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# Computation of covariances of derivatives of the anomalous gravity potential in a rotated reference frame

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**Abstract:** The computation of covariances of derivatives of the anomalous gravity potential in an arbitrary rotated reference frame requires that all covariances of derivatives of a specific order are available simultaneously. This has required a change of the earlier used computational strategy, where "horizontal" and "vertical" derivatives were computed separately.
General formulae for covariances of up to second order derivatives are derived, totally 85 expressions. It is described how the computational effort has been reduced by only computing some of the derivatives with respect to the radial distance and obtaining the remaining as linear combinations of these. The algorithms have been tested taking advantage of the pointwise harmonicity of the covariance functions.

## 1. Introduction.

The computation of covariances of quantities related to the anomalous gravity field has been discussed in many papers and technical reports (Tscherning and Rapp, 1974), (Tscherning, 1972, 1976, 1976a, 1983), (Krarup and Tscherning, 1984), (Milbert, 1988). There was always used a computational strategy, where horizontal derivatives were treated seperately from vertical derivatives. This was justified, because the observation functionals had the same characteristics, i.e. deflections of the vertical and torsion balance observations were horizontal derivatives and gravity was a vertical derivative. Today, however, we expect general measurements of linear combinations of derivatives of the gravity potential in space to be performed in the future. We may have observational quantities, which are the gravity vector components in the direction between two satellites in quite different orbits, for example.
It has therefore been necessary to revise the broadly used subroutine COVAX (Tscherning, 1976), so that it returns a matrix of covariances of all quantities of a given order. When all these quantities are available (3 x 3 x 3 x 3 = 81 for second order derivatives), then it is very simple to perform the necessary multiplications with the rotation matrix which defines the relationship between the usual local geographical system (east, north, up) and the given system.
In the following we will write down the many necessary equations, and we will then finally describe how the changes have been implemented and tested.

## 2. General formula for the covariances.

We work in a local geographical system, which in spherical approximation corresponds to a system with the radius vector being the third axis. Here the first axis points east and the second north. (Note that the equations will be the same if we use spherical coordinates without spherical approximation).

$$\frac{\partial}{\partial x_1} = \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda} \tag{1}$$

$$\frac{\partial}{\partial x_2} = \frac{1}{r}\frac{\partial}{\partial\phi} \tag{2}$$

$$\frac{\partial}{\partial x_3} = \frac{\partial}{\partial r} \tag{3}$$

where $\phi$ is the latitude, $\lambda$ the longitude and r the radial distance.

For the second order derivatives we have (cf. Tscherning, 1976b):

$$\frac{\partial^2}{\partial x_1 \partial x_1} = \frac{1}{r}\frac{\partial}{\partial r} - \frac{\tan\phi}{r^2}\frac{\partial}{\partial \phi} + \frac{1}{r^2\cos^2\phi}\frac{\partial^2}{\partial \lambda^2} \tag{4}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} = \frac{1}{r^2}[\frac{\partial}{\partial \phi}(\frac{1}{\cos\phi \cdot}\frac{\partial}{\partial \lambda})] \tag{5}$$

$$\frac{\partial^2}{\partial x_1 \partial x_3} = \frac{\partial}{\partial r}(\frac{1}{r\cos\phi}\frac{\partial}{\partial \lambda}) = \frac{1}{r\cos\phi}[\frac{\partial^2}{\partial \lambda \partial r} - \frac{1}{r}\frac{\partial}{\partial \lambda}] \tag{6}$$

$$\frac{\partial^2}{\partial x_2 \partial x_1} = \frac{\partial^2}{\partial x_1 \partial x_2}$$

$$\frac{\partial^2}{\partial x_2 \partial x_2} = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \tag{7}$$

$$\frac{\partial^2}{\partial x_2 \partial x_3} = \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial \phi}) = \frac{1}{r}[\frac{\partial^2}{\partial \phi \partial r} - \frac{1}{r}\frac{\partial}{\partial \phi}] \tag{8}$$

$$\frac{\partial^2}{\partial x_3 \partial x_1} = \frac{\partial^2}{\partial x_1 \partial x_3}, \quad \frac{\partial^2}{\partial x_3 \partial x_2} = \frac{\partial^2}{\partial x_2 \partial x_3}, \quad \frac{\partial^2}{\partial x_3 \partial x_3} = \frac{\partial^2}{\partial r^2} \tag{9}$$

The basic covariance function, C, is a function of two points in space P, Q with coordinates $\phi$, $\lambda$, r and $\phi'$, $\lambda'$, r', respectively. Local coordinates will be denoted $x_i$ and $y_i$ for P, Q, respectively. C may be regarded as a function of only 3 variables, the spherical distance $\psi$ between P and Q, and the radial distances r, r', see section 3.

Instead of $\psi$ we will use the variable $t = \cos\psi$. Putting: $cp = \cos\phi$, $sp = \sin\phi$, $cq = \cos\phi'$, $sq = \sin\phi'$, $cd = \cos(\lambda'-\lambda)$ and $sd = \sin(\lambda'-\lambda)$ we have:

$$t = sp \cdot sq + cp \cdot cq \cdot cd.$$

Since we will have to write down many equations we will introduce a compact notation:

$$t_{i0} = r \cdot \frac{\partial t}{\partial x_i}, \quad t_{0j} = r'\frac{\partial t}{\partial y_j}, \quad t_{ij} = rr'\frac{\partial^2 t}{\partial x_i \partial y_j}$$

This notation has the advantage that if a certain covariance expression has been obtained, we get the term $t_{ij}$ in the expression with P, Q interchanged by simply interchanging the subscripts.

When P and Q are interchanged, then r and r' are interchanged. They occur in a symmetric manner in the basic covariance expression, but differentiation produces factors 1/r and 1/r'.

Expression $C_{ij}$, where i is the order of differentiation with respect to t and j is the order of differentiation with respect to r and r' are defined so that they are symmetric in the latter variables as follows.

For the radial derivatives of first order in either P or Q we use:

$$r\frac{\partial^i}{\partial t^i}\frac{\partial C}{\partial r} = r'\frac{\partial^i}{\partial t^i}\frac{\partial C}{\partial r'} = C_{i1} \tag{10}$$

Since quantities involving first derivatives in both P and Q are different from quantities involving second order derivatives in either P or Q we use in this special case the superscript 1:

$$rr'\frac{\partial^2 C}{\partial r \partial r'} = C^1_{02} \tag{11}$$

When dealing with higher order derivaties in either P or Q we will use ($i \geq 0$)

$$r^2\frac{\partial^i}{\partial t^i}\frac{\partial^2 C}{\partial r^2} = (r')^2\frac{\partial^i}{\partial t^i}\frac{\partial^2 C}{\partial r'^2} = C_{i2} \tag{12}$$

$$r^2 r'\frac{\partial^i}{\partial t^i}\frac{\partial^3 C}{\partial r^2 \partial r'} = (r')^2 r\frac{\partial^i}{\partial t^i}\frac{\partial^3 C}{\partial r \partial r'^2} = C_{i3} \tag{12a}$$

and finally

$$(rr')^2 \frac{\partial^4 C}{\partial r^2 \partial r'^2} = C_{04} \tag{13}$$

Then we also need:

$$r'r \frac{\partial^2}{\partial r \partial r'} \frac{\partial^i}{\partial t^i} C = rr' [C_{i2} + C_{i1}] \tag{14}$$

For the general expression we need 4 subscripts:

$$C_{ijnm} = cov\left(\frac{\partial^2 T}{\partial x_i \partial x_j}, \frac{\partial^2 T}{\partial y_n \partial y_m}\right) \quad , \tag{15}$$

where the subscripts i, j are exchanged for n, m if P and Q are exchanged. Due to the identitiy of the mixed second order derivatives we also have $C_{ijnm} = C_{jinm} = C_{ijmn} = C_{jimn}$. We use the same shorthand expression for zero or first order derivatives, so that the subscripts j and/or m are zero if we have a first order derivative in P and/or Q. The subscripts i, n will be zero if we have no derivative in P and/or Q, i.e. $C_{0000} = C$. Here we also have $C_{ionm} = C_{iomn}$, $C_{ijno} = C_{jino}$, $C_{00nm} = C_{00mn}$.

For the covariances between T(P) and the first order derivatives in Q we then have with i < 3:

$$C_{ooio} = \frac{\partial}{\partial y_i} C = t_{0i} C_{10}/r' \tag{16}$$

$$C_{0030} = \frac{\partial}{\partial y_3} C = C_{01}/r' \tag{17}$$

and then for the derivative in P,

$$C_{i000} = t_{i0} C_{10}/r \tag{18}$$

$$C_{3000} = C_{01}/r \tag{19}$$

With one derivative in both P and Q, and observing the convention eq. (11) we get (i, j < 3):

$$C_{i0j0} = (t_{ij} C_{10} + t_{i0} t_{0j} C_{20})/(rr') \tag{20}$$

$$C_{30j0} = t_{0j} C_{11}/(r'r) \tag{21}$$

$$C_{i030} = t_{i0} C_{11}/(rr') \tag{22}$$

$$C_{3030} = C^1_{02}/(rr') \tag{23}$$

For second order derivatives a few preparations are useful. We see (cf.eq. (10), (12)),

$$C_{1100} = \frac{\partial^2}{\partial x_1^2} C = \frac{1}{r^2}[C_{01} - \tan\phi (t_{10} C_{10}) + \frac{1}{\cos\phi} \frac{\partial}{\partial \lambda} (t_{10}) C_{10} + (t_{10})^2 C_{20}] = \frac{1}{r^2}[C_{01} - t C_{10} + (t_{10})^2 C_{20}] \tag{24}$$

$$C_{1200} = \frac{\partial^2}{\partial x_1 \partial x_2} C = \frac{1}{r^2} t_{10} t_{20} C_{20} \tag{25}$$

because

$$\frac{\partial}{\partial \phi} t_{10} = 0$$

Then

$$C_{2200} = \frac{\partial^2}{\partial x_2^2} C = \frac{1}{r^2} [C_{01} + \frac{\partial}{\partial \phi} (t_{20}) C_{10} + (t_{20})^2 C_{20}] = [C_{01} - tC_{10} + (t_{20})^2 C_{20}]/r^2 \tag{26}$$

$$C_{i300} = \frac{\partial^2}{\partial x_i \partial x_3} C = \frac{1}{r^2}[t_{i0}(C_{11} - C_{10})] \tag{27}$$

$$C_{3300} = \frac{\partial^2}{\partial x_3^2} C = C_{02}/r^2 \tag{28}$$

The corresponding derivatives in Q are easily derived by symmetry interchanging the first and second subscript on t and substituting r by r'. For example:

$$C_{00i3} = \frac{\partial^2}{\partial y_3 \partial y_i} C = t_{0i}\ (C_{11} - C_{10})/(r')^2$$

For 1 derivative in P and 2 in Q we then have (k < 3, i < 3):

$$C_{k0ii} = \frac{\partial}{\partial x_k} \frac{\partial^2}{\partial y_i \partial y_i} C = (t_{k0}(C_{11} - C_{10}) - tC_{20} \tag{29}$$
$$+ (t_{0i})^2\ C_{30}) + 2\ t_{ki}\ t_{0i}\ C_{20}]/(rr'^2)$$

$$C_{k012} = \frac{\partial}{\partial x_k} \frac{\partial^2}{\partial x_1 \partial x_2} C$$
$$= [(t_{k1}\ t_{02} + t_{01}\ t_{k2})\ C_{20} \tag{30}$$
$$+ t_{01}\ t_{02}\ t_{k0}\ C_{30}]/(rr'^2)$$

$$C_{k03i} = \frac{\partial}{\partial x_k} \frac{\partial^2}{\partial y_3 \partial y_i} C = [t_{ki}(C_{11} - C_{10}) + \tag{31}$$
$$t_{k0}\ t_{0i}(C_{21} - C_{20})]\ (rr'^2)$$

$$C_{k033} = \frac{\partial}{\partial x_k} \frac{\partial^2}{\partial y_3^2} C = t_{k0}\ C_{12}/(r(r')^2) \tag{32}$$

$$C_{3033} = \frac{\partial}{\partial x_3} \frac{\partial^2}{\partial y_3^2} C = C_{03}/(r(r')^2) \tag{33}$$

The equations with 2 derivatives in P and 1 in Q are easily obtained again interchanging the subscripts of t and r, r'.

For the equations with 2 derivatives in both P and Q we naturally get something more complicated. We use here some observations:

$$r \frac{\partial}{\partial x_2} (t_{20}) = r' \frac{\partial}{\partial y_2} (t_{02}) = -t$$

$$-\tan\phi'\ t_{02} + r' \frac{\partial}{\partial y_1} (t_{01}) = -t$$

$$-\tan\phi\ t_{20} + r \frac{\partial}{\partial x_1} (t_{10}) = -t$$

and after some reordering one will see that:

$$-\tan\phi'\ t_{10}\ r \frac{\partial}{\partial x_1} (t_{02}) - (t_{01})^2 + r \frac{\partial}{\partial x_1}(t_{01})$$
$$+ t_{10}\ r' \frac{\partial}{\partial y_1} (t_{11}) = -(cp^2 + cq^2)sd^2 + cd^2$$

and

$$(t_{10})^2\ (-\tan\phi'\ t_{02} + r' \frac{\partial}{\partial y_1} (t_{01})) - t(t_{01})^2$$
$$+ 4\ t_{11}\ t_{10}\ t_{01}$$
$$= -t\ (cp^2 + cq^2)\ sd^2 - 4cd\ sd\ cp\ cq$$

where also the symmetries for P and Q can be used. Then using eq. (14),

$$C_{1111} = \frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial y_1^2} C = [C_{02} + C_{01} - t\ C_{11}$$
$$+ (t_{10})^2\ C_{21} - \tan\phi'[t_{02}(C_{11}$$
$$- C_{10} - tC_{20}) + 2\ t_{12}\ t_{10}\ C_{20}$$
$$+ (t_{10})^2\ t_{02}\ C_{30} + t_{11}(C_{11} - tC_{20} - C_{10})$$
$$+ (t_{01})^2\ (C_{21} - (2\ C_{20} + t\ C_{30}))$$
$$+ 2\ (\frac{\partial}{\partial y_1}\ t_{11}\ t_{10} + (t_{11})^2)C_{20} \tag{34}$$
$$+ (4\ t_{11}\ t_{10}\ t_{01} + (t_{10})^2 \frac{\partial}{\partial y_1}(t_{01}))C_{30}$$
$$+ (t_{01})^2\ (t_{10})^2 C_{40}]/(rr')^2$$
$$= [(C_{02} - C_{01} - 2\ t\ C_{11} + ((t_{01})^2 + (t_{10})^2)\ C_{21}$$
$$+ t\ C_{10} + C_{20}(t^2 + 2(cd^2 - cp^2 - cq^2))$$
$$+ C_{30}(-t(cq^2 + cp^2)\ sd^2 - 4\ sd\ cd\ cp\ cq)$$
$$+ C_{40}\ (t_{01})^2(t_{10})^2]/(rr')^2$$

$$C_{1222} = \frac{\partial^2}{\partial x_1 \partial x_2} \frac{\partial^2}{\partial y_2^2} C = [t_{10}\ t_{20}$$

$$(C_{21} - 2C_{20} - tC_{30} \tag{35}$$

$$+ (t_{01})^2\ C_{40} + 2(t_{11}\ t_{21}\ C_{20}$$

$$+ (t_{01}\ t_{20}\ t_{11} + t_{21}\ t_{01}\ t_{10})C_{300}]/(rr')^2$$

$$C_{i3kk} = \frac{\partial^2}{\partial x_i \partial x_3} \frac{\partial^2}{\partial y_k^2} C = [t_{io}(C_{12} - (C_{11} - C_{10})$$

$$-t(C_{21} - C_{20})) + 2\ t_{ik}\ t_{0k}(C_{21} - C_{20}) \tag{36}$$

$$+ (t_{0k})^2\ t_{io}(C_{31} - C_{30})]/(rr')^2$$

and the symmetric:

$$C_{kki3} = \frac{\partial^2}{\partial x_3^2} \frac{\partial^2}{\partial y_i \partial y_3} C = [t_{0i}(C_{12} - (C_{11} - C_{10})$$

$$-t(C_{21} - C_{20})) + 2\ t_{ki}\ t_{ko}\ (C_{21} - C_{20}) \tag{37}$$

$$+ (t_{ko})^2\ t_{0i}(C_{31} - C_{30})]/(rr')^2$$

Then for i < 3, and again using eq. (14),

$$C_{22ii} = \frac{\partial^2}{\partial x_2^2} \frac{\partial^2}{\partial y_i^2} C = (\frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2})$$

$$(C_{01} - tC_{10} + (t_{0i})^2\ C_{20})/(r')^2$$

$$= [C_{02} + C_{01} - tC_{11} + (t_{0i})^2\ C_{21}$$

$$+ \frac{\partial}{\partial \phi}[t_{20}\ C_{11} - t_{20}\ C_{10} - t\ t_{20}\ C_{20}$$

$$+ 2\ t_{2i}\ t_{0i}\ C_{20} + (t_{0i})^2\ t_{20}\ C_{30}]]/(rr')^2$$

which after recording becomes:

$$= [C_{02} + C_{01} - 2t\ C_{11} + ((t_{0i})^2 + (t_{20})^2) \cdot$$

$$C_{21} + tC_{10}(-2((t_{20})^2 + (t_{0i})^2$$

$$-(t_{2i})^2) + t^2)\ C_{20} + (4t_{2i}\ t_{0i}\ t_{20} \tag{38}$$

$$-t((t_{20})^2 + (t_{0i})^2)\ C_{30}$$

$$+ (t_{20})^2(t_{0i})^2\ C_{40}]/(rr')^2$$

The covariances involving the derivatives with respect to ($x_1$, $x_2$) or ($y_1$, $y_2$) have slightly simpler expressions because:

$$\frac{\partial^2}{\partial x_1 \partial x_2} t = \frac{\partial^2}{\partial y_1 \partial y_2} t = 0$$

With i < 2 we have:

$$C_{12ii} = \frac{\partial^2}{\partial x_1 \partial x_2} \frac{\partial^2}{\partial y_i \partial y_i} C$$

$$= \frac{\partial}{\partial x_2} (t_{10}[C_{11} - C_{10}]$$

$$+ 2\ t_{1i}\ t_{oi}\ C_{20} + t_{10}\ (t_{0i})^2\ C_{30})/((r')^2 r)$$

$$= [t_{10}\ t_{20}\ [C_{21} - C_{20} + (t_{0i})^2\ C_{40}] \tag{39}$$

$$+ 2\ t_{1i}[t_{2i}\ C_{20} + t_{0i}\ t_{20}\ C_{30}]$$

$$+ t_{10}\ 2\ t_{2i}\ t_{0i}\ C_{30}]/(rr')^2$$

$$= [t_{10}\ t_{20}[C_{21} - C_{20} + (t_{0i})^2\ C_{40}]$$

$$+ 2\ [t_{1i}\ t_{2i}\ C_{20} + (t_{0i}(t_{20}\ t_{1i} + t_{10}\ t_{2i})$$

$$C_{30}]/(rr')^2$$

Using the symmetry in P, Q we easily obtain the expression for

$$C_{ii12} = \frac{\partial^2}{\partial x_i \partial x_i} \frac{\partial^2}{\partial y_1 \partial y_2} C.$$

The expression involving $x_1$, $x_2$, $y_1$, $y_2$ is simple.

$$C_{1212} = \frac{\partial^2}{\partial x_1 \partial x_2} \frac{\partial^2}{\partial y_1 \partial y_2} C$$

$$= \frac{\partial}{\partial x_2} ((t_{12}\ t_{01} + t_{11}\ t_{02})\ C_{20}$$

$$+ t_{10}\ t_{01}\ t_{02}\ C_{30})/((r')^2 r) \tag{40}$$

$$= [(t_{22}\ t_{11} + t_{12}\ t_{21})\ C_{20}$$

$$+ (t_{01}\ t_{20}\ t_{12} + t_{02}\ t_{20}\ t_{11}$$

$$+ t_{10}\ t_{01}\ t_{22} + t_{10}\ t_{02}\ t_{21})\ C_{30}$$

$$+ t_{01}\ t_{02}\ t_{10}\ t_{20}\ C_{40}]/(rr')^2$$

Then for the derivatives involving radial differentiation we get with $ev_P$ the evaluation at P, using eq. (12)

$$C_{12i3} = \frac{\partial^2}{\partial x_1 \partial x_2} \frac{\partial^2}{\partial y_i \partial y_3} C = \frac{\partial}{\partial y_i} (\frac{\partial}{\partial r'} - ev_Q)$$

$$[t_{10}\ t_{20}\ C_{20}]/r^2$$

$$= \frac{\partial}{\partial y_i} [t_{10}\ t_{20}\ (C_{21} - C_{20})]/(r'r^2) \tag{41}$$

$$=[(t_{10}\ t_{2i} + t_{1i}\ t_{20})\ [C_{21} - C_{20}]$$

$$+ t_{10}\ t_{20}\ t_{0i}[C_{31} - C_{30}]]/(rr')^2$$

$$C_{i3j3} = \frac{\partial^2}{\partial x_i \partial x_3} \frac{\partial^2}{\partial y_j \partial y_3} C = \frac{\partial}{\partial x_i} [\frac{1}{r} \frac{\partial}{\partial r}$$

$$- ev_P](t_{0j}\ (C_{11} - C_{10}))$$

$$= \frac{\partial}{\partial x_i}[t_{0j}(C_{12} - (C_{11} - C_{10}))] \tag{42}$$

$$= [t_{i0}\ t_{0j}(C_{22} - (C_{21} - C_{20}))$$

$$+ t_{ij}\ (C_{12} - (C_{11} - C_{10}))]/(rr')^2$$

$$C_{33jj} = \frac{\partial^2}{\partial x_3^2} \frac{\partial^2}{\partial y_j^2} C \tag{43}$$

$$= (C_{03} - t\ C_{12} + (t_{0j})^2\ C_{22})/(rr')^2$$

$$C_{33j3} = \frac{\partial^2}{\partial x_3^2} \frac{\partial^2}{\partial y_j \partial y_3} C \tag{44}$$

$$= t_{0j}\ (C_{13} - C_{12})/(rr')^2$$

$$C_{3333} = \frac{\partial^2}{\partial x_3^2} \frac{\partial^2}{\partial y_3^2} C = C_{04}/(rr')^2 \tag{45}$$

$$C_{3312} = \frac{\partial^2}{\partial x_3^2} \frac{\partial^2}{\partial y_1 \partial y_2} C = t_{01}\ t_{02}\ C_{22}/(rr')^2 \tag{46}$$

The corresponding equations where P and Q are interchanged we again obtain by interchanging the subscripts on t.

We now have all necessary basic equations. Note, that there are already implied computational savings, since we only need to calculate $C_{ik}$, $C^1_{ik}$, and not **all** derivatives.

**3. Computation of covariances in a rotated reference frame.**

In section 2 all expressions for auto and cross-covariances of zero, first and second order derivatives were derived in the usual spherical reference frame (east, north and up). As may be inferred from the notation used, the covariances are considered as being elements of matrices of dimension i x j x n x m, where i, n are equal to 1, if the functionals are associated with the evaluation functional in P, Q respectively and equal to 3 in all other cases. j and m will be zero for the evaluation functional and for first order derivatives and 3 for second order derivatives.

In a rotated reference frame, obtained by applying the rotations matrices $R_P$, $R_Q$ in P, Q, respectively, to go **from** the spherical to the rotated frame, we may obtain the new covariances using covariance propagation. A new vector (i.e. the gradient vector) is obtained in the new frame by pre-multiplication with $R_P$, $R_Q$, respectively. A new 3 x 3 matrix (i.e. the matrix of second order derivatives) is obtained by pre-multiplication with $R_P$ or $R_Q$, and postmultiplication with the respective transposed matrices, $R_P^T$ or $R_Q^T$.

Suppose we then have the 3 x 0 x 3 x 0 = 3 x 3 matrix of covariances of first order derivatives where the quantities in P are associated with the rows and the quantities in Q with the columns. Then the matrix of covariances associated with the two new reference frames in P and Q are obtained by pre-multiplying the equivalent 3 x 3 matrix by $R_P$ and postmultiplication with $R_Q^T$.

For the case where we have a 3 x 3 x 3 x 3 matrix of covariances of second order derivatives we then must multiply each 3 x 3 matrix with the two last subscripts fixed (totally 9) with $R_P$ and post multiply with its transposed. In the new 3 x 3 x 3 x 3 matrix we must then multiply each 3 x 3 matrix with the first two subscripts fixed with $R_Q$ and postmultiply with its transposed.

If we define the 9 3 x 3 matrices $C_{ij}$ as having elements $C_{ij}(n,m) = C(i,j,n,m)$ and the 9 3 x 3 matrices $C_{nm}$ with elements $C_{nm}(i,j) = C(i,j,n,m)$, then the two step transformation for second order derivatives in both P and Q is done as follows (remember each $C_{ij}$ and $C_{nm}$ are 3 x 3 matrices !)

$$C_{ij}^R = R_Q * C_{ij} * R_Q^T.$$

from which we have 9 new matrices $C_{nm}^R$. They are then rotated in P,so that

$$C_{nm}^{RR} = R_P * C_{nm}^{R} * R_P^{T}.$$

Every one of the 81 elements have then been multiplied with 4 numbers, 2 from $R_P$ and 2 from $R_Q$.

## 4. Implementation considerations and tests.

We have to calculate up to 4 derivatives with respect to t. Fortunally they are easily computed using recursion, see (Tscherning and Rapp, 1974, section 9), (Tscherning, 1976, pp. 21–27). For the vertical derivatives we have the cases $C_{ij}$, j = 1, 2, 3, 4, $C_{ij}^1$, j = 2.
Using the general covariance expression,

$$C(t,r,r') = \sum_{k=2}^{\infty} \sigma_k \left(\frac{R^2}{rr'}\right)^{k+1} P_k(t) ,$$

where $\sigma_k$ are the so-called degree-variances, R the Bjerhammar sphere radius, and $P_k(t)$ the Legendre polynominals, we see that we need the corresponding series with coefficients

$\sigma_k\,(k+1)^2$ for $C^1_{i2}$

and

$$\left.\begin{array}{l}\sigma_k\,(k+1)\ ,\ \sigma_k\,(k+1)(k+2)\ ,\ \sigma_k(k+1)^2(k+2)\ ,\\ \sigma_k\,(k+1)^2(k+2)^2\end{array}\right. \quad \text{for } C_{ij}$$

(and subsequent division with powers of r and r').
We then need the functions as seen from Table 1.

| Order of derivative in P | 0 | 1 | 1 | 2 | 1 | 2 |
|---|---|---|---|---|---|---|
| order of derivative in Q | 1 | 0 | 1 | 1 | 2 | 2 |
| $C_{10}$, $C_{01}$ | x | x | x | x | x | x |
| $C_{02}^1$ | | | x | | | |
| $C_{20}$, $C_{11}$ | | | x | x | x | x |
| $C_{30}$, $C_{21}$, $C_{12}$, $C_{03}$ | | | | x | x | x |
| $C_{40}$, $C_{31}$, $C_{13}$, $C_{22}$, $C_{04}$ | | | | | | x |

Table 1. Derivatives needed for various orders of differentiation.

The testing of the algoritms are eased by the fact, that the functions are harmonic in P and Q, i.e.

$$\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\left(\frac{\partial^2}{\partial y_j \partial y_k} C\right) = \sum_{i=1}^{3} \frac{\partial^2}{\partial y_i^2}\left(\frac{\partial^2}{\partial x_j \partial x_k} C\right) = 0,$$

see Table 2, where the result of one test is shown.

Also a comparison with earlier tested equations

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right) \text{ or } \left(\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2}\right)$$

(see Krarup and Tscherning, 1984), has been usefull. Otherwise only numerical differentiation is available as a check of the analytic expressions. A number of such checks have been executed, but errors are always left. They will be detected, when the covariances are used for prediction by least squares collocation.

## 5. Conclusion.

We have here developed expressions between all covariances associated with zero, first and second order derivatives with respect to local spherical coordinates. Simple rotations of the vectors or matrices containing these quantities will furnish us with covariances in arbitrarity rotated coordinate systems in P or Q.
Further simplifications are possible, if specific degree-variance models are used. If, for example, $\sigma_i$ contain the factors:

1/(i+1) or 1/(i+2)

then the corresponding closed expressions of derivatives will include the simple function:

$$\frac{1}{L} = \sum_{i=0}^{\infty} \left(\frac{R^2}{rr'}\right)^{i+1} P_i\,(t)\ ,\ \text{where}$$

$$L = \left(1 - 2\left(\frac{R^2}{rr'}\right) t + \left(\frac{R^2}{rr'}\right)^2\right)^{1/2}$$

and its derivatives with respect to t. This may be advantageous, considering that many quantities are computed as differences between nearly equal quantities. A recommended covariance function model will therefore be:

$$C(r,r',t) = \sum_{i=2}^{N} \sigma_{ei} \left(\frac{R^2}{rr'}\right)^{i+1} P_i(t) + \sum_{i=N+1}^{\infty} \frac{A}{(i-1)(i+1)(i+2)} \left(\frac{R^2}{rr'}\right)^{i+1} P_i(t)$$

where $\sigma_{ei}$ are error-degree-variance of the used reference model and A a constant in units of $(m/s)^4$.

Table 2. Result of evaluating the components $\partial^2/\partial x_i^2$ of the Laplace equation in P on different covariance functions with $R^2 * \sigma_2 = 7.5$ mgal$^2$,

$\sigma(i) = A/((i+24)(i-1)(i-2))$, R/RE = 0.999808 and

RE = 6371 km, A = $R^2$ x 425.12 mgal$^2$. As a check the sum (which should be zero) is shown.

Covariances between second order derivatives in P and zero first and second order derivatives in Q:

| In P: | $x_1^2$ | $x_2^2$ | $x_3^2$ | sum |
|---|---|---|---|---|
| In Q: | $y_3$ | $y_3$ | $y_3$ | |
| ψ | | | | |
| o , | | EU$^2$ * mgal | | |
| 0 0 | 848.27011 | 848.27011 | −1696.54023 | −0.00001 |
| 0 30 | 61.26028 | −3.55901 | −57.70127 | 0.00000 |
| 1 0 | 28.59828 | −3.11583 | −25.48245 | 0.00000 |
| 1 30 | 18.01497 | −2.21557 | −15.79940 | 0.00000 |

| In Q: | $y_1^2$ | $y_1^2$ | $y_1^2$ | sum |
|---|---|---|---|---|
| o , | | EU$^4$ | | |
| 0 0 | 2659.37673 | 888.23232 | −3547.60905 | 0.00000 |
| 0 30 | 17.34368 | −5.60214 | −11.74154 | 0.00000 |
| 1 0 | 4.24152 | −1.47842 | −2.76309 | 0.00001 |
| 1 30 | 1.81071 | −0.65085 | −1.15986 | 0.00000 |

| IN Q: | $y_2^2$ | $y_2^2$ | $y_2^2$ | sum |
|---|---|---|---|---|
| o , | | EU$^4$ | | |
| 0 0 | 888.23232 | 2659.37673 | −3547.60905 | 0.00000 |
| 0 30 | −5.60214 | −6.61141 | 12.21356 | 0.00001 |
| 1 0 | −1.47842 | −1.57747 | 3.05589 | 0.00001 |
| 1 30 | −0.65085 | −0.66114 | 1.31199 | 0.00000 |

| In Q: | $y_3^2$ | $y_3^2$ | $y_3^2$ | sum |
|---|---|---|---|---|
| o , | | EU$^4$ | | |
| 0 0 | −3547.60905 | −3547.60905 | 7095.21810 | 0.00000 |
| 0 30 | −11.74154 | 12.21356 | −0.47202 | 0.00000 |
| 1 0 | −2.76309 | 3.05589 | −0.29280 | 0.00000 |
| 1 30 | −1.15986 | 1.31199 | −0.15212 | 0.00001 |

| In Q: | geoid | geoid | geoid | sum |
|---|---|---|---|---|
| o , | | EU$^2$ * m | | |
| 0 0 | −11.59987 | −11.59987 | 23.19974 | 0.00000 |
| 0 30 | −7.36669 | −5.56428 | 12.93098 | 0.00001 |
| 1 0 | −6.10139 | −4.35588 | 10.45727 | 0.00000 |
| 1 30 | −5.38322 | −3.72342 | 9.10665 | 0.00001 |

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# Application of inequality constraint least squares to GPS navigation under selective availability

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**ABSTRACT.** The least squares problem with inequality constraints (LSI) is first reviewed. The necessary and sufficient conditions for a LSI solution are given based on the general Kuhn-Tucker theorem. A procedure for solving the LSI problem is derived by using the least distance programming (LDP) solution. An application to single point GPS navigation under Selective Availability (SA) is then presented for the case when *a priori* knowledge of the height component is available. This is the case for marine and vehicular GPS navigation where the height can be constrained within known error bounds. The use of a least-squares inequality constraint is shown to be very effective in limiting height variations within the assigned error bounds. A significant increase of accuracy in the horizontal components also occurs. The advantage of the inequality constraint method, as compared to the weighted parameter constraint method, is illustrated numerically.

## INTRODUCTION

The least squares problem with *inequality* constraints on parameters may arise either from (a) the nature of the problem itself, (b) the requirements which have to be met or (c) the partial knowledge of the parameters. In single point GPS navigation, when a ship sails in the ocean or a vehicle moves along a relatively flat road, the height component is normally known to within a few metres, provided the ocean tide and geoid are taken into account. In this case, we can add an inequality constraint, namely $| h | \leq$ a few m, in the GPS navigation solution model. In other cases such as curve fitting, we may require that the functional value of the fitted curve evaluated at a specified point is less (or larger) than a given constant value, or we may also require that the first derivative of the curve is less (or larger) than a given constant depending on the physical process that the curve describes. These requirements also lead to inequality constraints on the parameters.

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In mathematical terminology, the least squares problem with inequality constraints can be written as:

LSI

*Minimize* $\quad \| \mathbf{Ex} - \mathbf{f} \|$ (1)

*subject to* $\quad \mathbf{Gx} \geq \mathbf{h}$ , (2)

where **E** is the design matrix, **f** is the misclosure vector, **x** is the parameter vector, **G** is the constraint matrix and **h** is the constant vector of the inequality constraints.

As we know, a linear *equality* constraint defines a *hyperplane* in the space. The least squares solution with *equality* constraints is then confined on the intersections of all the hyperplanes. For example, considering the following two-dimensional *least distance programming* problem

$$\text{minimize} \quad \| \mathbf{x} \| , \quad \text{subject to } x_1 + x_2 = 1, \tag{3}$$

the solution is confined on the straight line $x_1 + x_2 = 1$, as shown in Fig. 1. Clearly, the solution with the minimum distance from the origin to the straight line is $\mathbf{x}^T = (0.5, 0.5)$, which is the perpendicular from the origin to the straight line.

Instead of defining a hyperplane, a linear *inequality* constraint, on the other hand, defines a feasible half-space. The least squares solution is then obtained within the common feasible region (if it exists) defined by all the inequality constraints. Again using the above two-dimensional least distance programming problem as an example, but we change the equality constraint to an inequality constraint, i.e.

$$\text{minimize} \quad \| \mathbf{x} \| , \text{subject to } x_1 + x_2 \geq 1. \tag{4}$$