point P will end up *inside* the reference topography after the reduction, and T^c is *not* a harmonic function.

The non-harmonicity of the reduced potential below the reference height surface is a major theoretical problem with the RTM method when defined through a mean height surface. The problem may be circumvented through modifying the definition, representing the reference topography by a multipole masslayer expansion on the geoid, or it may be reformulated as a frequency domain method (Vermeer and Forsberg, 1992).

In the "conventional" RTM view a special correction is required after the terrain reduction computations, in order to change the value of a computed quantity $L(T^c)$ into the value $L(T^{c*})$, i.e. the value corresponding to the downward continued *outer* potential. Luckily this *harmonic correction* is quite straightforward if the reference topography is long-wavelength, so that the reference topography above the computation point P may be approximated by a Bouguer plate of density $-\rho$.

For downward continuation through a Bouguer plate, the harmonic correction for geoid and deflections of the vertical will be zero, whereas the harmonic correction for gravity will be

$$\Delta g_P^* - \Delta g_P = 4\pi G\rho(h_{ref} - h_P) \tag{23}$$

This follows from a gravity Prey reduction. If the approximation (20) is used to compute the RTM gravity anomaly the harmonic correction is automatically taken into account.

2.5. Direct or indirect use of terrain reductions

In the typical situation of geoid prediction from gravity anomalies using gridded data, the direct use of terrain reductions thus involves the following steps:

1. *Make terrain reduction* $\Delta g^c = \Delta g - \Delta g_m$
2. *Grid reduced gravity data* $\Delta g^c \rightarrow \Delta g^{grid}$
3. *Predict reduced geoid* $\zeta^c = S(\Delta g^c)$
4. *Restore terrain effects* $\zeta = \zeta^c + \zeta_m$

$S(\cdot)$ is the Stokes' operator, typically implemented by FFT methods. In addition to the above scheme a reference field like OSU91A would also be removed/restored. The advantage of the terrain remove/restore scheme is that the reduced gravity anomalies are smooth and with low variability, easy to grid, and further errors in the geoid computation step are mininized, especially the "Molodensky" errors due to the non-level surface the gravity observations refer to.

The above scheme is, however, in practice often modified to an "indirect" use of terrain reductions, so that terrain reductions are only applied in the gravity anomaly gridding process:

1. Terrain reduction $\Delta g^r = \Delta g - \Delta g_m$
2. Grid reduced gravity data $\Delta g^r \rightarrow \Delta g^{r,grid}$
3. Restore free-air anomalies $\Delta g^{grid} = \Delta g^{r,grid} + \Delta g_m^{grid}$
4. Predict final geoid $\zeta = S(\Delta g)$

In this scheme the terrain reduction used is nearly always the complete Bouguer reduction

$$\Delta g_m = -2\pi G \rho h_P - c_P \tag{24}$$

In the restore step (3.) only the simple Bouguer anomaly term ($2\pi G\rho h^{grid}$) is usually restored, resulting in a grid of *Faye* anomalies (Δg+c). This is done partly because it is difficult to compute average terrain corrections in a grid with sufficient accuracy, and partly because the Faye anomalies enter into the Helmert condensation approximation for the direct prediction of the geoid, cf. section 2.7.

The advantage of using the second "indirect" scheme is that large, national gravity grids can be directly utilized, and the use of terrain reductions is "hidden" in the generation of this grid. The drawback of the approach is that the full variablility of the gravity anomaliy field has to be handled in the geoid prediction, making the use of Molodensky corrections terms or high-order Helmert condensation terms more necessary.

2.6. Terrain reductions and Molodensky's theory

Molodensky's theory handles the problem of gravity anomalies referring to an non-level surface. It does not in any way remove the effects of topography, nor provide a smoothing of the gravity field allowing interpolation of sparse gravity measurements in mountainous areas. In the basic theory using harmonic continuation, the gravity field is found at a level surface by a sum of terms where

$$\Delta g^* = \sum_{n=0}^{\infty} g_n \,, \qquad g_n = -\sum_{r=1}^{n} z^r L^r(g_{n-r}), \; g_o = \Delta g \tag{25}$$

z is the height difference to the level surface, and L the "upward continuation" operator, applicable to any surface function (not just a harmonic function) through the Poisson integral

$$L(T) = \frac{\partial T}{\partial z} = -\frac{T}{r} + \frac{r^2}{2\pi}\iint_o \frac{T-T_P}{l_o^3} d\sigma \qquad (26)$$

At the level surface the usual Stokes operator may be applied to find the downward continued quasigeoid $\zeta^* = S(\Delta g^*)$, which is finally upward continued to the topographic surface by a similar sum

$$\zeta = \zeta^* + \sum_{n=1}^{\infty} \frac{1}{n!} z^n L^n \zeta^* \qquad (27)$$

For details see (Moritz, 1980) or Sideris (1987). To first order the Molodensky theory breaks down to a very simple scheme:

1. Predict vertical gravity gradient T_{zz} from Δg (e.g., by FFT)
2. Downward continue $\Delta g^* = \Delta g - T_{zz}h$
3. Apply Stokes operator $\zeta^* = S(\Delta g^*)$
4. Upward continue to surface by the vertical gradient of the height anomaly (i.e., the gravity anomaly)
$\zeta = \zeta^* + \Delta g^* z$

Obviously this scheme is much more stable if terrain-reduced quantities (Δg^c, T_{zz}^c etc.) are used. The whole Molodensky theory may in principle be applied to terrain-reduced data just as well as to the original free-air data, yielding much smaller Molodensky terms g_n^c. Molodensky theory and terrain reductions are therefore complementary, and both should in principle be applied for optimal results, cf. Forsberg and Sideris (1989).

The above first-order Molodensky scheme has been implemented in the GRAVSOFT FFT geoid prediction program GEOFOUR. In general the g_1^c is quite small when RTM anomalies are used, and may often be completely neglected. Without terrain reductions the g_1 integral becomes essentially an integral over the heights squared (similar to the terrain correction integral), and may often give large corrections (tens of mgal).

2.7. Helmert condensation

Helmert condensation is the "classical" solution to the problem of the non-level reference surface. In its conventional application it is a kind of mix of "Molodensky correction" and some elements of terrain reduction. The method is important because several major continental geoid models have been computed using the must be removed.

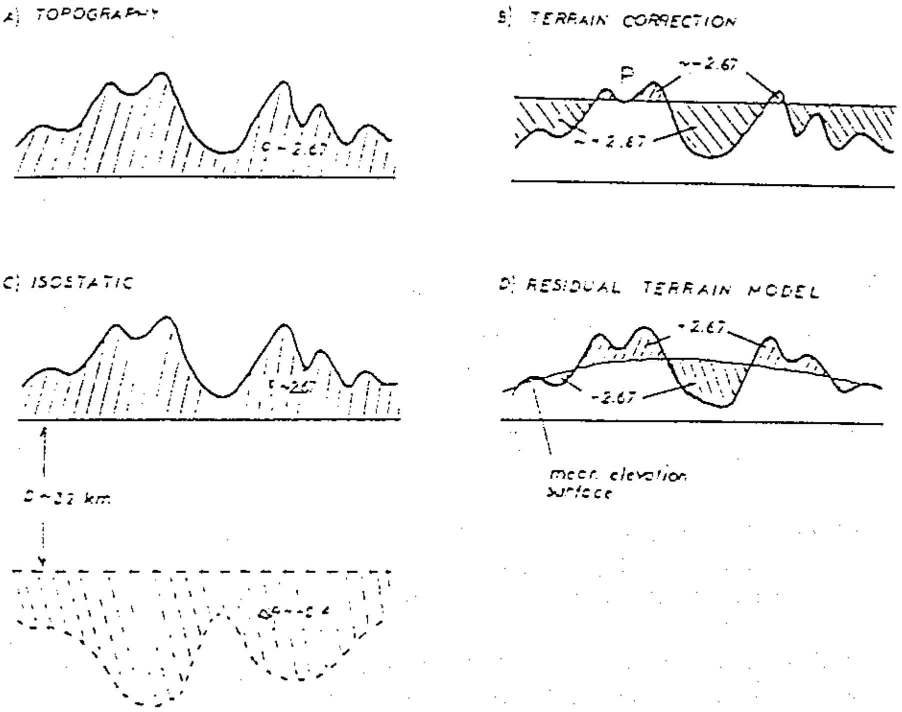

Fig. 5. Examples of density distributions associated with different terrain effects

This may in practice be done using a *residual terrain model (RTM)*. In the RTM model a smooth *mean elevation surface* $h_{ref}(\phi,\lambda)$ is used to define an implicit reference density model ρ_{ref} which has crustal density (e.g., 2.67 g/cm³) up to the reference level h_{ref}. The reference surface could e.g. be defined as a surface corresponding to T_{ref} (e.g. a spherical harmonic expansion of global topography to same degree and order), but could as well be defined "freely" through a suitable filtering of local terrain heights.

If the "resolution" of the mean surface is corresponding to the spherical harmonic method, e.g. in the US (Milberts' Geoid-90) and Canada (e.g. GSD-solutions by Verenneau and Mainville). In Europe the joint European geoid model (currently in prep. by Denker) is computed as a quasigeoid.

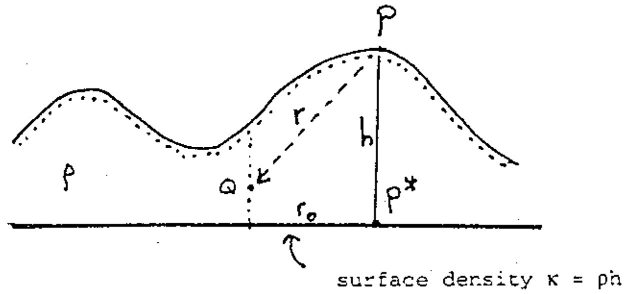

Fig. 6. Helmert condensation mass body

The Helmert condensation consists of shifting the topographic masses to a mass layer on the geoid of surface density $\kappa = \rho h$ (fig. 6). The "mass shift" is carried out with a downward continuation of the gravity anomalies in the following scheme:

1. *Remove complete Bouguer effect:* $\Delta g^B = \Delta g - (2\pi G\rho h - c)$
2. *Downward continue* $\Delta g^{B*} \approx \Delta g^B$ *(assume* T_{zz}^B *is zero)*
3. *Restore condensed topography* $\Delta g^* = \Delta g + 2\pi G\rho h$

The outcome of this process is the Faye anomaly (Δg +c), which is not smooth at all (the Faye
anomaly actually shows an even more perfect local correlation with height than the free-air anomalies), but it may still be viewed as the result of a terrain reduction of a density anomaly consisting of the topography (density ρ) and a masslayer on the geoid (density $-\kappa$), combined with a downward continuation assumption. This downward continuation assumption is reasonable because T_{zz}^B is much smoother than T_{zz} itself (there is currently in geodesy quite a debate on how exactly to interpret the Helmert reduction, and some theoretically inclined authors prefer not to apply any downward continuation assumption, making the method more theoretically correct, but less useful in practice).

In the above "classical" Helmert scheme the "terrain reduction" should be applied on the *full* gravity operator

$$\Delta g = -\frac{\partial T}{\partial r} - 2\frac{T}{r} \tag{28}$$

and not the just the gravity disturbance. So this leaves a contribution from the second term, called the *"indirect effect on gravity"*. To compute this small term (neglecting the downward continuation contribution) we need the Helmert potential terrain effect at sea level, i.e.

$$T_m(P^*) = G\rho \int_{-\infty}^{\infty}\int\int_{0}^{h} \frac{1}{r} dxdydz + G\rho h \int_{-\infty}^{\infty}\int \frac{1}{r_o}\, dxdy \approx -\pi G \rho h_P^2 \qquad (29)$$

$$r_o = \sqrt{(x-x_P)^2+(y-y_P)^2},\ r = \sqrt{r_o^2+z^2}$$

The approximative formula is obtained by a second-order expansion of r^{-1}. The "full" Helmert-condensed gravity anomaly at sea level Δg^{c*} may thus be converted to the quasigeoid at sea level ζ^{c*} by Stokes formula. By restoring the Helmert topography at sea-level the geoid undulation is now formally obtained (since the point P* after the restore correction will be located inside the mass), in other words

$$N = S(\Delta g + c - 2\frac{T_m}{\gamma r}) + \frac{T_m}{\gamma} \qquad (30)$$

The last term is often called the *"first order indirect effect"*. The Helmert condensation method thus gives the geoid directly, but the theory itself is approximative. Higher-order expansions can be carried out (see e.g. Sideris, 1990), but if refined expansions are used for the downward continuation as well, consistency is lost with the conventional Helmert orthometric heights. It is therefore (in my view) preferable as far as possible to work with height anomalies, and only at the end of a geoid computation shift back to formal geoid undulations. However, the Helmert condensation method is very simple and straightforward to apply, and will in most cases yield a sufficient accuracy.

2.8. Reference fields in rough topography

All of the above theory is in practice applied relative to a reference field expansion by spherical harmonics, e.g. OSU91A

$$T_{ref}(r,\phi,\lambda) = \frac{GM}{r} \sum_{n=2}^{n_{max}} \sum_{m=0}^{n} (\frac{R}{r})^n \, [C_{nm}\cos m\lambda + S_{nm}\sin m\lambda] \, P_{nm}(\sin\phi) \qquad (31)$$

Such an expansion is by its nature a *function in space*, and reference gravity at a point P should be evaluated at the correct elevation $r = R+h_P$. In practice reference effects are often computed in grids at a constant elevation. A 3-dimensional interpolation should therefore in principle be carried between at least reference two grids computed at different elevations (a "sandwich grid" interpolation). However, in many cases the effects of the removing a "sandwich" grid and restoring it in the final predictions is quite small, and can be neglected.

3. THE PRACTICAL COMPUTATION OF TERRAIN EFFECTS

Historically terrain corrections or -effects were computed using overlays on maps, subdivided in concentric circles and radial sectors. In each sector mean elevations were read from maps, and terrain corrections summed up using either tables or simple calculations based on the closed gravitational formulas for a cylindar, cf. Heiskanen and Moritz (sec. 3-1). The cylindrical overhead systems had names like "Hammer zones" or "Hayford zones". The Hammer system, used widely in geophysical prospecting, computed terrain corrections in a number of rings ranging in radius from zone B (2-16 m, 4 sectors) to zone K (15-22 km, 16 sectors). The Hayford system, used for regional work, had a similar system, in principle extending globally, but in practice extended only to zone "0_2" (167 km).

The Hayford terrain correction zone system often included a special correction - Bullard's term - taking into account the spherical earth, making a correction for a "spherical cap" rather than the usual planar Bouguer plate. It is very important when using "spherical" terrain-corrected Bouguer or isostatic anomalies, to be well-aware what has been done, e.g. to which distance have computations been carried out. Basically the full global "spherical" terrain correction is a quantity which is not usable at all: it is way too big, and stongly dependent on the *distance* out to which it is computed.

An example for illustration: consider a gravity point e.g. at the top of a large flat plateaux, with a negligible "planar" terrain correction (the edge of the plateaux is assumed to be far away), and no other topography on the earth. The complete topographic effect will in this case be $-2\pi G\rho h$. The spherical Bouguer correction corresponding to a spherical shell of thickness h is $-4\pi G\rho h$, so the global spherical terrain correction must be $2\pi G\rho h$.

The spherical "problems" can be completely avoided by using RTM-reductions, or by direct spherical computation of the complete topographic/isostatic attraction using e.g. prisms. The integration of terrain effects by prisms is nowadays the method of choice for space-domain computations, augmented by the much faster, but sometimes more approximative, FFT methods.

3.1 Terrain effect integration by prisms

The rectangular prism of constant density is a useful "building block" for numerical integrations of the basic terrain effects of form e.g. (17) or (19). Closed formulas for the gravitational potential and all derivatives exist, see e.g. Forsberg (1984), but the formulas are quite complicated. For a point P at the origin of the coordinate system, the gravity disturbance of a prism of density ρ in coordinate interval x_1-x_2, y_1-y_2, z_1-z_2 will be

$$\delta g_m = G\rho \left| x \log(y+r) + y \log(x+r) - z \arctan\frac{xy}{zr} \right|_{x_2}^{x_1}\Big|_{y_2}^{y_1}\Big|_{z_2}^{z_1}, \qquad r = \sqrt{x^2-y^2+z^2} \tag{32}$$

The corresponding formula for the potential (and hence the geoid height) is even more complicated

$$T_m = G\rho \Big| xy \log(z+r) + xz \log(y-r) + yz \log(x+r) - \frac{x^2}{2}\arctan\frac{yz}{xr} - \frac{y^2}{2}\arctan\frac{xz}{yr} - \frac{z^2}{2}\arctan\frac{xy}{zr} \Big|_{x_2}^{x_1}\Big|_{y_2}^{y_1}\Big|_{z_2}^{z_1} \tag{33}$$

It should be pointed out that in principle the computation of both (32) and (33) is required in order to rigorously compute the gravity anomaly, given by the functional

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r}T \tag{34}$$

When computing topographic/isostatic effects the second term in (34) is called the *indirect effect on gravity* (again), and is often explained using the concept of gravity reduction from the geoid to the "cogeoid" (the cogeoid is corresponding to the terrain-reduced potential $T-T_m$). The "indirect" term may often be neglected in connection with RTM-reductions, where the associated geoid effects T_m are small (typically less than 1 m, corresponding to a 0.3 mgal "indirect effect").

The gravitational formulas for the rectangular prisms are slow and numerically unstable at large distances (they involve a number of small differences between large numbers, corresponding to the corners of the prisms). Therefore approximative formulas are useful at larger distances. Such formulas are based on an expansion of the prism field in spherical harmonics, which gives a surprisingly simple expansion of form (McMillan, 1958)

$$T_m = G\rho \; \Delta x \Delta y \Delta z \; [\frac{1}{r} + \frac{1}{24r^5}[(2\Delta x^2-\Delta y^2-\Delta z^2)x^2 + (-\Delta x^2+2\Delta$$

$$+(-\Delta x^2-\Delta y^2+2\Delta z^2)z^2] + \frac{1}{288r^9}[\alpha x^4+\beta y^4+....] + ...],$$

$$\Delta x=x_2-x_1, \Delta y=y_2-y_1, \Delta z=z_2-z_1 \qquad (35)$$

This equation is easily differentiated for other gravity field quatitities. In the GRAVSOFT "TC"-program, such approximative formulas are automatically used in the prism integration when accuracy permits to obtain reasonable computation speeds. Approximative formulas are also used for the prism potential itself (corresponding to a mass-plane).

To further increase computation speed, and to allow use of less detailed, remote topography, it is practical to use a coarse/detailed grid system. This is fully implemented in TC. The detailed grid is used out to a minimum distance, and the coarse grid is used for the remainder of the topography. In a small innerzone of 3 x 3 grid points just around the computation point the topographic data are densified using a bicubic spline interpolation, so that a "finer" more smooth set of prisms is used to integrate the often large effects of the innerzone (fig. 7). This densification of the innerzone is essential to avoid a computation P being located at the edge of a prism, giving rise to artificial terrain effects from the "edges" of a prism.
Since gravity terrain effects are strongly dependent on the height of the computation point (through the $2\pi G\rho h$-term), a special precaution is necessary when the height of the computation point does not agree with the interpolated height from the DTM. Either the computation point can be "forced" to match the interpolated topography level, or the topography can be modified locally to give the "right" value at P. The modificiation in the TC program is done using a "smooth" correction in the innerzone (fig 8). The discrepancy between DTM and station heights will always

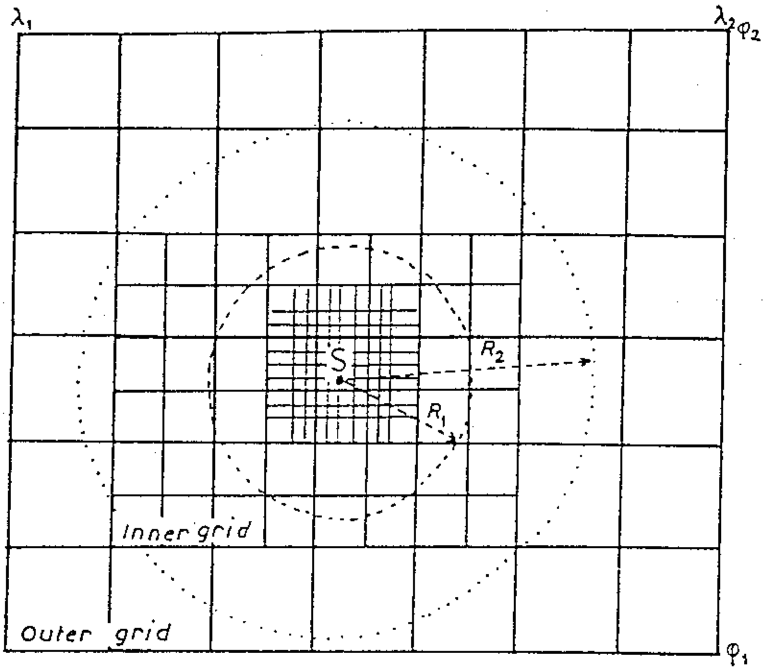

Fig. 7. Use of innerzone, detailed and coarse height grids in "TC"

be present, since the DTM's will hardly ever have sufficient resolution to represent all features in rugged topography.

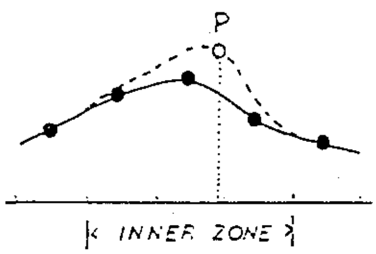

Fig. 8. Modification of terrain heights in innerzone

In the TC program the terrain grids to be used are assumed to be given as regular grids in either geographical coordinates or a UTM-zone. Grids are stored rowwise from north to south, and an additional grid must be provided by the user for the smooth mean height surface if RTM anomalies are to be computed. Mean height grids and smoothed reference surface grids (using a moving-average filter) may be computed by an auxillary program "TCGRID".

3.2. Terrain effects by Fourier transformation methods

Since digital heights or bathymetry is typically given in regular grids, Forurier transform methods is an obvious choice for speeding up computations of large sets of terrain effects, especially if a complete grid of terrain effects is required. The basis of most Fourier methods can be formulated in terms of *convolutions*, which can be evaluated very fast by Fast Fourier Transform methods.

A two-dimensional convolution integral is an integral over the x-y plane of form

$$g(x,y) = \int\int_{-\infty}^{\infty} f(x-x',\ y-y')\ h(x',y')\ dx'dy' = f * h(x,y) \qquad (36)$$

In terms of signal analysis terminology f is the input function, h the convolution kernel, and g the output. Using the two-dimensional Fourier transform

$$\mathscr{F}(f) = F(k_x,k_y) = \int\int_{-\infty}^{\infty} f(x,y) e^{-i(k_x x + k_y y)} dx dy$$

$$\mathscr{F}^{-1}(F) = f(x,y) = \frac{1}{4\pi^2} \int\int_{-\infty}^{\infty} F(k_x,k_y) e^{i(k_x x + k_y y)} dk_x dk_y \qquad (37)$$

the convolution f*h is readily obtained by the convolution theorem by a multiplication in the spectral domain

$$f * h = \mathscr{F}^{-1}[F \cdot G] \qquad (38)$$

The convolution is in practice carried on gridded data over a final domain using FFT. Since FFT assumes the function grids to be periodic in x and y, the grids must be extended by a border zone of zeros (zero padding), or results discarded close to the grid edges. These topics are treated in more detail in Schwarz et al.(1990).

Most terrain effects are characterized by being unlinear integrals, so series expansions are typically required to make terrain computations by FFT. However, in many cases just one or two terms are sufficient. In the sequel I will illustrate how convolutions are obtained in a few examples.

Terrain corrections.

In the planar terrain correction integral

$$c_P = G\rho \int_{-\infty}^{\infty}\int\int_{h_P}^{h} \frac{z-h_P}{r^3} dxdydz, \quad r = \sqrt{(x_P-x)^2+(y_P-y)^2+(h_p-z)^2} \tag{39}$$

the unlinear kernel may for small surface slopes be approximated by

$$\frac{1}{r^3} \approx \frac{1}{r_o^3}, \quad r_o = \sqrt{(x_P-x)^2+(y_P-y)^2} \tag{40}$$

This approximation is termed the *linear approximation* to the boundary value problem. In this approximation the z-integration of the integral yields

$$c_P = \frac{1}{2}G\rho \int_{-\infty}^{\infty}\int \frac{(h-h_P)^2}{r_o^3} dxdy \tag{41}$$

This formula is not a convolution, but by writing the integrand in terms, a set of convolutions in h and h^2 is obtained

$$c_P = \frac{1}{2}G\rho[h^2*f - 2h_P(h*f) - h_P^2 f_o], \; f = (x^2+y^2)^{-\frac{3}{2}} \tag{42}$$

In this formula f_o is a singular integral

$$f_o = \int_{-\infty}^{\infty}\int \frac{1}{r_o^3} dxdy \tag{43}$$

which, when evaluated with FFT over a finite domain, turns into a constant, readily obtained from the 0 (DC) value of the Fourier transform of the kernel. Similarly the singularity of r_o^{-3} at zero distance presents no problem in practice when discrete data are used, as the expression (42) will be independent of the actual value at zero of the kernel. For more details see Sideris (1985) or Forsberg (1985).

The computation of terrain corrections by FFT is of widespread use, and may be refined with higher order terms, see e.g. Harrison and Dickinson (1989). Based on terrain corrections RTM anomalies may be obtained by (20).

Isostatic, bathymetric or airborne terrain effects.

When gravity anomalies are wanted at a constant level above some topography or

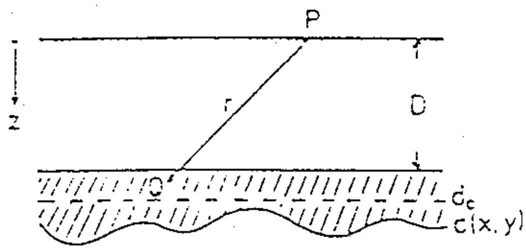

Fig. 9

bathymetry, a convolution series expansion of Fourier transforms is obtained, equivalent to the frequency domain Parker's formula. The general situation illustrated in fig. 9 can be used to compute isostatic effects, the effect of bathymetry on marine gravity, or - by changing the sign of d - to obtain gravity terrain effects in airborne gravimetry.

For the isostatic case, the gravity effect of the compensation mass body at sea level will be

$$g_m = G\Delta\rho \int_{-\infty}^{\infty}\int \int_{D}^{D+d} \frac{z}{r^3} dxdydz \qquad (44)$$

The kernel may be evaluated in a power series around a suitable reference level d_o as

$$\frac{z}{r^3} = \frac{d_o}{r_o^3} + \frac{r_o - 3d_o}{r_o^5}(z-d_o) + \ldots \ , \ r_o = \sqrt{(x-x_P)^2+(y-y_P)^2+d_o^2} \qquad (45)$$

and the integration be carried out with respect to z, yielding to second order a set of convolutions in d and d^2

$$g_m = G\Delta\rho[d*f_1 + d^2*f_2]$$

$$f_1 = \frac{d_o}{r_o^3} + \frac{r_o^2-3d_o^2}{r_o^5}(D-d_o), \quad f_2 = \frac{r_o^2-3d_o^2}{2r_o^5} \tag{46}$$

A similar set of expansions may be obtained for geoid heights, using an expansion of the "geoid" kernel

$$\frac{1}{r} \approx \frac{1}{r_o} - \frac{d_o}{r_o^3}(z-d_o) \tag{47}$$

yielding to second order

$$\zeta_m = \frac{G\Delta\rho}{\gamma}[d*t_1 + d^2*t_2],$$

$$t_1 = \frac{1}{r_o} - \frac{d_o(D-d_o)}{r_o^3}, \quad t_2 = -\frac{d_o}{2r_o^3} \tag{48}$$

The above second-order formulas are efficiently implemented in FFT programs by obtaining simultanously the transforms of d and d^2 in one complex transform.

RTM geoid effects.

The above formulas for the geoid effects do not apply to RTM geoid effects evaluated at the surface of the topography. However, since RTM geoid effects are quite small, a simple condensation approximation will in most cases be sufficient, i.e.

$$\zeta_m = \frac{G\rho}{\gamma}\int\int_{-\infty}^{\infty}\int_{h_{ref}}^{h}\frac{1}{r}dxdydz \approx \frac{G\rho}{\gamma}(h-h_{ref})*\frac{1}{r_o} \tag{49}$$

Such geoid effects are often computed over large regions (opposed to gravity terrain corrections typically computed in local blocks), and spherical FFT methods may with advantage be used.

In the GRAVSOFT system the FFT terrain effect computations have been implemented in the planar "TCFOUR" program, or - to a less complete degree - in the spherical FFT program SPFOUR.

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