1210

Proceedings of the Interdisciplinary Inversion Workshop 1, Aarhus 1992,
editor: B.H. Jacobsen, GeoSkrifter 41, 5-8.

Analytical and Discrete Inversion Applied to Gravity Data

C.C. Tscherning
Geophysical Institute, University of Copenhagen
Haraldsgade 6, DK-2200 Copenhagen N, Denmark

1. Introduction.

Let the Earth's density distribution ρ, be split into a normal part ρ_o and a disturbing part $\delta\rho$, so that $\rho = \rho_o + \delta\rho$. The potential of the disturbing part is the anomalous potential, T, with associated disturbing gravity δg or gravity anomaly Δg. The process of finding the anomalous density distribution from data such as gravity anomalies is called gravity inversion. In its most general setting data may be values of any linear functional applied on T. It is well known, that the inversion problem is non-unique. A large class of "density" functions has zero potential, see Sanso' et al., 1986. There are at least two ways to achieve uniqueness. We restrict a-priori the solution space, so that uniqueness is obtained, or we select a solution between many possible requiring a certain minimum property to be fulfilled. In the following recent results in these two directions will be reviewed.

2. Analytic methods for inversion.

The idea used in analytic inversion comes from the theory of boundary value problems for elliptic partial differential equations. Here we know that if the differential equation, the boundary data and the boundary is reasonably regular, then there is a unique relationship between the boundary data and the solution. A good example is the disturbing potential T itself, which is harmonic (fulfills the Laplace equation), and which is uniquely determined from gravity data on the Earths surface.

A space of solutions to an elliptic partial differential equation with support inside a closed surface like the Earth will have a countable number of base functions. It will when equipped with an appropriate inner product be a so-called Hilbert space with reproducing kernel. Here we may easily solve the last part of the inversion problem, namely the construction of a solution from a finite number of observations. The collocation method (Moritz, 1980) will here give and answer, and also provide error estimates.

The problem of analytic inversion is then which operator to chose ? Here it would be nice, if we could use an operator so that there were a direct connection between the Fourier coefficients of the gravity anomaly regarded as a function on a sphere, and the Fourier coefficients of the anomalous density expanded in series inside the same sphere.

This is achieved if we require that the anomalous density fulfills a kind of radially weighted Laplace equation, in which case we call the function quasi-harmonic,

$$\Delta\ (P(r)\ *\ \delta\rho)\ =\ 0,$$

where P(r) is a polynomial in the radial distance r (with no zero-points between 0 and the Earths surface) and Δ is the Laplace operator. In this case there is a simple linear relationship between the coefficients of the spherical harmonic expansion of T (or Δg) and the expansion in internal spherical harmonic functions divided by P(r), see Tscherning & Suenkel, 1981.

The analytic inversion method was tried with various quasi-harmonic solution spaces with rather disappointing results, see Hein et al. 1989. The reason was that the solution space was too restrictive. Nearly all mass was put close to the surface of the earth, and the use of various polynomials P(r) did not give essentially different results. In the future it may be of interest to investigate less restrictive solution spaces such as these formed by solutions to the bi-harmonic equation.

3. Discrete inversion methods.

In the gravity inversion problem there are two kinds of non-uniqueness: the one which has a physical origin (Newtons law), and the one due to the availability of a finite amount of data. The discrete methods solves both problems simultaneously. First the geological structure which is supposed to have created the anomalous gravity is discretized, i.e. divided in a finite number of blocks with constant density. The location and size of these blocks may be changed step by step in order to achieve a suitable solution. Here I will only consider a single step.

The relation between the discrete geological model and the data vector **y** with m elements is written as

$$\mathbf{y}\ =\ \mathbf{A}\ \mathbf{x},$$

where the elements x_i of the n-vector **x** are the densities of the i'th block. **A** is a n*m matrix with elements A_{ij} equal to the gravity at the j'th measurement point caused by the i'th block with density equal to 1. The dimension of **x** may be larger than, equal to or less than the dimension of **y**. Solutions are obtained in the various situations by minimizing the square-sum of the difference **y-Ax** combined with the requirement that **x** has a minimum norm.

The various situations, $n > m$, $n = m$, $n < m$ are closely related, and may all be seen as equivalent to the case $n = m$, where either linear combinations of the data or of the model elements are used. Sometimes the problem is even reduced to a situation where the number of parameters which are determined is less than both n and m. (Singular value decomposition, where the smallest eigenvectors are removed).

4. Introduction of statistical information.

When constructing the solution, it would be optimal, if it would have a smoothness like empirically observed. For an analytic solution, this may be achieved by using an inner product, so that reproducing kernel resembles the co-called auto-covariance function, see Moritz (1980). In the most primitive discrete case, where the norm of **x** is minimalized, for example, then this is equivalent to the introduction of a hypothesis where the elements of the geological model are considered uncorrelated if they do not overlap. This is naturally an erroneous hypothesis, except if the model spans two different geological provinces.

One of the reasons for having the possibility of introducing correlations in the analytic model is that the base functions have overlapping support. In Tscherning (1991) it was therefore tried to construct simple case functions with overlapping support. This kind of construction requires that the basefunctions subsequently are orthonormalized, which is a rather large computational effort. Simple rules were obtained for the relationship between block overlap and covariance function shape parameters. However, this line of thought has not been followed up due to lack of time.

Another method to obtain correlations is to use minimalization of norms derived from inner products which operates on several blocks. Such norms minimalize e.g. the second order difference between the densities of neighboring blocks, see Knudsen, 1991 and Knudsen & Strykowski, 1991. This kind of norm gives a geologically reasonable method of weighting the blocks, and very satisfactory results have been obtained.

5. Conclusion.

The theory of gravity inversion has still many unsolved problems: Which base functions, which norms are really the best to use. How do we include information from seismics, magnetics and geological structural analysis ? The use of an analytic method, where solutions to a partial differential equation are used, will give us solutions which fulfill certain minimum energy principles. Is it possible to find a "geological minimum energy principle" and thereby obtain an optimal setting for an analytic inversion procedure ?

Also, basic mathematical concepts like inner products are not easily introduced in a consistent manner, so that we are sure that the equations we try to solve really are solvable. Also we need besides the density error estimates. How do we obtain reliable error estimates. From simulations ? Or from a good theory ? The best would be to have a test area like the one used in Strykowski (1989) from the North Sea, but where the data quality should be much better.

References.

Hein,G., F.Sanso', G.Strykowsky and C.C.Tscherning: On the Choice of Norm and Base Functions for the Solution of the Inverse Gravimetric Problem. Ricerche di Geodesia Topografia Fotogrammetria, Vol. 5, pp. 121-138, CLUP, Milano, 1989.

Knudsen, P.: A smoothness constraint for the representation of correlations in a discrete earth model. Proc. 9. Int. Math. Geophys. Seminar on Model Optimization in Exploration Geophysics, Berlin, Feb. 1991.

Knudsen,P. and G.Strykowski: 3D inversion of gravimetric data in the Horn Graben area, Danish North sea. Proc. 9. International Math. Geophys. Seminar on Model Optimization in Exploration Geophysics, Berlin, Feb. 1991.

Moritz, H.: Advanced Physical Geodesy. H.Wichmann Verlag, Karlsruhe, 1980.

Sanso',F., R.Barzaghi and C.C.Tscherning: Choice of Norm for the Density Distribution of the Earth. Geoph.Jour.Royal Astr. Soc., Vol. 87, pp. 123-141, 1986.

Strykowski, G.: A study of statistical properties of the Earth's Density Distribution based on Density Logs from the North sea. Proc. II Hotine-Marussi Symp. on Mathematical Geodesy, Pisa, June, 1989, pp. 377-420, Milano, 1989.

Tscherning, C.C.: Some Simpel Methods for the Unique Assignment of a Density Distribution to a Harmonic Function. Reports of the Department of Geodetic Science No. 213, The Ohio State University, Columbus, Ohio, 1974.

Tscherning, C.C.: Density-Gravity Covariance Functions Produced by Overlapping Rectangular Blocks of Constant Density. Geophysical Journal International, Vol. 105, pp. 771-776, 1991.

Tscherning, C.C. and G.Strykowski: Quasi-harmonic inversion of gravity field data. In: A.Vogel (Ed.), Model Optimization in Exploration Geophysics, Vol. 2, pp. 137-154, 1987.

Tscherning, C.C. and H. Suenkel: A Method for the Construction of Spheroidal Mass Distributions consistent with the harmonic Part of the Earth's Gravity Potential. Manuscripta Geodaetica, Vol. 6, pp. 131-156, 1981.