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Achievable accuracy for geoid height differences across the Strait of Gibraltar.

by

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Abstract:

The use of space positioning methods enables the precise computation of ellipsoidal height differences for points located around the Strait of Gibraltar. However, in order to permit a comparison with traditional orthometric or normal heights, geoid height differences must be subtracted from the ellipsoidal height differences.

A comparison of gravity data in the Gibraltar area with gravity values computed from the spherical harmonic expansions OSU81, GPM2, OSU86F and IFE88E2 showed that the IFE88E2 model gave the best agreement, namely a r.m.s.v. value equal to 24 mgal for the difference observed minus computed, as compared to a r.m.s.v. value of the free-air gravity anomalies of 47 mgal. The IFE88E2 model is complete to degree 360.

Using a 1 km * 1 km digital height model the topographic effects were computed, and subtracted from the free-air anomalies and the IFE88E2 anomalies. While the r.m.s.v. of the topographically reduced free-air gravity anomalies remained at 47 mgal, it increased for the IFE88E anomalies. This shows, that the gravity variation primarily is caused by deeper mass anomalies, and not by the topographic variations.

The IFE88E-anomalies were used to compute an empirical covariance function. Based on this function an analytic expression for the covariance function was developed, from which auto-and cross covariance functions with geoid heights could be computed. These showed, that the IFE88E2 geoid heights in the Gibraltar area have an error of only 0.45 m, if possible regional biases are disregarded.

Using the developed covariance model, error estimates may be computed using the mean error expressions of least squares collocation, for varying data configurations. This was used to compute the achivable accuracy of geoid height differences for points around the Strait, assuming various (but realistic) distributions of gravity anomaly data, and with one height at the coast held fixed. Furthermore, the contribution of the deflections of the vertical, observed at points close to the Strait, was studied.

The results show, that geoid height differences may be computed with a standard deviation of 0.4 cm per km, with a gravity data spacing of 5 km. Using this spacing, the observation of one pair of deflections of the vertical will not contribute to an improved precision of the estimated geoid height differences.

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1. Introduction

The use of space positioning methods enables the precise computation of ellipsoidal height differences. For distances below 100 km, height differences have been determined with a precision better than 1 ppm. This means for distances relevant for the Gibraltar Strait crossing 1-2 cm. However, these height differences must be compatible with these obtained from levelling, i.e. we must subtract the geoid undulation differences from the height differences,

$$\Delta H_{PQ} = H_P - H_Q = h_P - h_Q - (N_P - N_Q) \tag{1}$$

where P, Q are two points, H_P, H_Q their orthometric heights, h_P, h_Q, their ellipsoidal heights and N_P, N_Q their geoid heights. (If normal heights are used, height anomalies must be used instead of geoid heights).

The geoid height difference must be determined with a standard deviation of 2-3 cm, in order to fulfil the engineering requirements of knowing the height difference to within 5 cm. But an even higher precision would be necessary, if space-geodetic and levelling methods are to be used for the study of geodynamic phenomena.

The purpose of this paper is to investigate which amount and which kind of gravity field data will be needed in order to achieve a given precision. A method for carrying out such an investigation has been described in (Tscherning, 1975, 1983), and it will be briefly reviewed in section 2. The method requires that the gravity field variability and correlation is estimated for the Strait of Gibraltar area, and that it is expressed in an analytic form. This estimation and modelling is described in section 3, and in section 4 the result of simulation computations, are presented.

The reader may then ask, whether a precision of a few cm really is achivable for geoid height differences over distances from 0-30 km. Fortunally this has been demonstrated several times, see Engelis et al. (1984, 1985), Denker and Wenzel (1987), Torge et al. (1988), Forsberg and Solheim (1988). It is very satisfactory, that these results have been obtained using several different methods.

2. Estimating the achievable accuracy

Let W be the gravity potential, U a corresponding normal or reference potential, and T = W-U the anomalous gravity potential, which is harmonic outside the Earth's surface. The so-called empirical covariance function C(P,Q) expresses the correlation of the values of T in two points P, Q. It is rigously defined globally (in spherical approximation), and may with some caution be used locally, see (Goad et al. 1984). It may be expressed as the sum of a Legendre series,

$$C(P,Q)=\sum_{i=2}^{\infty} \sigma_i{}^2 \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\psi) \tag{2}$$

with

$$\sigma_i{}^2 = \left(\frac{GM}{R}\right)^2 \sum_{j=o}^{i} (\overline{C}_{ij}^{\,2} + \overline{S}_{ij}^{\,2}). \tag{3}$$

Here ψ is the spherical distance between P and Q, r, r′ are the distances of P, Q from the origin, respectively, R is the mean radius of the Earth, P_i are the Legendre polynomials, and $\sigma_i{}^2$ the so-called (potential) degree-variances. They are equal to the square-sum of the coefficients of T in an expansion in fully normalized Legendre functions, multiplied by the square of the product of the mass of the Earth, M, and the gravity constant G, divided by R.

If the covariance function is known, the value of a functional, L, applied on the anomalous potential, T, L(T), may be estimated from the values of other functionals (observations),

$$x_i = L_i(T) + e_i, \; i=1,\ldots, n$$

where e_i is the error and x_i the observed quality. The estmation may be made using least-squares collocation (LSC),

$$L(\tilde{T}) = \{C_{Pi}\}^T \{\overline{C}_{ij}\}^{-1} \{x_j\}, \tag{4}$$

$$\overline{C}_{ij} = L_i L_j (C(P,\, Q)) + n_{ij}, \tag{5}$$

and

$$C_{Pi} = LL_i C(P,\, Q_i).$$

n_{ij} is the error covariance, C_{Pi} the covariance between the observations and L(T) and $L_i L_j C$ the covariances between the observations.

For our purpose the most important is that we also may obtain an estimate of the error associated with $L(\tilde{T})$,

$$\sigma^2 (L(\tilde{T})) = C_{PP} - \{C_{Pi}\}^T \{\bar{C}_{ij}\}^{-1} \{C_{Pj}\} \tag{6}$$

where C_{PP} = LLC(P, P), the variance of L(T). Using eq. (6) with systematically varying area coverage, data spacing and data noise characteristics, it is possible to establish a relationship between these parameters and the accuracy of $L(\tilde{T})$. Unfortunately, the statistical distribution of $\sigma^2(L(T))$ is not well defined, see Tscherning (1986). However, if the gravity field data in an area have a normal distribution, we may expect the error estimate to have the usual χ^2-distribution.

3. Estimation of the empirical covariance function for the Strait of Gibraltar area.

In order to study the statiscal characteristics of the gravity field in the Strait of Gibraltar area, a set of 2170 gravity values were made available by the Instituto Geografico Nacional, Madrid. All the values were located on land on the northern side of the Strait.

In order to make the gravity data in this local area as homogeneous as possible, the long wavelength information must be subtracted from the data (see Goad et al. (1984)). Also gravity field variations caused by local topographic variations are generally removed. The remaining gravity signal will then, generally, be far more homogeneous and isotropic then the original signal, see e.g. Forsberg (1986).

The subtraction of the long-wavelength field is done by calculating the length of the gravity vector from a spherical harmonic expansion, and then subtracting this quantity from the observed value. The residual field will have a much smaller variation, and the correlation distances will decrease considerably, see e.g. (Tscherning, 1985). Consequently, only data in a small area surrounding the area of interest need to be taken into account. The size of the surrounding area depends on the maximal degree and order of the spherical harmonic expansion, the effect of which has been subtracted.
Today several spherical harmonic expansions of maximal degree larger than 180

are available. The best ones are the OSU81 and the GPM2 models (Rapp, 1981, Wenzel, 1985) complete to degree 180 and 200, respectively, and the OSU86F (Rapp and Cruz, 1986) and IFE88E2 (H. Denker, private communication, 1989), both complete to degree 360.

The contributions from these four fields were calculated and subtracted from the observed gravity values. The results are given in Table 1.

Table 1. Mean and standard deviations (mgal) of the 2170 free-air gravity anomalies without and with the contribution of four spherical harmonic expansions.

	Free-air anomalies	Anomalies with contributions subtracted: OSU81	GPM2	OSU86F	IFE88E2
Mean	-19.9	-0.9	-7.5	-5.0	-1.8
St.dev.	43.7	43.9	44.3	29.7	25.9
Highest degree used:		180	180	360	360

The results show that the new IFE88E2 model gives the best fit, and we have then used the residuals with respect to this field in the following. Fig. 1 shows the geoid derived from the IFE88E2 coefficients.

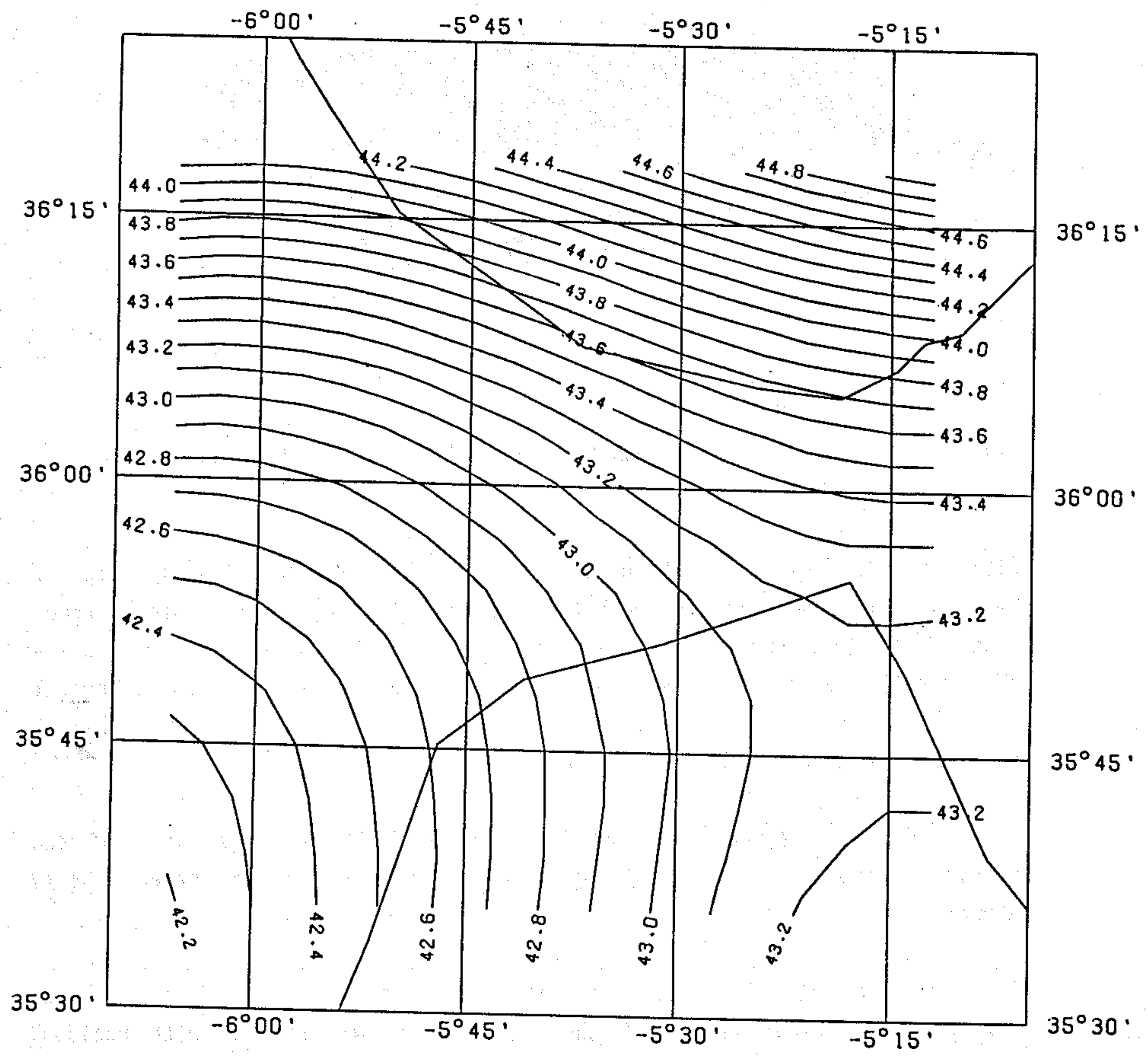

Fig. 1. Geoid heights derived from the IFE88E2 coefficient set in GRS 80. Contour interval 0.1 m.

The smoothed field also let 3 grosserrors stand out. They were not used in the following. However, several other observations seems to have large errors, but their identification requires access to the original data material.

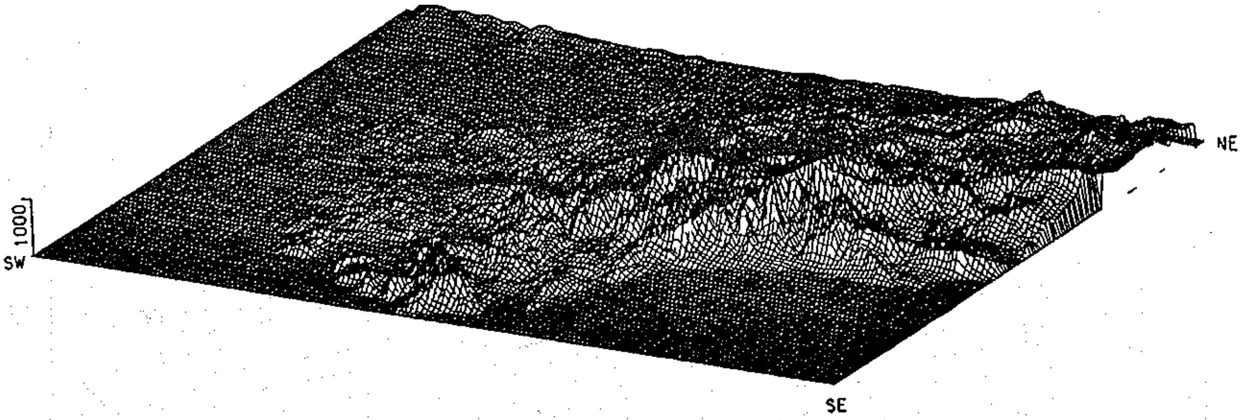

Fig. 2. Digital height model used for the calculation of topographic effects.

With the gravity data, a 1 km digital terrain model was supplied, see Fig. 2. Based on this model, and the heights of the gravity observations, the attraction of the topography was calculated and subtracted, (Forsberg and Tscherning, 1981). (Since topographic effects are also contained in the spherical harmonic expansion used, the heights were referred to a 0.5° mean height surface). The results are given in Table 2.

The results are surprising, and indicates that the primary cause of the gravity variation not is the topography, but probably deeper seated density anomalies.

Table 2. Mean, standard deviation, root mean square variation (r.m.s.v.) and maximal and minimal values of the gravity data without and with residual topographic effects subtracted. (2067 obs). (All units mgal).

	Free-air anomalies	Topograph. effects	Topograph. removed	IFE88E2 anomalies	Topography removed
Mean	-23.6	-2.6	-21.0	-2.4	0.2
St. dev.	40.7	20.9	41.6	24.3	30.3
r.m.s.v.	46.8	21.1	46.6	24.4	30.3
Min.	-82.9	-38.7	-88.5	-67.2	-79.4
Max.	124.2	93.7	141.3	85.9	106.3

The empirical (gravity anomaly) covariance function was the estimated from the IFE88E2 anomalies by calculating mean values of products of anomalies associated with points having a spherical distance falling within a certain interval, here of size 1′. (For details, see (Tscherning, 1985)). The graph of the function is shown in Fig. 3.

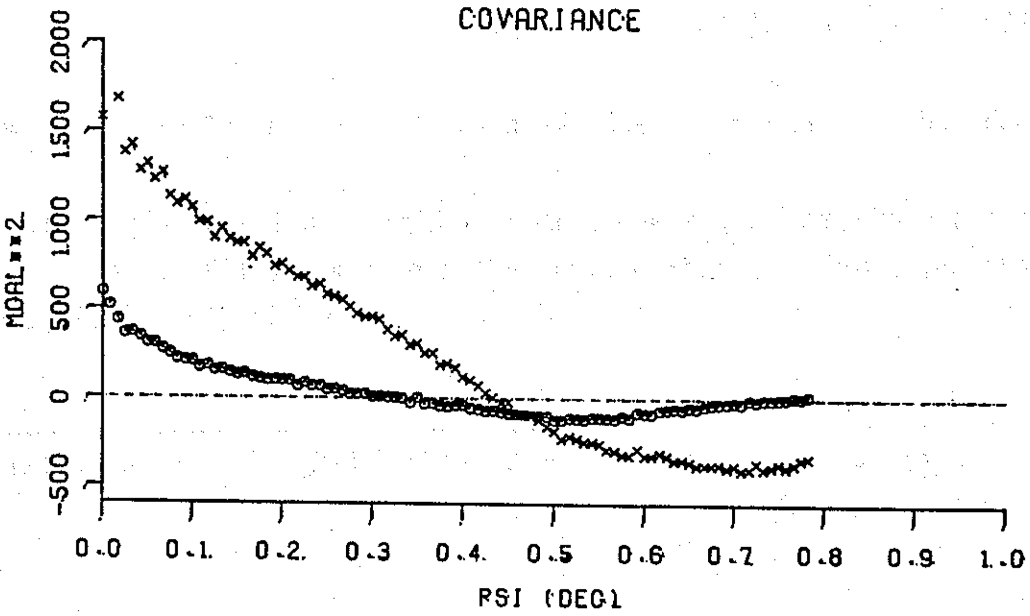

Fig. 3. Empirical gravity anomaly covariances. (x) indicate values calculated using free-air gravity anomalies and (o) indicates anomalies from which the contribution from IFE88E2 has been subtracted.

The gravity anomaly covariance function is obtained from the one given in eq. (2), by multiplying each term with $(i-1)^2/(rr')$, i.e.

$$C\,(\Delta g_P,\ \Delta g_Q) = \sum_{i=2}^{\infty} \sigma_i^2\ \frac{(n-1)^2}{rr'} \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\psi) \qquad (7)$$

An analytic model is determined from the estimated values, using

$$\sigma_i^2 = a \cdot \epsilon_i^2\ ,\ i = 2,\ldots,N \qquad (8)$$

$$\sigma_i^2 = A/((n-1)(n-2)(n+B))\ \left(\frac{R}{R_B}\right)^{2i+2},\ i > N \qquad (9)$$

ϵ_i^2 are so-called error-degree-variances, which express the error in the coefficients of the spherical harmonic expansion, B and N are integers, and R_B

is a quantity which is less than R, called the radius of the Bjerhammar sphere. We will here use B = 4, but try various values for N. Since we have no estimates available for the IFE88E2 coefficients, two parametric models were used

$$\text{(I)} \qquad \epsilon_i^2 = R^2 \cdot 0.01 * i / (i-1)^2$$

$$\text{(II)} \qquad \epsilon_i^2 = R^2 \cdot 0.0001 \cdot i^2 / (i-1)^2 \qquad (10)$$

For both models, the gravity anomaly error for degree 100 will be 1 $mgal^2$.

The free parameters, R_B, a and A were estimated using an algorithm developed in (Knudsen, 1987) and incorporated in the program COVFIT. Several models were determined using different values of N and both models (I) and (II), see Table 3.

Table 3. Estimates of the parameters C_o, a and $R-R_B$. (C_o is the total variance of the gravity anomalies).

Model	I	II	II	II
N	360	360	320	280
a	0.1155	0.0572	0.0301	0.0354
C_o ($mgal^2$)	559.9	560.1	560.0	559.8
$R-R_B$ (m)	464	408	466	487

Using the analytic expression (2), auto- and cross-covariance functions with geoid heights were derived using the functional relationship between the quantities. Fig. 4 shows the fit to the empirical gravity anomaly covariance function, and Fig. 5 and 6 the auto- and cross-covariance function with the geoid heights, respectively.

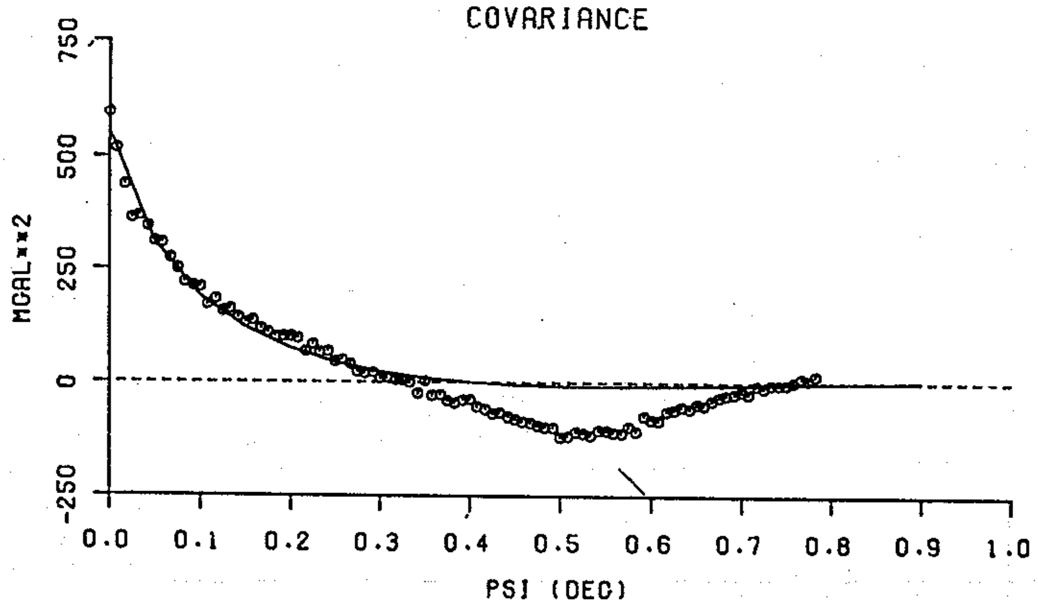

Fig. 4. Empirical and model covariance function. (Values in first column of Table 3 used here and in the following two figures).

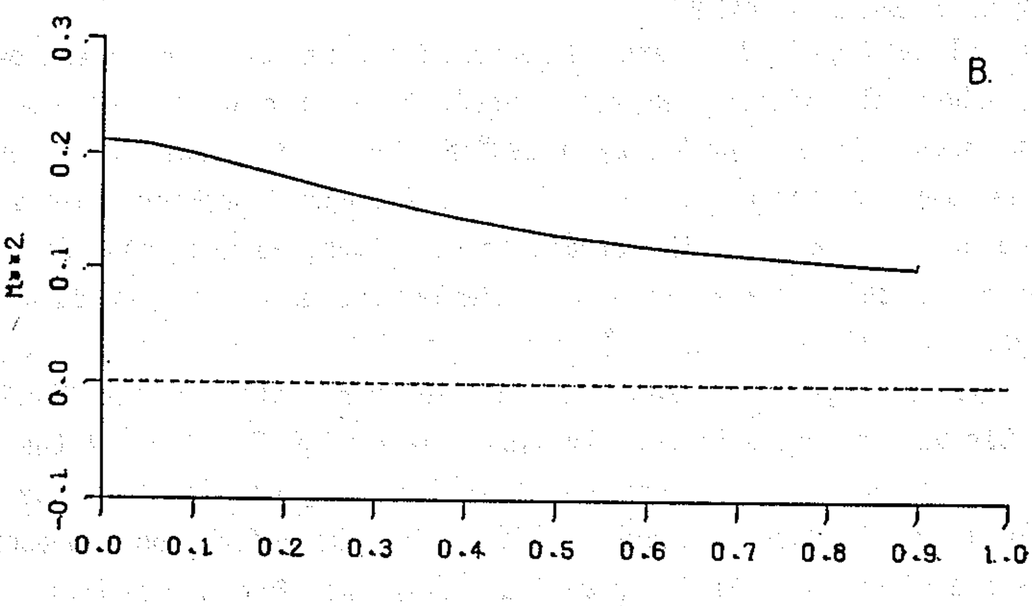

Fig. 5. Model geoid height auto-covariance function (for IFE88E2 residual quantities).

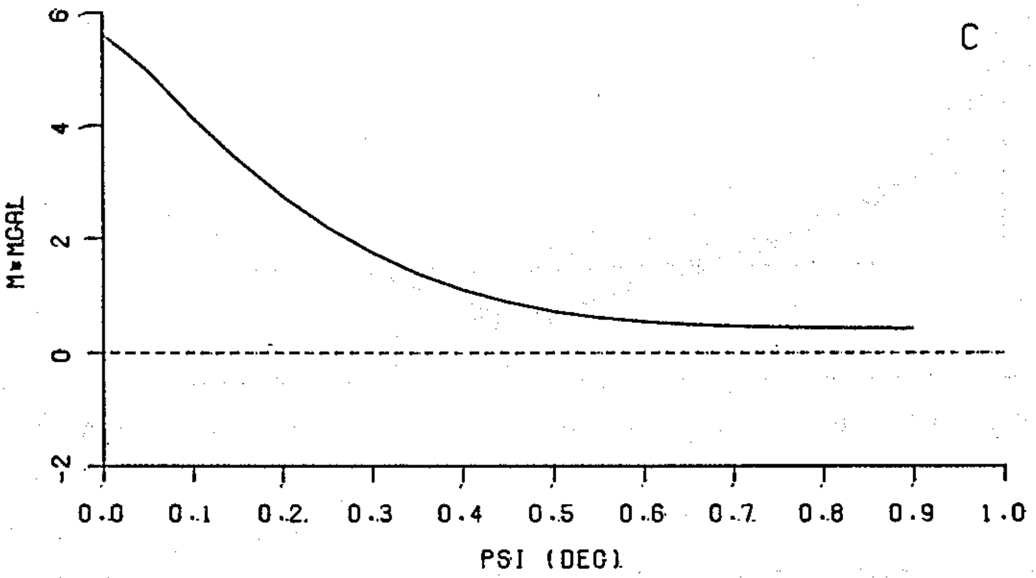

Fig. 6. Model geoid height-gravity anomaly cross-covariance function.

4. Results of the simulations.

The minimal data collection area depends on the maximal degree and order (N) of the spherical harmonic expansion used. As a rule of thumb, the area of interest should be surrounded by a 180°/N wide border zone, i.e. here 0.5°. Since we may be interested in height differences between points with a distance up to 20 km from the Strait of Gibraltar, we must consider a 1.5° x 1.5° area. In this area we then made simulations using gravity data in grids with spacing 0.05°, 0.1° and 0.2°, i.e. maximally 961 data points. We introduced one geoid height observation at the Northern coast of the Strait as an extra observation, and subsequently also a pair of deflections of the vertical in the same point. A signal noise of 2 mgal was used for the gravity anomalies, 0.01 m for the geoid height, and 0.3" for the deflection components. The results are given in Table 4 for various covariance function models.

Table 4. Result of simulations. Parameters from Table 3 used, except for Model II, N = 360, where a = 0.1155 was used.

Model:	I	II	II	I	I	III	I	II
N	360	360	320	360	320	360	360	360
Geoid St.dev.(m)	0.46	0.44	0.28	0.46	0.29	0.44	0.46	0.44
Data spacing (degree)		0.05°			0.1°		0.2°	
Distance from "observed" point (degr.)				Estimated error (cm)				
0.05	2	2	2	4	4	4	7	6
0.10	3	3	3	8	8	7	11	9
0.15	4	3	3	8	8	7	14	12
0.20	4	4	4	10	10	9	17	14
0.25	5	4	4	10	9	8	19	16
0.30	6	5	4	11	11	10	19	17
0.35	6	6	5	11	10	9	20	17
0.40	7	7	5	13	11	11	21	18

The values for the simulations using one deflection pair are identical to these given in Table 4, i.e. this additional information gives no new information.

In Table 4 is also given the expected geoid height error, using only the information in the spherical harmonic expansion. Even for the largest value of this error (0.46 m), we see that a geoid height difference may be determined with a standard deviation of 2-3 cm, using a 0.05° data spacing. According to Bureau Gravimetrique International (1988), there are in each 1° x 1° square surrounding the Strait of Gibraltar more than the required 400 gravity points. However, the gravity data spacing may be irregular, as indicated in Bureau Gravimetrique International (1986). Consequently, data "holes" may have to be filled in, e.g. using sea-bottom gravimeters near the coast, where gravity information normally is missing.

5. Conclusion.

Since the gravity data distribution in the Strait of Gibraltar seems to be denser than one point per 0.05°, an accuracy of 2-3 cm should be achievable for points spaced up to 10 km apart. This, however, requires that the data is spaced resonably regularily. One should also take into account, that the error estimates are statistical in nature. In case heights are needed at locations of extreem gravity variation, a denser gravity spacing may be needed.

If higher precision is needed, hydrostatic levelling between GPS points located on each ride of the Strait may contribute very much. This also holds for GPS points on land connected by precise levelling. Such GPS-levelling points may be used to detect tilts introduced in the geoid, originating from long-wavelength errors in the spherical harmonic expansion used.

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