Lecture 5. **Gravity field –modelling - sphere.**

We use the development for T and integrate over sphere:

$$T(\varphi,\lambda,r)=\sum_{i=2}^{\infty}\frac{GM}{r}\left(\frac{R}{r}\right)^{i}\sum_{j=-i}^{i}Y_{ij}(\varphi,\lambda)\frac{i-1}{i-1}\left(\frac{R^2}{R^2}\right)\Delta\overline{C}_{ij}$$

$$=\sum_{i=2}^{\infty}\frac{1}{r}\left(\frac{R}{r}\right)^{i}\sum_{j=-i}^{i}Y_{ij}(\varphi,\lambda)\frac{R^2}{i-1}\frac{1}{4\pi}\iint\Delta g(\varphi',\lambda',R)Y_{ij}(\varphi',\lambda')dS$$

$$=\frac{R}{4\pi}\iint\left[\Delta g(\varphi',\lambda',R)\sum_{i=2}^{\infty}\frac{1}{i-1}\left(\frac{R}{r}\right)^{i+1}\sum_{j=-i}^{i}Y_{ij}(\varphi,\lambda)Y_{ij}(\varphi',\lambda')\right]dS$$

Stokes formula

We get:

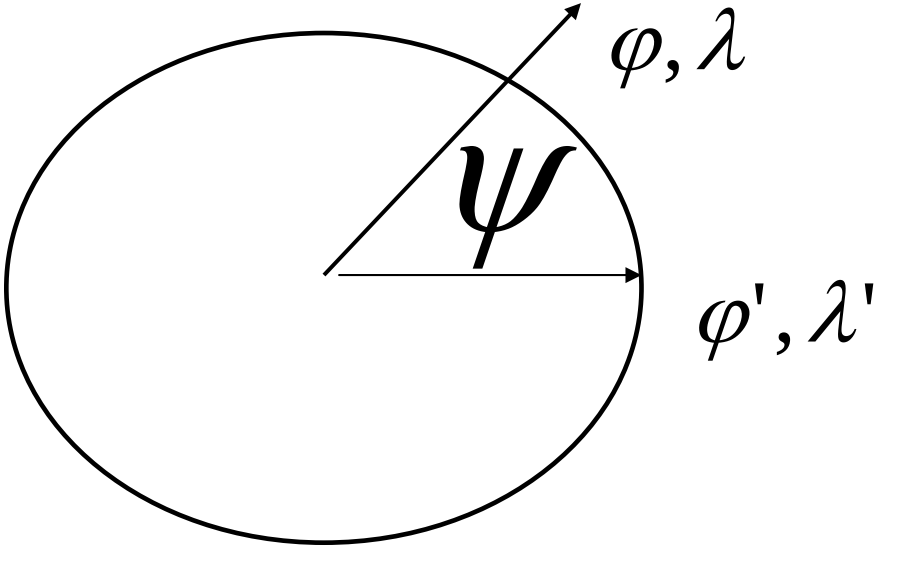
$$T(\varphi,\lambda,r) =$$

$$\frac{R}{4\pi}\iint_S \Delta g \cdot \left(\frac{R}{r}\right)^{i+1} \sum_{i=2}^{\infty} \frac{2i+1}{i-1} P_i(\cos\psi)\, d\sigma$$

Gravity field-modelling.

We use the development in Legendre-polynomials

$$P_i(\cos\psi) = \frac{1}{2i+1}\sum_{j=-i}^{i} Y_{ij}(\varphi,\lambda)Y_{ij}(\varphi',\lambda')$$



Stokes formula, 1849, Pizetti, 1911.

We may obtain closed expression because:

$$\frac{1}{l} = \sum_{i=0}^{\infty} \frac{1}{r} \left(\frac{R}{r} \right)^{i} P_i(\cos\psi)$$

$$S(\psi, r) = \left(\frac{R}{r} \right)^{i+1} \frac{2i+1}{i-1} P_i(\cos\psi),$$

so on the sphere : $R / r = 1$

$$S(\psi, R) = \frac{1}{\sin(\psi/2)} + 15\cos\psi - 6\sin(\psi/2)$$

$$-3\cos\psi \cdot \ln(\sin(\psi/2) + \sin^2(\psi/2))$$

Stokes formula.

We have solved a boundary value problem for an elliptic partial differential equation, but

Note singularity:

$$\frac{1}{\sin(\psi / 2)} \to \infty$$

when

$$\psi \to 0$$

Integration must be done with special method.

Use of Stokes formulae.

Height-anomaly/geoid height:

$$\zeta(\varphi,\lambda,r)=\frac{R}{\gamma\cdot 4\pi}\iint_S \Delta g\cdot S(\psi,r)dS$$

Deflections of the vertical, Vening-Meinesz, 1928:

$$\xi(\varphi,\lambda,r)=\frac{1}{\gamma\cdot 4\pi}\iint_S \Delta g\cdot\frac{\partial}{\partial\varphi}S(\psi,r)dS$$

$$\eta(\varphi,\lambda,r)=\frac{1}{\gamma\cdot 4\pi\cos\varphi}\iint_S \Delta g\cdot\frac{\partial}{\partial\lambda}S(\psi,r)dS$$

Stokes formula.

Due to the singularity most of the contribution comes from data with a short distance from the computational-point.

A subtraction of a spherical harmonic model (EGM) to N=360, then only very small area is needed.

$$\zeta(\varphi,\lambda,r) =$$

$$\zeta_{EGM}(\varphi,\lambda,r) + \frac{R}{\gamma \cdot 4\pi}\iint_{S_0}(\Delta g - \Delta g_{EGM}) \cdot S(\psi,r)\,dS$$

Note: EGM must be added back later !

Stokes formula, plane Earth.

If we want to compute the integral locally, we may regard the Earth as plane.

Integral may be computed using Fourier-transformation (convolution) in the plane, where the spectral relation (i-1)/r is used.

$$\zeta(x,y,R) = \zeta(z,R)_{EGM} + \int \Delta g(t) S(z-t)\,dt$$

Developed by R.Forsberg and M.Sideris in 1986.

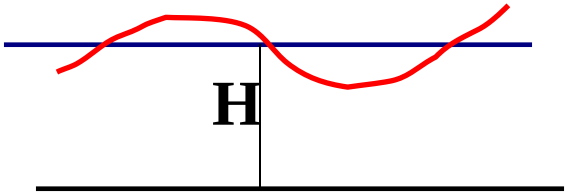
R.Forsberg/CCT: subtract (and add) also local topography

Stokes on the sphere.

Remove masses T_M over altitude 0 (adder back later)

Gravity anomali must then be computed in altitude zero 0 ("downward continuation")

Simpel metod to remove: Bouger-plate removed in each point:

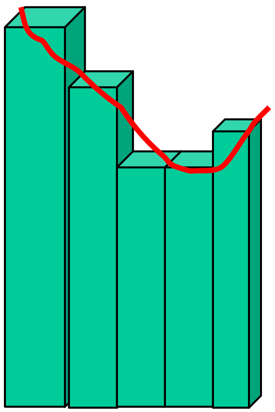


$$2\pi G\rho H \approx 0.11 \quad mgal/m$$

Not good because $T-T_M$ not harmonic anymore.

Removal of residual topography, Torge 6.5.

Rectangular prisms better:



May be refined with sloping boundaries – more details in R.Forsberg’s lectures.

Boundary value problem for T.

For Harmonic functions, the functionen may be determined from its values on the boundary

Chaucy's boundary value problem

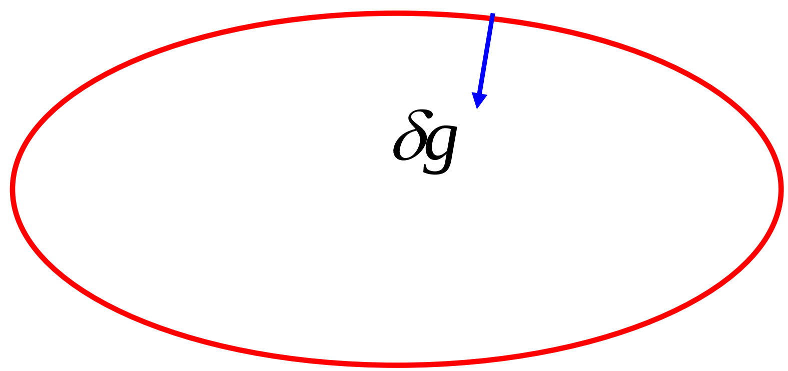
Or from the normal-derivative on the surface

Neumanns-randværdi-opgave

Side condition:

$$\rightarrow 0 \quad for$$

$$r \rightarrow \infty$$



Pizettis formula.

We use:

$$\frac{GM}{R}\Delta\overline{C}_{ij} = \frac{1}{4\pi}\iint_S T(\varphi',\lambda',R)\cdot Y_{ij}(\varphi',\lambda')dS$$

$$T(\varphi,\lambda,r) = \sum_{i=2}^{\infty}\frac{GM}{r}\left(\frac{R}{r}\right)^{i}\sum_{j=-i}^{i}Y_{ij}(\varphi,\lambda)\Delta\overline{C}_{ij}$$

$$= \sum_{i=2}^{\infty}\frac{1}{r}\left(\frac{R}{r}\right)^{i}\sum_{j=-i}^{i}Y_{ij}(\varphi,\lambda)\frac{1}{4\pi}\iint T(\varphi',\lambda',R)Y_{ij}(\varphi',\lambda')dS$$

$$= \frac{1}{4\pi}\iint\left[T(\varphi',\lambda',R)\sum_{i=2}^{\infty}\left(\frac{R}{r}\right)^{i+1}\sum_{j=-i}^{i}Y_{ij}(\varphi,\lambda)Y_{ij}(\varphi',\lambda')\right]dS$$

Poissons Integral.

We use Legendre-polynomials expressed using Legendre-functions, and the summation formula for 1/(distance):

$$T(\varphi,\lambda,r)=\frac{R^2(r^2-R^2)}{4\pi}\iint_S \frac{T(\varphi',\lambda',R)}{l^3}dS$$

Solves Cauchys boundary value-problem for sphere for harmonic function, regular at infinity.

In the same way are $r\cdot\Delta g$ harmonic and may be upward continued or gives integral-equation so that gravity on the sphere can be determined.

Evaluation of the integrals.

(1) T or Δg known or "estimated"= "Predicted" everywhere on the sphere.

Collocation useful metod !

(2)Possible to use functional connection between data and T for a direct estimation of T using collocation

$$T(P) = \sum_{i=1}^{N} a_i \cdot COV(T(P), observation(i)) =$$

$$\{observation(i)\}^T \{COV(obs(i), obs(j) + \sigma_{ij}\}^{-1}$$

$$\{COV(T(P), observation(i))\}$$

Solution in Hilbertspace.

Hilbertspace of harmonic functions has reproducing kernel (identity-mapping):

$$T(P) = \left(T(Q), K(P,Q)\right),$$

$$\left(\ldots,\ldots\right) = inner\ \ product\ in\ space$$

$$If \quad F_i\ orthonormal\ basis, then$$

$$T(P) = \sum_{i=0}^{\infty} C_i \cdot F_i(P)$$

$$K(P,Q) = \sum_{i=0}^{\infty} F_i(P) \cdot F_i(Q)$$

Solution in Hilbertspace.

Equivalent with statistical Collocation there is a functional-analytic formulation, COV(T(P),T(Q))=K(P,Q).

We want to find an approximation, which agree with noise-free data:

$$L_i(T) = L_i(\tilde{T}) = observation(i) \quad \text{(without noise)}$$

Solution with minimum norm in the Hilbert-space:

$$\tilde{T}(P) = \sum_{i=1}^{N} b_i L_i(K(P,\cdot))$$

$$\{b_i\} = \{L_i L_j K(\cdot,\cdot)\}^{-1} \{observation(j)\}$$

Isotropic inner product

R - rotation of the full space. Inner product isotropic if

(F,G)=(F(R),G(R)). Reproducing kernel then like covariance function:

$$K(P,Q) = K(r, r', \psi) =$$

$$\sum_{i=0}^{\infty} \sigma_i \left(\frac{R^2}{rr'} \right)^{i+1} P_i(\cos \psi)$$

Isotropic inner product, L_2 on the sphere.

Basis-functions: Normalized spherical harmonics.

$$K(P,Q) = K(\varphi, \lambda, r, \varphi', \lambda', r') =$$

$$\sum_{i=0}^{\infty} \left(\frac{R^2}{rr'} \right)^{i+1} \sum_{j=-i}^{i} Y_{ij}(\varphi, \lambda) Y_{ij}(\varphi', \lambda') =$$

$$\sum_{i=0}^{\infty} \left(\frac{R^2}{rr'} \right)^{i+1} (2i+1) P_i(\cos\psi)$$

Not usefull, because infinite on the sphere for distance zero.

= variancen of potentialet is infinite on the sphere.

Closed expression for kernl possible !

Degree-variancers which give finite values of potential and gravity variance on sphere:

$$\sigma_i = \left(\frac{R_B^2}{R^2}\right)^{i+1} \frac{A}{(i-1)(i-2)(i+24)}$$

$R_B < R$, radius for Bjerhammar-sphere

Collocation with parameters.

Data may be biased – i.e. a constant or linear function has been added to data due to errors in data-collection or because local height datum has been used.

$$l = \text{observation vector}, \quad X \ \text{parameter vector}$$

$$A = \text{linearized observation} - \text{matrix}$$

$$s = \text{signal (gravity field)},$$

$$L(s) \ \text{observation vektor}, \ n = \text{noise}$$

$$l = AX + L(s) + n$$

Collocation with parameters.

Covariances or functionals applied on Kernel:

$$\overline{C} = \left\{ COV(obs_i, obs_j) + \sigma_{ij} \right\}, n \times n \text{ matrix}$$

$$C_P = \left\{ COV(T(P), obs_i) \right\}, n \text{ vector}$$

$$\tilde{X} = (A^T \overline{C}^{-1} A)^{-1} (A^T \overline{C}^{-1} l)$$

$$\tilde{s}(P) = \tilde{T}(P) = C_P^T \cdot \overline{C}^{-1} \cdot (l - AX)$$

Error-estimates may also be computed !

Integrated geodesy.

One may mix gravity data with geometrical data.

Geodetic Earth-models

If only finite number of coefficients used (N) for T and M other data,

then we get traditional least-squares solution withN+M unknowns. Typisk N=73*73 eller 361*361. M=300*3+100 tide-parametre+GM, a, f +datum-shift.

WGS84: NIMA (US DoD),

GEM: Goddard (NASA) Earth Model serie

GRIM: French/German model series

EIGEN: German/French/US

Only gravity: OSU91, EGM96, EGM2008.